ABSOLUTE VALUE – THE BASICS

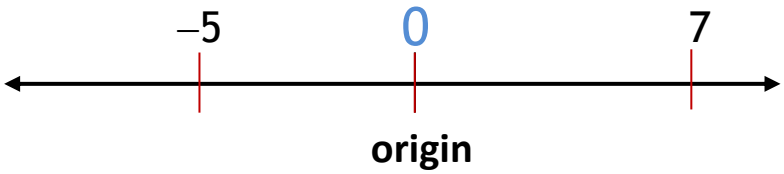


Sometimes the "size" of a number is more important than the "sign" of the number, that is, whether it's positive, zero, or negative. To measure the size of a number (without regard to its sign), we consider the ***absolute value*** of that number. There are two equivalent approaches to the meaning of absolute value.

❑ *TWO WAYS OF LOOKING AT ABSOLUTE VALUE*

1) Geometry

On a number line, the ***absolute value*** of a number (any kind of real number) is the ***distance*** from that number to **0** (the origin) on the line.



What is the distance from the number **7** to 0? It's 7. So we say that the absolute value of 7 is **7**.

How far from the origin is the number **–5**? It's 5 units away, and thus the absolute value of –5 is **5**.

What is the distance from the origin to **2π**? The distance is 2π, and we conclude that the absolute value of 2π is **2π**.

What is the distance between the number **0** and the origin? It's 0; conclusion: the absolute value of 0 is **0**.

Notes on Distance:

First, we can calculate the distance from any number to 0 (the origin). In other words, the distance between the origin and any point on the line always exists, even if it's hard or impossible to calculate.

Second, distance must be greater than or equal to 0; that is, it's never negative. If ***d*** represents distance, then $\boldsymbol{d \geq 0}$.

2) Arithmetic

If a number is positive, the *absolute value* of the number is the same number. For example, the absolute value of 9 is **9**.

If a number is negative, the *absolute value* of the number is the opposite of the number. For example, the absolute value of –3 is **3**.

And the *absolute value* of 0 is simply **0**.

Put more simply, the absolute value of a number greater than or equal to 0 is the same number; if the number is negative, just take the opposite of the number and you've got the absolute value.

❑ *NOTATION*

Instead of writing the words *absolute value* all the time, we do what we always do in math: condense the concept into symbols. To represent the absolute

In computers, the *absolute value* of *n* would be written

abs(*n*)

value of a number, we put a vertical bar on each side of the number. Thus,

The absolute value of 9 is 9: $|9| = \mathbf{9}$

The absolute value of –3 is 3: $|-3| = \mathbf{3}$

The absolute value of 0 is 0: $|0| = \mathbf{0}$

Just for repetition: If a number is greater than or equal to 0, then its absolute value is that same number. If a number is less than 0 (which means it's a negative number), then its absolute value is the opposite of that number (which will then turn into a positive number). We can write this in the following way; it looks confusing, but it's totally valid:

If $x \geq 0$, then $|x| = x$

If $x < 0$, then $|x| = -x$

This definition ensures that the absolute value of a quantity is never negative.

Here are some more examples of absolute value:

$|17.5| = 17.5$ $\quad |0| = 0$ $\quad |-13| = 13$

$|\pi| = \pi$ $\quad |-\sqrt{7}| = \sqrt{7}$ $\quad |-3\pi| = 3\pi$

Bottom Line:

The ***absolute value*** of a quantity is either positive or zero:

If x is ANY quantity, then $|x| \geq 0$.

In other words, the absolute value of a quantity is never negative.

For example, consider the following expression:

$$\left|\sin^2(\pi/6) - \ln(e-1)\right|$$

Even though you may not be able to calculate it until Pre-calculus, you should still be able to understand that the answer to this problem — whatever it is — is greater than or equal to zero. Equivalently, the answer is not negative.

EXAMPLE 1: **Evaluate each expression:**

A. $|7-2| = |5| = \mathbf{5}$

B. $|3^2-15| = |9-15| = |-6| = \mathbf{6}$

C. $-|2-7| = -|-5| = \mathbf{-5}$

D. $|-3-4|-|10-2| = |-7|-|8| = 7-8 = \mathbf{-1}$

Homework

1. The absolute value of any number is _____.

2. If x represents any number, then

   a. $|x|>0$ b. $|x|<0$ c. $|x|\geq 0$ d. $|x|\leq 0$

3. Which one of the following inequalities has NO solution?

   a. $|x|>0$ b. $|x|<0$ c. $|x|\geq 0$ d. $|x|\leq 0$

4. True/False:

   a. Every number has an absolute value.
   b. There is a number whose absolute value is 0.
   c. There is a number whose absolute value is negative.
   d. There are two different numbers whose absolute value is 9.

5. Simplify each expression:

   a. $|4-14|$ b. $|2(-3)-7|$ c. $-|-9|$ d. $|2^0-\pi^0|$

6. Simplify each expression:

   a. $|7\pi|$ b. $|-7\pi|$ c. $|\sqrt{67}|$ d. $|-\sqrt{2.7}|$ e. $|0^{3\pi}|$

7. What's the best statement you can make regarding the expression

$$\left|\frac{\tan x-\sin x}{e^{\int \sec x\,dx}}\right| ?$$

8. Consider the statement: $|a\cdot b| = |a|\cdot|b|$

   a. Give three examples where the statement is true, using

      i) two positive numbers.
      ii) two negative numbers.
      iii) two numbers of opposite sign.

   b. Do you think the statement is true or false?

9. Consider the statement: $|x+y| = |x|+|y|$

   a. Give an example where the statement is true.

   b. Give an example where the statement is false.

   c. So, is the statement true or false?

10. Consider the statement: $|x+y| < |x|+|y|$

   a. Give an example where the statement is true.

   b. Give an example where the statement is false.

   c. Is the statement true or false?

11. [Hard] Try to use the previous two problems to come up with a statement that is TRUE.

The *absolute value* of a positive number is *itself*.

The *absolute value* of 0 is 0.

The *absolute value* of a negative number is its *opposite*.

Solutions

1. "greater than or equal to 0"
OR "never negative"
OR " ≥ 0 "

2. c.

3. b.

4. a. T b. T c. F d. T

5. a. 10 b. 13 c. –9 d. 0

6. a. 7π b. 7π c. $\sqrt{67}$ d. $\sqrt{2.7}$ e. 0

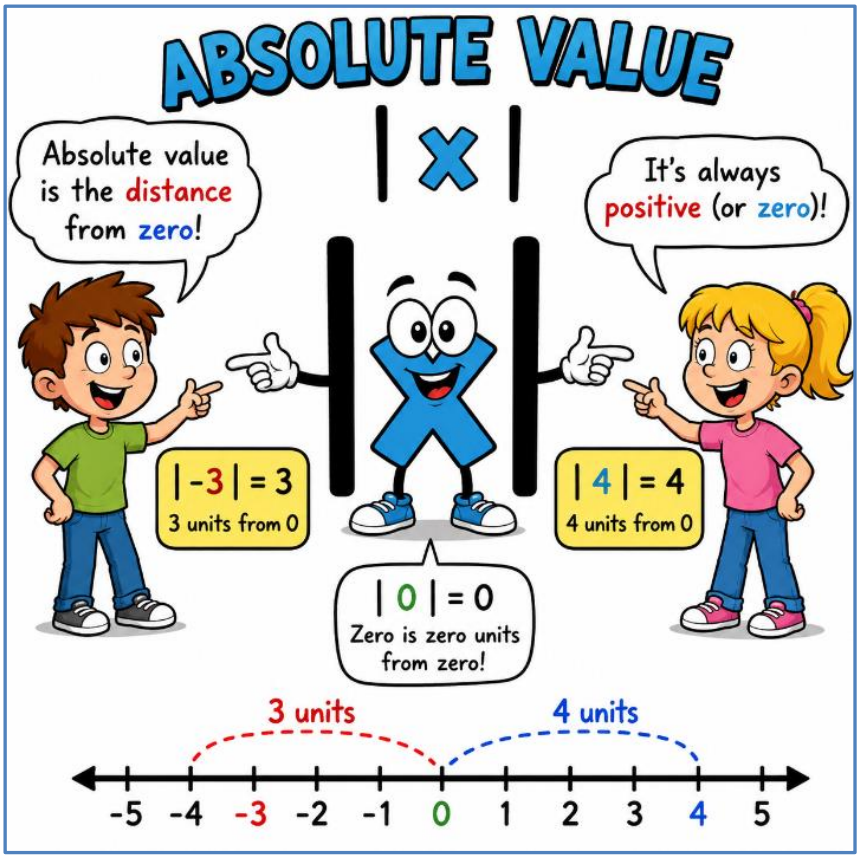
7. If you don't know enough algebra and trig, the best you could say is that, whatever legal value of x you place in the expression, the result WON'T BE NEGATIVE. That is, the expression represents something that is GREATER THAN OR EQUAL TO 0.

8. Although a whole slew of examples does not prove that the statement is true, this particular statement is always true.

9. Recall that if a statement fails in just one case, it's considered a false statement.

10. Again, if a statement fails in just one case, it's considered a false statement.

11. I will not give the answer away, but it's called the *Triangle Inequality*, and is one of the most important rules in math.



"Who questions much,
shall learn much,
and retain much."

– Francis Bacon



ABSOLUTE VALUE EQUATIONS

An ***absolute value equation*** is just what it sounds like: an equation where the variable is contained within an absolute value sign. An example is the equation $|z+5| = 11$.



But notice that the equation $x+|3| = 99$ (whose solution is $x = 96$) is <u>not</u> an absolute value equation because the variable, *x*, is not contained within the absolute value sign.

What does it mean for something to be a *solution* of an absolute value equation? Just what it's always meant: When that something is substituted into the equation, both sides of the equation result in the same number; both sides balance.

Homework

1. Is $x = 3$ a solution of the absolute value equation $|x| = 3$?
2. Is $n = -9$ a solution of the absolute value equation $|n| = 9$?
3. Is $c = -9$ a solution of the absolute value equation $|c| = -9$?
4. Is $a = 2$ a solution of the absolute value equation $|a+7| = 9$?
5. Is $w = -10$ a solution of the absolute value equation $|w-7| = 17$?
6. Is $y = -3$ a solution of the absolute value equation $|y-5| = 2$?
7. Show that $x = 5$ <u>and</u> $x = -5$ are both solutions of $|x| = 5$.

8. Show that $z = 6$ and $z = -16$ are both solutions of $|z+5| = 11$.
9. Explain why there is NO solution of the equation $|q-6| = -10$.
10. Show that both $n = 3$ and $n = -3$ are solutions of $|n|-10 = -7$.

❑ *EXAMPLES OF ABSOLUTE VALUE EQUATIONS*

EXAMPLE 1: **Solve for x: $|x| = 12$**

Solution: What can you take the absolute value of, and get a result of 12? Well, the absolute value of 12 is 12, so x could be 12. But -12 also has an absolute value of 12, so x could be -12, also. In other words, since $|12| = 12$ and $|-12| = 12$, it appears that this equation has two solutions, 12 and -12.

$x = 12, -12$

EXAMPLE 2: **Solve for n: $|n| = -5$**

Solution: This equation is a statement that the absolute value of some number is -5. But the absolute value of any number is greater than or equal to zero; that is, the absolute value of any number can never be negative. [See: Absolute Value, the Basics] Thus, there is no number n that will work in this equation. Our conclusion:

No solution

EXAMPLE 3: **Solve for w: $|w| = 0$**

Solution: What number has an absolute value of 0? There's only one such number, and it's 0: $|0| = 0$. Therefore,

$$\boxed{w = 0}$$

EXAMPLE 4: **Solve for x: $|8-2x| = 20$**

Solution: Here's what we ask ourselves: "What has an absolute value of 20?" There are two numbers that have an absolute value of 20, and they are 20 and –20. This means that the entire quantity inside the absolute value sign must be either 20 or –20. In other words, the quantity $8 - 2x$ must be either 20 or –20.

This gives us two equations to solve:

$8 - 2x = 20$	$8 - 2x = -20$
$-2x = 12$	$-2x = -28$
$x = -6$	$x = 14$

Our absolute value equation has two solutions:

$$\boxed{x = -6, 14}$$

Check:

$x = -6$: $|8-2x| = |8-2(-6)| = |8+12| = |20| = 20$ ✓

$x = 14$: $|8-2x| = |8-2(14)| = |8-28| = |-20| = 20$ ✓

Homework

11. Solve each absolute value equation:

a. $|t| = 4$ b. $|n| = 0$ c. $|R| = -1$

d. $|x+1| = 9$ e. $|x-3| = 5$ f. $|2w+8| = 0$

g. $|2-5x| = -3$ h. $|2x+8| = 10$ i. $|3y-6| = 9$

j. $|2a+1| = 19$ k. $|7x+\sqrt{7}| = -\frac{\pi}{2}$ l. $|n-4| = 9$

m. $|2x-17| = 0$ n. $|2a+3| = 21$ o. $|\sqrt{5}-\sqrt{7y}| = -\sqrt{\pi}$

The absolute value of a positive number is itself.

The absolute value of 0 is 0.

The absolute value of a negative number is its opposite.

Solutions

1. Yes

2. Yes

3. No; $|-9| = 9$, not –9

4. Yes

5. Yes; $|-10-7| = |-17| = 17$

6. No; letting $y = -3$ results in $8 = 2$.

7. Each of the numbers works in the equation.

8. Plug each one in (one at a time!), and the equation is satisfied.

9. The absolute value of any quantity is always ≥ 0.

10. Plug them in and see what happens.

11. a. $t = 4, -4$ b. $n = 0$ c. No solution

d. $x = 8, -10$ e. $x = 8, -2$ f. $w = -4$

g. No solution h. $x = 1, -9$ i. $y = 5, -1$

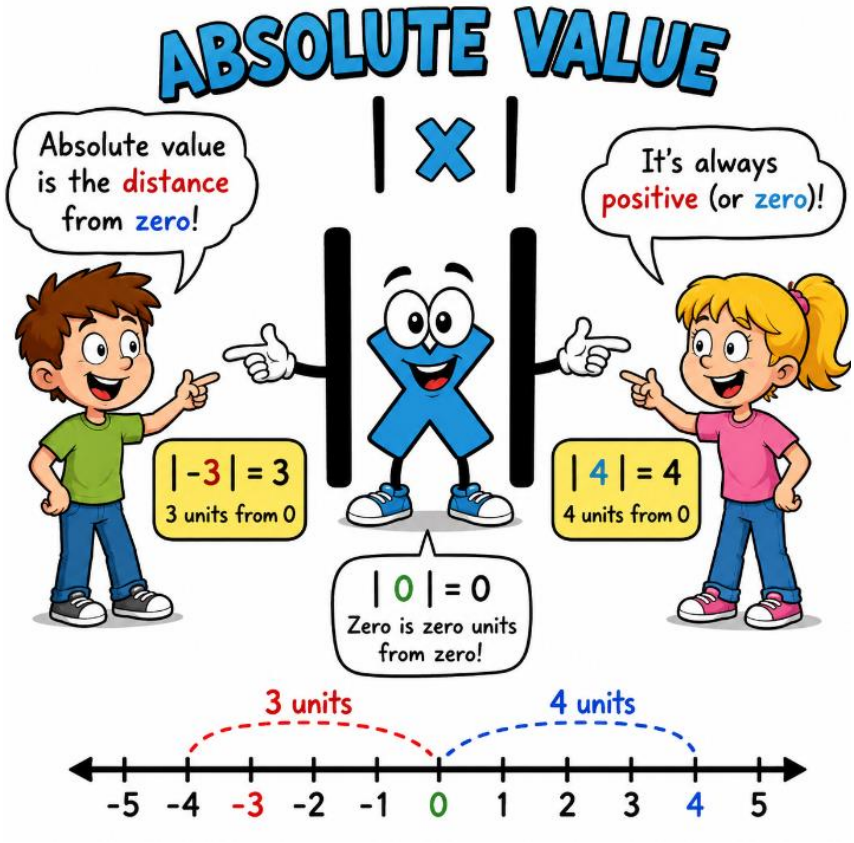
j. $a = 9, -10$ k. No solution l. $n = 13, -5$

m. $x = \frac{17}{2}$ n. $a = 9, -12$ o. No solution

The Official Definition of Absolute Value:

If $x \geq 0$, then $|x| = x$

If $x < 0$, then $|x| = -x$



Live as if you were to die tomorrow.

Learn as if you were to live forever."

Mahatma Gandhi (1869–1948)



GRAPHING ABSOLUTE VALUES



It's time to do more graphing — of course, this time there will be absolute values in the formula.

EXAMPLE 1: **Graph: $y = |x|$**

Solution: Pick some x-values, calculate the corresponding y-values, and plot the points:

$x = -2 \Rightarrow y = |-2| = 2 \Rightarrow$ **(–2, 2)**

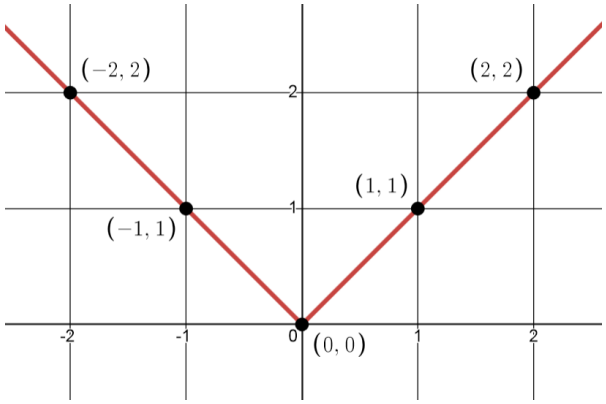
$x = -1 \Rightarrow y = |-1| = 1 \Rightarrow$ **(–1, 1)**

$x = 0 \Rightarrow y = |0| = 0 \Rightarrow$ **(0, 0)**

$x = 1 \Rightarrow y = |1| = 1 \Rightarrow$ **(1, 1)**

$x = 2 \Rightarrow y = |2| = 2 \Rightarrow$ **(2, 2)**

We've got five points, enough to make a decent picture:



$y = |x|$

Convince yourself that our graph also contains the points (165, 165) and (–499, 499).

EXAMPLE 2: **Graph: $y = |x - 3| + 2$**

Solution: Don't panic! Let's just do what we did in the first example: Pick x-values off the top of our head, calculate the associated y-values, plot the points (x, y), and see what we get.

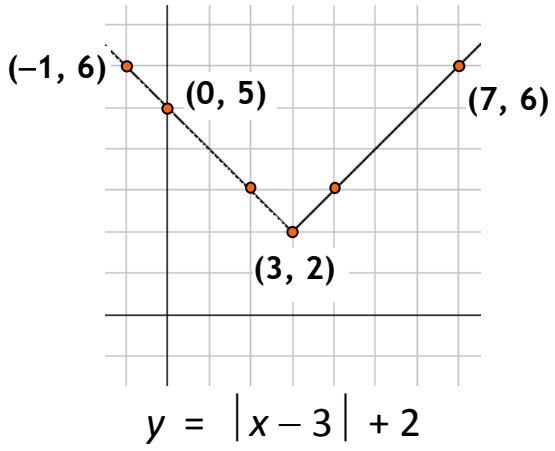
Setting x to 0 gives $y = |0 - 3| + 2 = |-3| + 2 = 3 + 2 = \mathbf{5}$.
We now have our first point on the graph: (0, 5).

Now set x to 7: $y = |7 - 3| + 2 = |4| + 2 = 4 + 2 = \mathbf{6}$.
Our new point on the graph is (7, 6).

One more — set x to –1: $y = |-1 - 3| + 2 = |-4| + 2 = 4 + 2 = \mathbf{6}$.
Our third point is (–1, 6).

We'll now construct an x-y table using the three points we've just calculated and some additional points; it's your job to verify the rest of the points.

x	y
–1	6
0	5
1	4
2	3
3	2
4	3
5	4
6	5
7	6

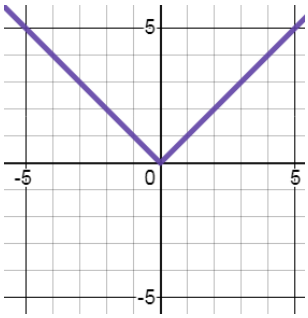


Items of Note:

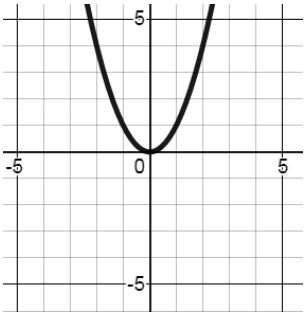
Later in your course, you will be asked what x-values are allowed in this formula. It may not be obvious, but x can take on any value; that is, x can be any real number.

As for y-values, the graph shows that y can't be any smaller than 2. That is, y has to be at least 2; we could even write $y \geq 2$.

The graph is certainly not a line; it's in the shape of a V. It is sharp at its bottom point (3, 2), not smooth and curvy like the graph of the parabola $y = x^2$ that you may have seen before.



$y = |x|$



$y = x^2$

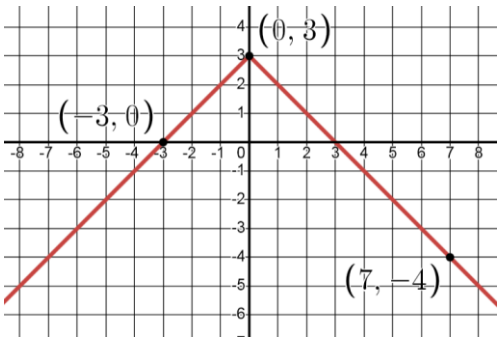
EXAMPLE 3: **Graph: $y = -|x| + 3$**

Solution: Setting x to 0 gives $y = -|0| + 3 = -0 + 3 = \mathbf{3}$.
We now have our first point on the graph: (0, 3).

Now set x to 7: $y = -|7| + 3 = -7 + 3 = \mathbf{-4}$.
Our new point on the graph is (7, –4).

One more: set x to –3: $y = -|-3| + 3 = -3 + 3 = \mathbf{0}$.
Our third point is (–3, 0).

Plotting these 3 points gives us a good idea of the graph (although more points would be a great idea).



Homework

1. Graph: $y = |x|$

2. Graph: $y = |x| + 3$

3. Graph: $y = |x| - 4$

4. Graph: $y = -|x + 2|$

5. Graph: $y = |x - 5|$

6. Graph: $y = -|x + 3| - 4$

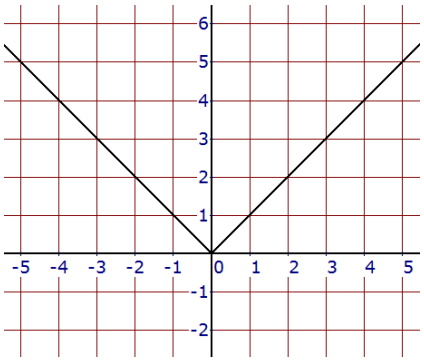
The absolute value of a positive number is itself.

The absolute value of 0 is 0.

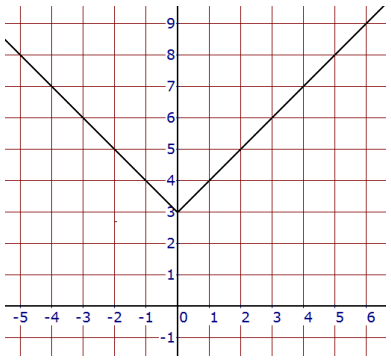
The absolute value of a negative number is its opposite.

Solutions

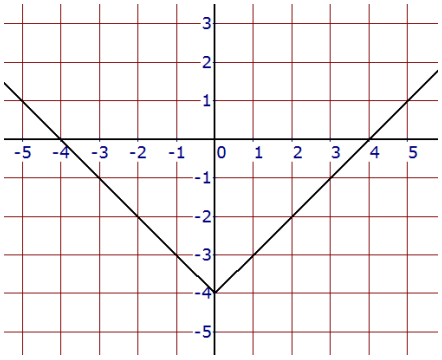
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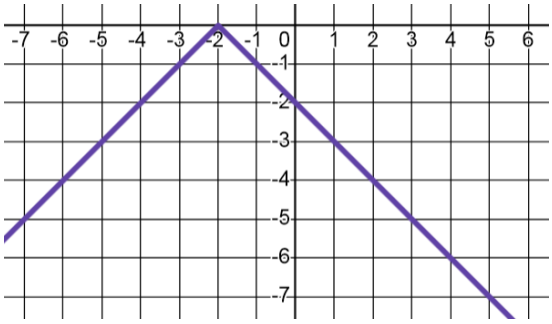
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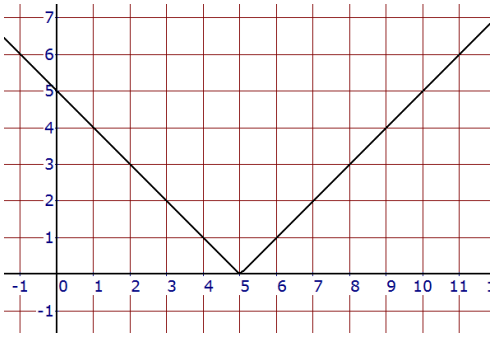
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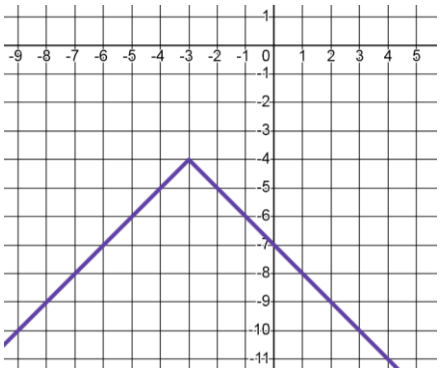
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5.



6.

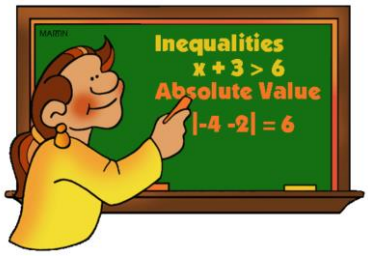


"The aim of education is the knowledge, not of facts, but of values."

— William S. Burroughs

ABSOLUTE-VALUE INEQUALITIES

For linear inequalities (like $7n + 5 < 2$), the Basic Principle of Inequalities is sufficient (If you multiply or divide by a negative number, switch the inequality sign). But what about an inequality like



$$|3x - 4| + 8 \leq 20?$$

We will use the Boundary Point Method, discussed in the chapter Quadratic Inequalities.

❑ *THE BOUNDARY POINT METHOD*

1) Change the inequality to an *equation* and solve the equation.

2) The solutions of that equation are called the *boundary points*.

3) Plot the boundary points on the real line. This will create one or more *intervals* on the line.

4) Now choose a *test point* from each interval (any convenient number inside the interval) and plug it into the original inequality. If the test point satisfies the inequality, then the entire interval in which the test point lies is a part of the solution of the inequality. If the test point does not satisfy the inequality, then the associated interval is not part of the solution.

5) Lastly, check the boundary points themselves in the inequality.

❑ *ABSOLUTE VALUE INEQUALITIES*

EXAMPLE 1: **Solve for x: $|2x - 6| - 8 \geq 0$**

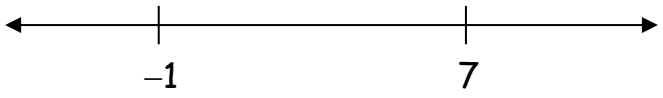
Solution: We do this by following the five steps of the Boundary Point Method.

1. Solve the associated equation:

$$|2x - 6| - 8 = 0$$
$$\Rightarrow \quad |2x - 6| = 8$$
$$\Rightarrow \quad 2x - 6 = 8 \quad \text{OR} \quad 2x - 6 = -8$$
$$\Rightarrow \quad 2x = 14 \quad \text{OR} \quad 2x = -2$$
$$\Rightarrow \quad \mathbf{x = 7 \quad OR \quad x = -1}$$

2. The boundary points are therefore 7 and –1.

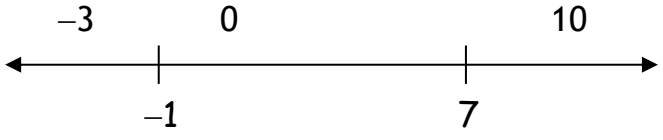
3.



Plotting the two boundary points produces three intervals:

$(-\infty, -1)$ $(-1, 7)$ $(7, \infty)$

4.



We have chosen three test points: **–3**, **0**, and **10**, one from each interval. We now test each point in the original inequality, $|2x - 6| - 8 \geq 0$:

$\lvert 2(\mathbf{-3}) - 6\rvert - 8 \geq 0$	$\lvert 2(\mathbf{0}) - 6\rvert - 8 \geq 0$	$\lvert 2(\mathbf{10}) - 6\rvert - 8 \geq 0$
$\lvert -6 - 6\rvert - 8 \geq 0$	$\lvert 0 - 6\rvert - 8 \geq 0$	$\lvert 20 - 6\rvert - 8 \geq 0$
$\lvert -12\rvert - 8 \geq 0$	$\lvert -6\rvert - 8 \geq 0$	$\lvert 14\rvert - 8 \geq 0$
$12 - 8 \geq 0$	$6 - 8 \geq 0$	$14 - 8 \geq 0$
$4 \geq 0$	$-2 \geq 0$	$6 \geq 0$
True	False	True
$(-\infty, 1)$ <u>is</u> part of the solution.	$(-1, 7)$ is <u>not</u> part of the solution.	$(7, \infty)$ <u>is</u> part of the solution.

5. Last, we check the boundary points themselves.

   If you substitute -1 into the inequality $\lvert 2x - 6\rvert - 8 \geq 0$, it should work out to $0 \geq 0$, which is true. Thus, -1 <u>is</u> part of the solution. Similarly, checking the boundary point 7 will also result in $0 \geq 0$, so 7 <u>is</u> part of the solution.

Our final answer comprises the two intervals from Step 4 and the two boundary points confirmed in Step 5.

$$\boxed{(-\infty, -1] \cup [7, \infty)}$$

In English, this means that any number which is either ≤ -1 **or** ≥ 7 will work in the original inequality.

<u>EXAMPLE 2:</u> **Solve for n: $2 - \lvert 8 - 2n\rvert > 0$**

<u>Solution:</u>

1. Solve the associated equation:

$$\begin{aligned} & 2 - \lvert 8 - 2n\rvert = 0 \\ \Rightarrow \quad & -\lvert 8 - 2n\rvert = -2 && \text{(subtract 2 from each side)} \\ \Rightarrow \quad & \lvert 8 - 2n\rvert = 2 && \text{(multiply each side by } -1\text{)} \end{aligned}$$

$\Rightarrow \quad 8 - 2n = 2 \quad \text{OR} \quad 8 - 2n = -2$

$\Rightarrow \quad n = 3 \quad \text{OR} \quad n = 5$

2. The boundary points are **3** and **5**.

3. The three intervals are therefore $(-\infty, 3)$, $(3, 5)$, and $(5, \infty)$.

4. Choose the test point $0 \in (-\infty, 3)$. Plugging 0 into the inequality $2 - |8 - 2n| > 0$ gives $-6 > 0$. Therefore, $(-\infty, 3)$ is <u>not</u> part of the answer.

   Choose the test point $4 \in (3, 5)$. Plugging 4 into the inequality $2 - |8 - 2n| > 0$ gives $2 > 0$. Thus, $(3, 5)$ <u>is</u> part of the answer.

   Choose the test point $9 \in (5, \infty)$. Plugging 9 into the inequality $2 - |8 - 2n| > 0$ gives $-8 > 0$, which implies that $(5, \infty)$ is <u>not</u> part of the answer.

5. You check the boundary points, 3 and 5, in the inequality. Neither of them works.

Our final answer consists of the middle interval <u>without</u> the boundary points. This can be written $\{n \in \mathbb{R} \mid 3 < n < 5\}$, or,

$$\boxed{(3, 5)}$$

Homework

Solve each absolute value inequality:

1. $|3a - 7| - 4 \geq 0$
2. $5 - |3 - c| > 0$
3. $-2 + |4y - 6| \leq 0$
4. $|3x - 9| - 18 \geq 0$

5. $|w + 7| \leq 10$

6. $|a + 1| - 4 > 0$

Review Problems

Solve each inequality:

7. $2(x - 5) - 9(2x + 1) > 0$

8. a. $|3x - 15| \geq 30$
   b. $|x - 5| > 0$
   c. $|9 - 3x| \leq 0$
   d. $|2x + 6| \geq 0$
   e. $|3 - x| < 0$
   f. $2|3x - 12| + 8 < 20$

9. $\frac{x}{a} > c$ Hint: We know a can't be 0 in the problem, but do we know for sure whether $a > 0$ or $a < 0$?

10. $|4x + 12| < 20$

11. True/False:

    a. When solving a linear inequality, one needs to reverse the inequality symbol only when dividing each side by a negative number.

    b. The solution of the inequality $ax \leq b$ is $x \leq \frac{b}{a}$.

    c. The solution of $|2g - 6| \geq 8$ is $(-\infty, -1] \cup [7, \infty)$.

    d. The solution of $|7 - 3x| < 0$ is $\mathbb{R}$.

    e. The solution of $|8y + 17| \geq 0$ is $\mathbb{R}$.

    f. The solution of $|x - 10| < 25$ is $[-15, 35]$.

Solutions

1. $(-\infty, 1] \cup \left[\frac{11}{3}, \infty\right)$ **2**. $(-2, 8)$ **3**. $[1, 2]$

4. $(-\infty, -3] \cup [9, \infty)$ **5**. $[-17, 3]$ **6**. $(-\infty, -5) \cup (3, \infty)$

7. $\left(-\infty, -\frac{19}{16}\right)$

8. a. $(-\infty, -5] \cup [15, \infty)$ b. $(-\infty, 5) \cup (5, \infty)$; or $\mathbb{R} - \{5\}$

c. $\{3\}$ d. $\mathbb{R}$ e. $\varnothing$ f. $(2, 6)$

9. Not solvable — it depends on whether a is positive or negative.

10. $(-8, 2)$

11. a. F b. F c. T d. F e. T f. F

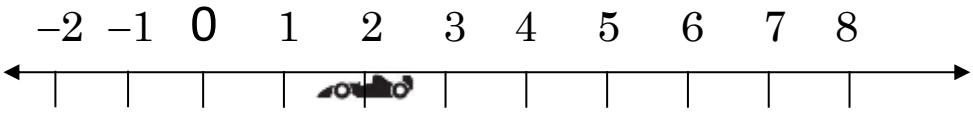
"Education breeds confidence.
Confidence breeds hope.
Hope breeds peace."

Confucius

AVERAGE VELOCITY

Now we look at a number line – a simple horizontal axis – and talk about some object moving left or right on the line.







It could be a racecar speeding down the backstretch or a subatomic proton hurtling through a linear accelerator.

❑ *NOTATION*

Throughout this chapter, we'll let ***t*** represent ***t****ime* and ***s*** represent the *po****s****ition* of the object on the line. We agree that $s > 0$ (positive) when the object is to the right of 0, and $s < 0$ (negative) on the left side of 0. It is also important that we consider the position, s, to be a function of time, t. In other words, as the time changes, the position of the object may also change. We can even write the position as ***s*(*t*)** [read: "***s*** of ***t***"] to express the view that the position, ***s,*** is a function of the time, ***t***.

In addition, to emphasize the math, we won't worry about units. But for your edification, if position is measured in meters from the origin (as either a positive number, negative number, or zero) and time is measured in seconds (common in physics), then the

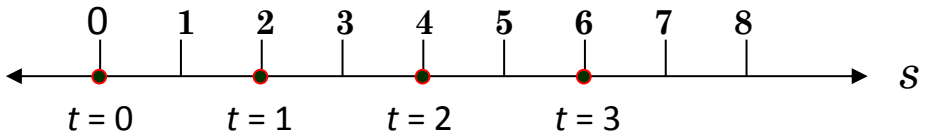
Speed v. Velocity

65 mph is *speed*.
65 mph To The Right is *velocity*.

unit of **velocity** is *meters per second*, or m/s, or $\frac{\text{m}}{\text{s}}$, followed by the direction "left" or "right."

❑ *TIME AND POSITION*

Consider the following real line (which we could call the ***s***-axis). Each dot on the line represents the **position** of an object at the moment of **time** specified, where t might be measured in seconds:



Here's what this little picture means. At the "beginning," when $t = 0$, the position is $s = 0$. One second later, when $t = 1$, the object is at $s = 2$. One second after that, when $t = 2$, s has the value 4. When $t = 3$, the object is at 6. Expressing these facts in function notation, we write:

$s(0) = 0$ $\quad$ $s(1) = 2$ $\quad$ $s(2) = 4$ $\quad$ $s(3) = 6$

[The notation ***s(t)*** is read "*s of t*."]

We can also express this number line picture with a table of values:

t	0	1	2	3
s	0	2	4	6

Our goal now is to predict where the object will be at some time in the **future.** For example, when $t = 10$, that is, 10 seconds after the object begins moving, where will it be? To do this, we ask the question: Given the data in the table, can we find a formula for the position of the object as a function of time? Looking either at the real line (the s-axis) or the table, it seems (at least from the data we have) that the position is always twice the value of time: ***s* is twice *t***. So we can write the formula

$$s(t) = 2t$$ [The position at time t is twice the t-value]

Thus, to predict the position s when $t = 10$, all we have to do is calculate $s(\mathbf{10})$, which is $2 \cdot \mathbf{10} = 20$. Therefore, after 10 seconds have passed, the object is at **position 20**. Notice that the object in this example never has a negative value of s, and that it moves to the right as the value of time increases.

Homework

1. The following table represents the position of a car at various moments in time. Predict the position of the car when $t = 850$.

t	1	2	3	4	5
s	4	7	12	19	28

❑ *AVERAGE VELOCITY*

Have you studied **delta** notation? A change in y, which might be written as the difference $y_2 - y_1$, can be expressed as Δy. In discussing velocity, we need the notion of a change in time, denoted by Δt, and as a consequence, a change in position, written Δs.

The symbol Δ is the Greek capital letter **delta**, and is a common symbol representing a **change** in something.

Now we're ready to define the concept of ***average velocity***:

average velocity = the change in position *divided by* the change in time.

It's a measure of the *net distance* traveled in a specified amount of time. We use the symbol $\bar{v}$ (v-bar) to represent average velocity:

$$\bar{v} = \frac{\Delta s}{\Delta t}$$

This formula is very similar to the formula involving distance, rate, and time:

$$r = \frac{d}{t}$$

Four Notes:

1. The change in position, Δs, does not take into account the actual distance traveled by the object. Suppose you and I are standing on the s-axis at the same spot, say $s = 10$. You run many miles to the left and then turn around and run back, ending at $s = 12$. Your change in position is $12 - 10 = \mathbf{2}$. I, being the lazy bum that I am, simply stroll directly from $s = 10$ to $s = 12$. My change in position is $12 - 10 = \mathbf{2}$, the same change in position as yours. We started at the same spot and we ended at the same spot, and so we each experienced the same change in position — our Δs's are the same (even though you ran a much farther distance than I did.)

2. The formula for average velocity may remind you of the definition of **slope**, $m = \frac{\Delta y}{\Delta x}$. This is no coincidence. Each of the concepts — *slope* and *average velocity* — is described by a ratio of changes, and each of them forms the foundation for the branch of math called *calculus*.

3. Just as the slope of a line can be negative, the average velocity may be negative. You'll see this in Example 2 below. By the way, this is one of the reasons that we refer to the theme of this chapter using the term *velocity* rather than speed.

4. You might notice at this point that your calculation for average velocity will involve two subtractions followed by a division. The order in which you perform the subtractions is up to you, but you MUST be consistent in that order. Nevertheless, tradition dictates that we subtract initial values from final values. For example, if an object is at $s = 3$ and then later is at $s = 10$, we agree to calculate $10 - 3$ rather than $3 - 10$. So our average velocity formula looks like

Velocity has direction; speed does not.

$$\bar{v} = \frac{\textit{final position} - \textit{initial position}}{\textit{final time} - \textit{initial time}}$$

EXAMPLE 1: **A bird takes off from position $s = 3$ when $t = 5$ and lands at position $s = 15$ when $t = 9$. Find the average velocity for the time interval $t = 5$ to 9.**

Solution: The average velocity is the change in position (the difference of the positions) divided by the change in time (the difference of the times):

$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{15-3}{9-5} = \frac{12}{4} = \boxed{3}$$

EXAMPLE 2: **A racecar is at position $s = 8$ when $t = 0$ and then drives to position $s = -10$ in 2 seconds. Find the average velocity during the 2-second interval.**

Solution: The racecar started when $t = 0$ and ended when $t = 2$. Using the formula for average velocity, we calculate

$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{-10-8}{2-0} = \frac{-18}{2} = \boxed{-9}$$

Why is the velocity negative?

Homework

2. A snail is at position $s = 22$ at $t = 4$ and moves to position $s = 50$ at $t = 18$. Find the average velocity, $\bar{v}$, for the time interval $t = 4$ to 18.

3. A rocket is at position $s = 1{,}000$ at $t = 20$ and rises to position $s = 5{,}500$ at $t = 65$. Find the average velocity, $\bar{v}$.

EXAMPLE 3: **Let $s(t) = t^2$. Calculate each of the following:**

a. $s(0)$ b. $s(20)$ c. $\bar{v}$ from $t = 5$ to $t = 15$

Solution:

a. $s(0)$ means let $t = 0$ and calculate s using the given formula:

$$s(0) = 0^2 = \mathbf{0}$$

b. s(20) is the position when $t = 20$:

$$s(20) = 20^2 = \mathbf{400}$$

t	s
0	0
5	25
15	225
20	400

c. i) $s(5) = 5^2 = 25$

ii) $s(15) = 15^2 = 225$

The average velocity, $\bar{v}$, is calculated as follows:

$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{225 - 25}{15 - 5} = \frac{200}{10} = \mathbf{20}$$

Homework

For each position function, calculate the average velocity, $\bar{v}$, during the given time interval:

4. $s(t) = 3t + 7$ $t = 3$ to $t = 10$

5. $s(t) = t^2 + 3t + 7$ $t = 1$ to $t = 5$

6. $s(t) = t^3$ $t = 3$ to $t = 5$

7. $s(t) = t^3 + t^2 + 3t + 7$ $t = 1$ to $t = 2$

8. $s(t) = \frac{1}{t}$ $t = 4$ to $t = 6$

9. $s(t) = \sqrt{t}$ $t = 9$ to $t = 36$

10. $s(t) = \frac{1}{\sqrt{t}}$ $t = 4$ to $t = 100$

11. $s(t) = 7$ $t = \pi$ to $t = 50$

12. For each position function, calculate the average velocity, $\bar{v}$, during the time interval from $t = 0$ to $t = 1$:

    a. $s(t) = at + b$

    b. $s(t) = at^2 + bt + c$

    c. $s(t) = at^3 + bt^2 + ct + d$

Solutions

1. 722,503

2. 2

3. 100

4. 3

5. 9

6. 49

7. 13

8. –0.04

9. 0.11

10. –0.004

11. 0

12. I want to see what you come up with.

"STRIVE FOR *progress,*
NOT *perfection.*"

BREAK-EVEN POINT – LINEAR FUNCTIONS



Certainly, the ultimate goal of a business is not merely to "break even." However, the **break-even point** is one of the most important concepts in business. It tells the business owner the point (in production, time, or investment) where losses have ended and profits will appear (or, unfortunately, the other way around). Every business that requires funding (either from investments, like selling stock, or the borrowing of money) requires a written business plan stating the projected break-even point.

❑ *BUSINESS TERMS*

A few business terms, with simplified definitions, will help us understand how algebra can represent the real world. We'll use the term ***Revenue*** to represent all the money that a company takes in through sales and services. The term ***Cost*** represents the money spent by the company to produce those sales and services. And we'll define ***Profit*** to be the <u>difference</u> between revenue and cost. We're ready to write a formula now:

Profit = Revenue – Cost

$$P = R - C$$

❑ *REVIEW OF INEQUALITIES*

The fact that 3 is **less than** 5 is written

$$3 < 5$$

If we want to say that x is **less than or equal to** 5, we write

$$x \leq 5$$

Similarly, we might write $10 > 8$ to say that 10 is **greater than** 8, and we write $n \geq 15$ to say that n is **greater than or equal to** 15. Notice that $7 \geq 7$ is a true statement because 7 is greater than or equal to 7. (Namely, it's equal to 7.)

Homework

1. a. Solve the profit formula $P = R - C$ for R.

   b. Solve the profit formula $P = R - C$ for C.

2. True/False:

a. $7 < 9$	b. $-2 < 7$	c. $-3 < -7$	d. $-9 < -1$
e. $8 \geq 8$	f. $0 > 9$	g. $0 > -9$	h. $-2 \leq -2$
i. $-2 < -2$	j. $0 < 0$	k. $0 \geq 0$	l. $-12 > -5$
m. $\pi > \sqrt{2}$	n. $2\pi \geq \pi$	o. $\sqrt{2} < \sqrt{3}$	p. $\sqrt{10} < \pi$

3. Consider the inequality $n \geq 7$. Which of the following values of n would make the statement true?

   a. 15 b. 7.001 c. 6.999 d. 7 e. 2 f. –3 g. –29 h. $\sqrt{40}$

4. Which of the following values of x will satisfy the inequality $x < -3$?

   a. -100 b. $-\pi$ c. -3 d. -2.999 e. 0 f. 3.14 g. 140

❑ *Definition of Break-Even*

There are two ways to define the ***break-even point***. One is to say that break-even occurs when revenue matches cost; that is, when $R = C$. On the other hand, if the revenue and cost are the same, then there's zero profit. (See that?) So break-even can also be defined as the point at which the profit is 0.

Either equation can be used to find the ***break-even point***:

$$\boldsymbol{R = C}$$

$$\boldsymbol{P = 0}$$

To prove that the profit must be zero when revenue = cost, we can use a little algebra. Assume that

	$R = C$	(one criterion for break-even)
$\Rightarrow$	$R - C = 0$	(subtract C from each side of the equation)
$\Rightarrow$	$\boldsymbol{P = 0}$	(substitute, since $P = R - C$)

❑ *USING A TABLE*

Assuming w represents the number of widgets a company manufactures and sells, let's assume the formula for revenue to be:

$$R = 4w$$

and a cost formula to go with it:

$$C = 2w + 8$$

The Revenue formula might come from the fact that we sell our widgets for $4 apiece.

The Cost formula may be the result of the fact that each widget costs $2 to manufacture, together with a fixed cost (salaries, rent, utilities, etc.) of $8.

Let's create a table with revenue, cost, and profit for various quantities of widgets sold, according to the given formulas.

w $\quad R = 4w \quad C = 2w + 8 \quad P = R - C$

Widgets	Revenue	Cost	Profit
0	$0	$8	–$8
1	$4	$10	–$6
2	$8	$12	–$4
3	$12	$14	–$2
4	**$16**	**$16**	**$0**
5	$20	$18	$2
6	$24	$20	$4

Break-Even Point →

Now for the important observations:

- ✓ As the number of widgets increases, so do the revenue and the cost. Also, the profit increases; that is, the numbers –8, –6, –4, –2, 0, 2, and 4 are getting larger.

- ✓ If we produce and sell anywhere from 0 to 3 widgets, we incur a loss (a negative profit). That is, when $w \le 3$, $P < 0$.

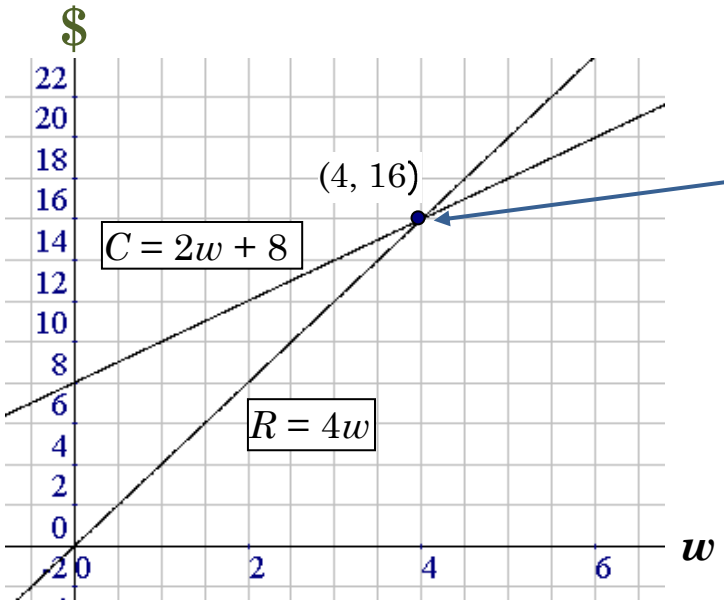
- ✓ At a production level of 4 widgets we break even. This is the point at which the revenue ($16) equals the cost ($16). But also note that this is where the profit is $0. Either way we look at it, according to the two definitions, $w = 4$ is the break-even point. When $w = 4$, $P = 0$.

- ✓ Any number of widgets beyond 4 (that is, 5 or more) produces a positive profit (we're making money!). Thus, if $w \geq 5$, then $P > 0$.

- ✓ In summary, a **loss** occurs when $w \leq 3$, the **break-even point** occurs when $w = 4$, and a **profit** results when $w \geq 5$.

❑ *USING A GRAPH*

We're going to make a picture of this situation of revenue, cost, profit, and the break-even point. Our grid will contain the graphs of both the revenue and cost formulas from the previous example. Note that the horizontal axis will certainly be w, the number of widgets, but the vertical axis will be the generic category money, since both revenue and cost are in units of money.



The ***break-even point*** occurs when $w = 4$, where both the revenue and the cost are $16.

Fewer than 4 widgets results in a loss — more than 4 produces a profit.

❑ *USING AN EQUATION*

The table and the graph were quite useful in determining and understanding the relationship among revenue, cost, profit, and break-even. But these take time to construct and they may give us only approximations. Let's use algebra to solve the same problem for the third time.

EXAMPLE 1: **The revenue formula is $R = 4w$ and the cost formula is $C = 2w + 8$. Find the break-even point.**

Solution: We recall that the break-even point occurs when the revenue equals the cost. Thus,

$R = C$	(to calculate break-even)
$4w = 2w + 8$	(substituting the given formulas)
$4w - \mathbf{2w} = 2w - \mathbf{2w} + 8$	(subtract $2w$ from each side)
$2w = 8$	(simplify each side)
$\frac{2w}{\mathbf{2}} = \frac{8}{\mathbf{2}}$	(divide each side by 2)
$w = 4$	(simplify)

Thus, the break-even point is | 4 widgets |

just as we saw with the table and the graph.

EXAMPLE 2: **The revenue formula is $R = 12w - 20$ and the cost formula is $C = 2w + 30$.**

a) Calculate the profit formula.

b) Find the break-even point using the profit formula.

Solution: a) The profit formula is $P = R - C$, so we can calculate the profit like this:

	$P = R - C$	(the profit formula)
$\Rightarrow$	$P = (12w - 20) - (2w + 30)$	(notice the parentheses!)
$\Rightarrow$	$P = 12w - 20 - 2w - 30$	(distribute)
$\Rightarrow$	$\boxed{P = 10w - 50}$	(combine like terms)

For part b) we find the break-even point by setting the profit formula to 0:

	$P = 0$	(one criterion for break-even)
$\Rightarrow$	$10w - 50 = 0$	(substitute the given profit formula)
$\Rightarrow$	$10w = 50$	(add 50 to each side of the equation)
$\Rightarrow$	$\boxed{w = 5}$	(divide each side of the equation by 10)

Homework

5. Regarding Example 2, part b), use the profit formula to show that we incur a loss if $w < 5$ and we enjoy a profit if $w > 5$. (A couple of examples will suffice.)

6. Suppose revenue and cost formulas are given by

$$R = 3w + 1 \qquad C = w + 5$$

a. Construct a table with columns for Widgets, Revenue, Cost, and Profit. Let w take the values from 0 to 4.

b. Use the table to determine the break-even point. Explain in <u>two</u> different ways how you arrived at your conclusion.

c. Graph both formulas on the same grid. Use the graphs to determine the break-even point.

d. Now find the break-even point by solving the formula equating Revenue and Cost.

7. Find the **break-even point** for the given revenue and cost formulas by solving an equation:

a. $R = 10w \qquad C = 7w + 18$

b. $R = 5w + 1 \qquad C = 4w + 50$

c. $R = 8w - 9 \qquad C = 3w + 6$

d. $R = 72w + 12 \qquad C = 50w + 100$

8. Find the **profit** formula for the given revenue and cost formulas:

a. $R = 30w + 90 \qquad C = 22w - 13$

b. $R = 22w - 5 \qquad C = 10w + 17$

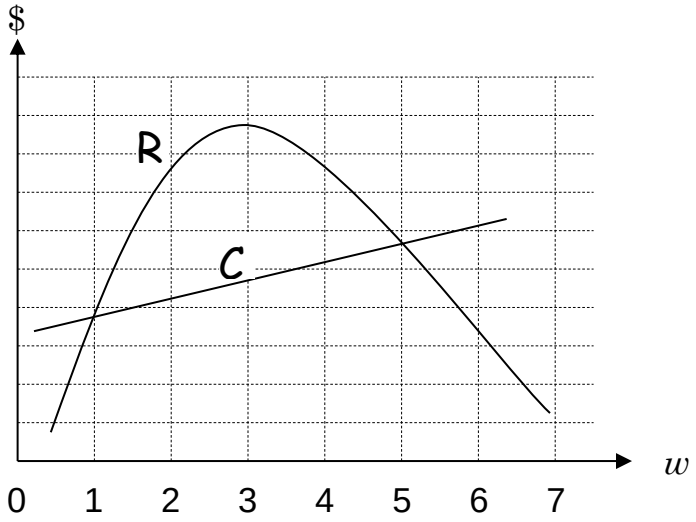
c. $R = w + 10 \qquad C = 8w - 14$

d. $R = 13w \qquad C = 2w - 5$

e. $R = 99w + 17 \qquad C = 9w$

f. $R = 8w - 1 \qquad C = 7w - 10$

9. Consider the graphs of the revenue and cost formulas:

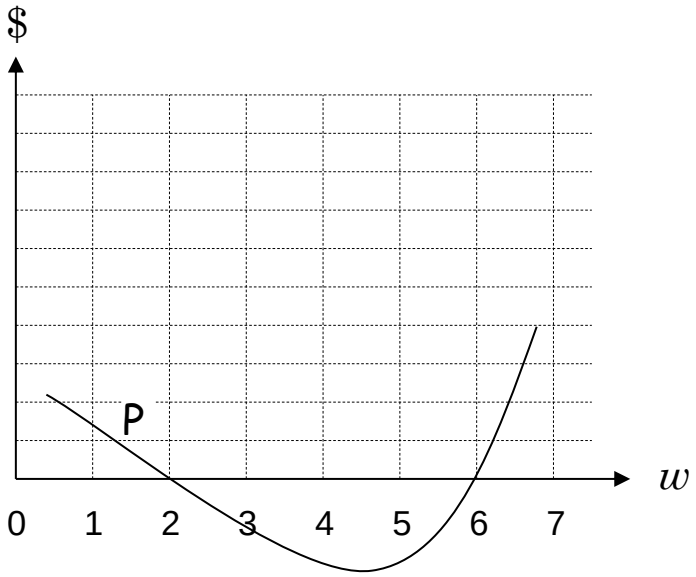


a. Find the two break-even points.

b. Is there a profit or loss when $w = 4$?

c. Is there a profit or loss when $w = 6$?

10. Consider the graph of the **profit** formula:



Find the two break-even points. What's happening between those two points?

11. Find the **break-even point** given each profit formula:

a. $P = 18w - 810$ b. $P = 2.5w - 300$

c. $P = 5.23w - 287.65$ d. $P = -7.6w + 410.4$

Review Problems

12. Let the revenue and cost formulas be given by $R = 7w + 1$ and $C = 5w + 11$. Construct a table with columns for widgets, revenue, cost, and profit, and let w take on the values from 3 to 6. Use the table to find the break-even point. Explain in two different ways how you arrived at your conclusion.

13. Use algebra (that is, solve an equation) to find the break-even point if the revenue and cost formulas are given by $R = 15w - 13$ and $C = 7w + 43$.

14. Find the profit formula if the revenue and cost formulas are given by $R = 10w + 13$ and $C = 6w - 5$.

15. Find the break-even point for the profit formula $P = 2.5w - 247.5$.

16. Graph $R = 3w - 4$ and $C = w + 1$ on the same grid, and then estimate the break-even point. How did you find that break-even point?

Solutions

1. a. $R = P + C$ b. $C = R - P$

2. a. T b. T c. F d. T e. T f. F
g. T h. T i. F j. F k. T l. F
m. T n. T o. T p. F

3. a, b, d **4**. a, b

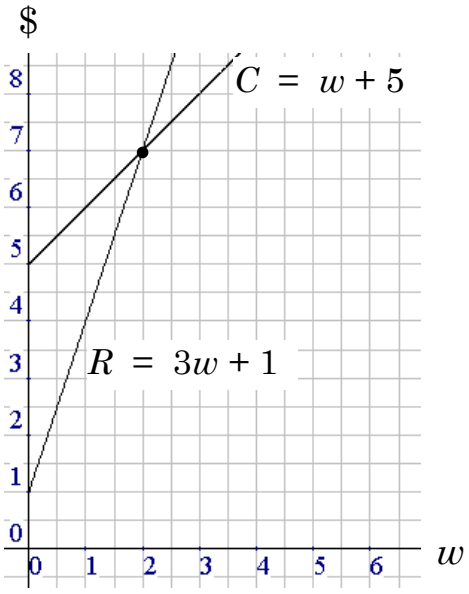
5. Suppose w is 3; then $P = 10(3) - 50 = 30 - 50 = -20$, a loss.
If w is 8, then $P = 10(8) - 50 = 80 - 50 = 30$, a profit.

6. a.

Widgets	Revenue	Cost	Profit
0	\$1	\$5	–\$4
1	\$4	\$6	–\$2
2	\$7	\$7	\$0
3	\$10	\$8	\$2
4	\$13	\$9	\$4

b. The break-even point is $w = 2$. It's the point where Revenue = Cost, and it's also the point where Profit = 0.

c.



d. The break-even point can be found by setting Revenue to Cost:

$R = C$	(to calculate break-even)
$3w + 1 = w + 5$	(substituting the given formulas)
$3w - \boldsymbol{w} + 1 = w - \boldsymbol{w} + 5$	(subtract w from each side)
$2w + 1 = 5$	(simplify)
$2w + 1 - \mathbf{1} = 5 - \mathbf{1}$	(subtract 1 from each side)
$2w = 4$	(simplify)
$\boldsymbol{w = 2}$	(divide each side by 2)

7. a. $w = 6$ b. $w = 49$ c. $w = 3$ d. $w = 4$

8. a. $P = R - C = (30w + 90) - (22w - 13) = 30w + 90 - 22w + 13$
$= 8w + 103$

b. $P = (22w - 5) - (10w + 17) = 22w - 5 - 10w - 17 = 12w - 22$

c. $P = -7w + 24$

d. $P = 11w + 5$

e. $P = 90w + 17$

f. $P = w + 9$

9. a. $w = 1$ and $w = 5$ (found by looking at the w-values where the graphs intersect)

b. profit, since the revenue graph is above the cost graph (i.e., the revenue exceeded the cost)

c. loss, since the cost graph is above the revenue graph (i.e., the cost exceeded the revenue)

10. $w = 2$ and $w = 6$ (found by looking at where the profit is zero; that is, where the profit graph crosses the w-axis, which is where $P = 0$)
The profit is negative (a loss) between the break-even points.

11. To find the break-even points, set each profit formula to zero:

a. $18w - 810 = 0 \Rightarrow 18w = 810 \Rightarrow w = 45$

b. $2.5w - 300 = 0 \Rightarrow 2.5w = 300 \Rightarrow w = 120$

c. $w = 55$

d. $w = 54$

12. $w = 5$. Why is this the break-even point? First, it's where $R = C$; second, it's where $P = 0$.

13. 7 widgets **14**. $P = 4w + 18$ **15**. 99 widgets

16. $w \approx 2.5$; it's the w-coordinate of the point of intersection

"Next in importance
to freedom and justice
is *education*, without
which neither
freedom nor justice
can be permanently
maintained."

James A. Garfield **(1831 - 1881)**

20th U.S. President

BREAK-EVEN POINT – QUADRATIC FUNCTIONS



A while back we discussed the notions of revenue (*R*), cost (*C*), profit (*P*), and the break-even point:

Profit: $P = R - C$

Break-even: $R = C$ or $P = 0$

Using either factoring or the Quadratic Formula, we can solve more break-even business problems; these problems will result in a **quadratic equation**.

❑ *BREAK-EVEN*

EXAMPLE 1: **Find the break-even point(s) if the profit formula is given by $P = 2w^2 - 31w + 84$, where w is the number of widgets sold.**

Solution: We find the break-even points by setting the profit formula to zero:

	$2w^2 - 31w + 84 = 0$	(set profit to 0)
$\Rightarrow$	$(2w - 7)(w - 12) = 0$	(factor)
$\Rightarrow$	$2w - 7 = 0$ or $w - 12 = 0$	(set each factor to 0)
$\Rightarrow$	$2w = 7$ or $w = 12$	(solve each equation)

$\Rightarrow \quad w = 3\frac{1}{2}$ or $w = 12$

Thus, the break-even points are

$3\frac{1}{2}$ widgets and 12 widgets

Of course, $3\frac{1}{2}$ widgets can't really exist, but it's good enough for this problem. But if w stood for wages, for instance, then $3\frac{1}{2}$ would make sense, since that number represents \$3.50, a perfectly legit answer.

EXAMPLE 2: **Find the break-even point(s) if revenue and cost are given by the formulas**

$$R = 3w^2 - 3w - 8$$
$$C = 2w^2 + 30w - 268$$

Solution: Recall that one of the two ways to describe the **break-even points** is by equating revenue and cost. Notice that we put the resulting equation in standard form by bringing all the terms on the right side to the left side so that the right side will be zero.

	$R = C$	(to find break-even)
$\Rightarrow$	$3w^2 - 3w - 8 = 2w^2 + 30w - 268$	(use the given formulas)
$\Rightarrow$	$w^2 - 3w - 8 = 30w - 268$	(subtract $2w^2$)
$\Rightarrow$	$w^2 - 33w - 8 = -268$	(subtract $30w$)
$\Rightarrow$	$w^2 - 33w + 260 = 0$	(add 268 $\Rightarrow$ standard form)
$\Rightarrow$	$(w - 20)(w - 13) = 0$	(factor)
$\Rightarrow$	$w - 20 = 0$ or $w - 13 = 0$	(set each factor to 0)
$\Rightarrow$	$w = 20$ or $w = 13$	(solve each equation)

And so the two break-even points are

20 widgets and 13 widgets

Homework

1. Find the **break-even points** for the given profit formula:

   a. $P = w^2 - 12w + 35$ b. $P = 2w^2 - 13w + 15$

   c. $P = w^2 - 25w + 150$ d. $P = 6w^2 - 31w + 40$

2. Find the **break-even points** for the given revenue and cost formulas:

   a. $R = 2w^2 + 8w - 20$ $C = w^2 + 24w - 75$

   b. $R = 2w^2 - 3w + 1$ $C = -w^2 + 16w - 29$

   c. $R = 5w^2 - 26w + 80$ $C = 3w^2 + w + 10$

Review Problems

3. If the profit formula is given by $P = w^2 - 57w + 350$, find the two break-even points.

4. If the revenue and cost are given by the formulas $R = 5w^2 - 30w + 100$ and $C = 4w^2 + 4w - 180$, find the break-even points.

Solutions

1. a. $w = 5, 7$ b. $w = \frac{3}{2}, 5$ c. $w = 10, 15$ d. $w = \frac{8}{3}, \frac{5}{2}$

2. a. $w = 5, 11$ b. $w = \frac{10}{3}, 3$ c. $w = \frac{7}{2}, 10$

3. $w = 7$ and $w = 50$ **4**. $w = 14$ and $w = 20$

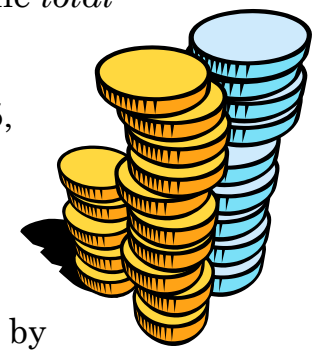
"There is one purpose to life and one only: to bear witness to and understand as much as possible of the complexity of the world – its beauty, its mysteries, its riddles. The more you understand, the more you look – the greater is your enjoyment of life and your sense of peace. That's all there is to it. If an activity is not grounded in '***to love***' or '***to learn,***' it does not have value."

– Anne Rice, American Author

COINS

❑ *INTRODUCTION*

Milo has 7 dimes. This means he has \$0.70 worth of dimes. We'll call the **7** the *number* of coins that Milo has, we'll call **\$0.10** (or 10¢) the *unit value*, and we'll call the **\$0.70** (or 70¢) the *total value* of Milo's coins.



If Sierra has 5 quarters, then the *number* of coins is 5, the *unit value* is \$0.25 (or 25¢), and the *total value* is \$1.25 (or 125¢).

In both examples, how was the total value of the coins calculated? By *multiplying* the number of coins by their unit value. For example, using Sierra's quarters:

5 quarters × \$0.25 per quarter = \$1.25

In general,

Number of Coins × Unit Value = Total Value

❑ *COIN PROBLEMS*

EXAMPLE 1: **Brennan has 3 times as many nickels as dimes. (That is, the number of nickels is 3 times the number of dimes.) If the total value of the coins is \$2.75, how many of each coin does Brennan have?**

Solution: The easiest way to solve these kinds of problems (so that we can avoid decimals) is to think of all money amounts in pennies (cents). For example, instead of saying that a quarter has a unit value of \$0.25, we'll say that its unit value is 25¢. And the total value of \$2.75 in this problem will be represented by the number 275¢. So, here we go.

First, we notice that we are asked for two different things: the number of nickels and the number of dimes. That's two variables we'll have to create -- let's declare n to represent the number of nickels, and let d equal the number of dimes. As you'll see, this means that we'll need two equations in order to find the values of the two variables.

Let's set up a chart to solve this problem. The central idea of the chart is that, for any given kind of coin, if we multiply the number of coins by its unit value, we get the total value of that coin. Study this chart carefully.

	Number of Coins × **Unit Value** = **Total Value**		
nickels	n	5	$5n$
dimes	d	10	$10d$
mixture	n/a	n/a	275

Since it's given that the number of nickels is 3 times the number of dimes, we get our first equation:

$$n = 3d$$

Also, the total value of the nickels ($5n$) plus the total value of the dimes ($10d$) should equal the total value of the mixture; that is,

$$5n + 10d = 275$$

Sure enough, we have two equations with two variables, just as we had predicted. And now we solve the set of two equations by rewriting the second equation, but using the fact that $n = 3d$:

$$5(\mathbf{3}\boldsymbol{d}) + 10d = 275 \qquad \text{(substitute } 3d \text{ for } n\text{)}$$

$\Rightarrow \quad 15d + 10d = 275$

$\Rightarrow \quad 25d = 275$

$\Rightarrow \quad d = 11$, so Brennan has 11 dimes

Since $d = 11$, it's pretty easy to calculate n:

$n = 3d = 3(\mathbf{11}) = 33$, so Brennan also has 33 nickels.

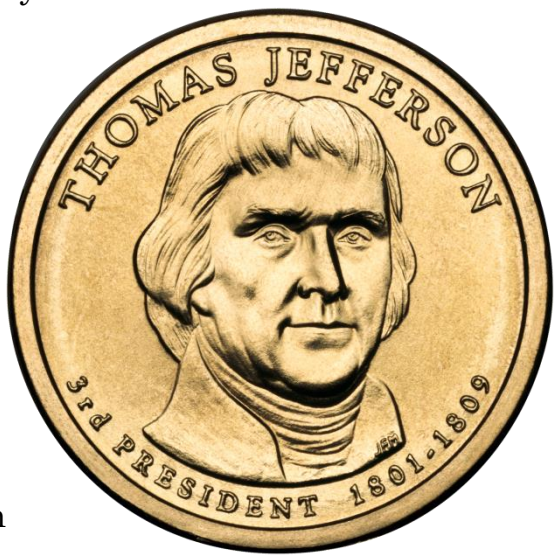
In summary, Brennan has

11 dimes and 33 nickels

Homework

1. Lisa has 7 times as many dimes as nickels. If the total value of the coins is \$3.75, how many of each coin does Lisa have?

2. Orion has 5 times as many quarters as nickels. If the total value of the coins is \$3.90, how many of each coin does Orion have?



3. Lauren has 4 times as many pennies as dimes. If the total value of the coins is \$0.98, how many of each coin does Lauren have?

4. Cal has 10 times as many nickels as quarters. If the total value of the coins is \$6.75, how many of each coin does Cal have?

EXAMPLE 2: **Alana has 3 more quarters than pennies. If the total value of the coins is $5.17, how many of each coin does Alana have?**

Solution: Letting q represent the number of quarters and p the number of pennies, here's the chart:

	Number of Coins × Unit Value = Total Value		
quarters	q	25	$25q$
pennies	p	1	$1p$
mixture	n/a	n/a	517

The first sentence of the problem, the relationship between the number of quarters and the number of pennies, can also be expressed this way: *The number of quarters is 3 more than the number of pennies.* This leads to the equation:

$$q = p + 3$$

For our second equation, we use the same logic as Example 1. The value of the quarters ($25q$) plus the value of the pennies ($1p$) should equal the total value of the coin mixture; that is,

$$25q + 1p = 517$$

Since $q = p + 3$ from the first equation, we now change the q in the second equation to $p + 3$:

$$25(\mathbf{p + 3}) + 1p = 517 \qquad \text{(since } q = p + 3\text{)}$$

$$\Rightarrow \quad 25p + 75 + p = 517 \qquad \text{(distribute)}$$

$$\Rightarrow \quad 26p + 75 = 517 \qquad \text{(combine like terms)}$$

$$\Rightarrow \quad 26p = 442 \qquad \text{(subtract 75 from each side)}$$

$$\Rightarrow \quad p = 17 \qquad \text{(divide each side by 26)}$$

So Alana has 17 pennies.

Thus, the number of quarters is

$$q = p + 3 = \mathbf{17} + 3 = 20$$

To conclude, Alana has

20 quarters and 17 pennies

Homework

5. Alana has 5 more quarters than nickels. If the total value of the coins is \$4.25, how many of each coin does Alana have?

6. Ben has 7 more dimes than pennies. If the total value of the coins is \$1.47, how many of each coin does Ben have?

7. Andrew has 2 more nickels than quarters. If the total value of the coins is \$5.20, how many of each coin does Andrew have?

8. Justus has 10 more pennies than dimes. If the total value of the coins is \$3.73, how many of each coin does Justus have?

EXAMPLE 3: **Amir has a total of 14 coins, some quarters and some dimes. If the total value of the coins is \$2.75, how many of each coin does Amir have?**

Solution: Since Amir has some unknown number of quarters and dimes, let's let q stand for the number of quarters and let d be the number of dimes. The following chart is very similar to the previous ones:

	Number of Coins ×	Unit Value =	Total Value
quarters	q	25	$25q$
dimes	d	10	$10d$
mixture	14	n/a	275

We're ready for our first equation:

$q + d = 14$ (14 coins altogether)

What else do we know? As before, the total value of the quarters plus the total value of the dimes must equal the total value of the coin mixture, which is 275¢:

$25q + 10d = 275$

Let's summarize what we have, and what we're looking for. We're now trying to solve two equations with two variables:

$q + d = 14$
$25q + 10d = 275$ (we want to find both q and d)

How about we solve for q in the first equation, $q + d = 14$. By subtracting d from each side of the equation, our system of equations becomes

$q = 14 - d$
$25q + 10d = 275$

Now change the q in the second equation to $14 - d$ from the first equation:

$25(14 - d) + 10d = 275$ (since $q = 14 - d$)

What has this maneuver accomplished? We now have an equation with just <u>one</u> variable in it -- this we can solve:

$\Rightarrow \quad 350 - 25d + 10d = 275$ (distribute)

$\Rightarrow \quad 350 - 15d = 275$ (combine like terms)

$\Rightarrow \quad -15d = -75$ (subtract 350 from each side)

$\Rightarrow \quad d = 5$ (divide each side by –15)

This means that Amir has 5 dimes, and since $q = 14 - d$, we calculate that the number of quarters is $14 - 5 = 9$. In summary, Amir has

9 quarters and 5 dimes

Homework

9. The total number of coins is 20, some nickels and some dimes. If the total value of the coins is $1.40, how many of each coin are there?

10. The total number of coins is 29, some quarters and some pennies. If the total value of the coins is $3.89, how many of each coin are there?

11. The total number of coins is 40, some nickels and some quarters. If the total value of the coins is $5.00, how many of each coin are there?

12. The total number of coins is 87, some dimes and some pennies. If the total value of the coins is $6.09, how many of each coin are there?

Solutions

1. 5 nickels and 35 dimes

2. 15 quarters and 3 nickels

3. 28 pennies and 7 dimes

4. 90 nickels and 9 quarters

5. 15 quarters and 10 nickels

6. 14 dimes and 7 pennies

7. 19 nickels and 17 quarters

8. 43 pennies and 33 dimes

9. 12 nickels and 8 dimes

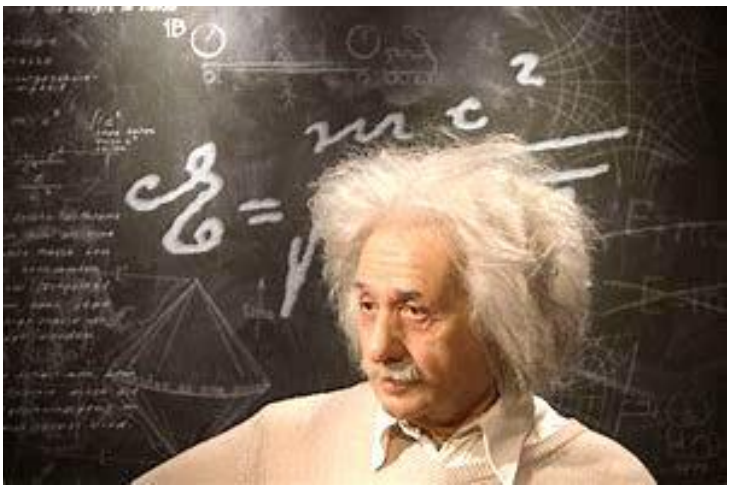
10. 15 quarters and 14 pennies

11. 25 nickels and 15 quarters

12. 58 dimes and 29 pennies

“Any fool can know. The point is to understand.”

– Albert Einstein

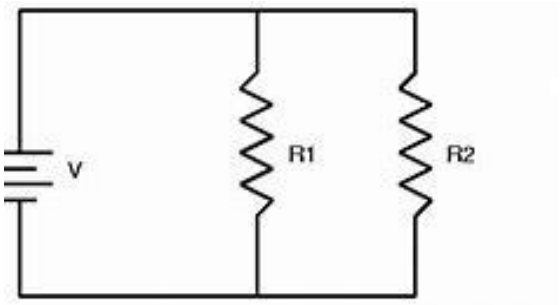


APPLICATIONS OF FRACTIONAL EQUATIONS

❑ *INTRODUCTION*

❑ *ELECTRONICS*

The electrical circuit shows a voltage source, along with two resistors wired in **parallel**. The TOTAL resistance, R_T, in the circuit is related to the resistors R_1 and R_2 by the formula



$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$$

EXAMPLE 1: **A parallel circuit contains two resistors, one with a resistance of 8Ω and the other with a resistance of 5Ω. Find R_T, the total resistance in the circuit.**

Resistance can be measured in a unit called the ***ohm***, (named after Georg Ohm) and abbreviated using the Greek capital letter omega, Ω.

Solution: Using the formula we can write

$$\frac{1}{R_T} = \frac{1}{8} + \frac{1}{5}$$

$$\Rightarrow \quad \frac{1}{R_T} = \frac{13}{40}$$

$$\Rightarrow \quad R_T = \frac{40}{13}$$ (take the reciprocal of each side of the equation)

$$\Rightarrow \quad \boxed{R_T \approx 3.08\,\Omega}$$

EXAMPLE 2: **Solve the total resistance formula for R_1.**

Solution: We start, of course, with the given formula:

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$$

The LCD is the product of all three denominators, which is $R_T R_1 R_2$, so we multiply each side of the formula by the LCD, which clears out the fractions:

$$\left[R_T R_1 R_2\right]\left(\frac{1}{R_T}\right) = \left[R_T R_1 R_2\right]\left(\frac{1}{R_1}\right) + \left[R_T R_1 R_2\right]\left(\frac{1}{R_2}\right)$$

Now simplify by cross-canceling:

$$R_1 R_2 = R_T R_2 + R_T R_1$$

$$\Rightarrow \quad R_1 R_2 - R_T R_1 = R_T R_2$$

$$\Rightarrow \quad R_1(R_2 - R_T) = R_T R_2$$

$$\Rightarrow \quad \boxed{R_1 = \frac{R_2 R_T}{R_2 - R_T}}$$

Homework

1. A parallel circuit contains two resistors, one with a resistance of 5Ω and the other with a resistance of 0.2Ω. Find R_T, the total resistance in the circuit.

2. The total resistance in a parallel circuit containing two resistors is $\mathbf{8}\Omega$**.** If one of the resistors has a resistance of $\mathbf{10}\Omega$, find the resistance of the other resistor.

3. Solve the total resistance formula for R_T.

4. Solve the total resistance formula for R_2.

❑ *More Motion Problems*

A while back, the key to solving motion problems was to use the formula $rt = d$: The product of the rate (speed) and the time is the distance. We can solve for t by dividing each side of the formula by r, thereby getting a formula for time:

$$\boxed{t = \frac{d}{r}}$$

This formula should make sense. For instance, if you travel a distance of 200 miles at an average rate of 50 miles per hour, your travel time is

$$t = \frac{d}{r} = \frac{200 \text{ miles}}{50 \text{ miles per hour}} = 4 \text{ hours}$$

This formula for *time* is the key to the next two examples.

A note for you science people: What justifies the above calculations where *miles* over *miles per hour* turned into hours? I will try to prove to you that miles divided by miles per hour must turn into hours:

$$\frac{\text{mi}}{\frac{\text{mi}}{\text{hr}}} = \frac{\frac{\text{mi}}{1}}{\frac{\text{mi}}{\text{hr}}} = \underbrace{\frac{\text{mi}}{1} \div \frac{\text{mi}}{\text{hr}} = \frac{\text{mi}}{1} \times \frac{\text{hr}}{\text{mi}}}_{\text{Multiply by the reciprocal}} = \frac{\cancel{\text{mi}}}{1} \times \frac{\text{hr}}{\cancel{\text{mi}}} = \frac{\text{hr}}{1} = \textbf{hr}$$

EXAMPLE 3: **Maria hikes up a 30-mile mountain trail. She then rents a pair of skis and skis back down the trail. Her skiing speed is 9 mph faster than her hiking speed. If Maria spent a total of 7 hours on the trail, what were her hiking speed and her skiing speed?**

Solution: Let's set up a table, with the top row stating our standard motion formula: Rate × Time = Distance. The next two rows are labeled "Hiking up" and "Skiing down."

	Rate ×	Time =	Distance
Hiking up			
Skiing down			

The first thing we can see is that the distance hiking up the mountain and the distance skiing down the mountain are the same, 30 miles. Also, since the speed skiing down the mountain is 9 mph faster (more than) the hiking speed up the mountain, we can let h represent the hiking speed, which implies that $h + 9$ is the skiing speed:

	Rate ×	Time =	Distance
Hiking up	h		30
Skiing down	$h + 9$		30

So far, so good. But what about the Time column? The problem says nothing about the time it took to hike up the mountain or the time it took to ski back down to the bottom. Here's where the time formula discussed earlier, $t = \frac{d}{r}$, comes to our rescue. For the "Hiking up" row, using the formula for t and the values of r and d in the table, we get

$$t = \frac{d}{r} = \frac{30}{h},$$

and for the "Skiing down" row we get

$$t = \frac{d}{r} = \frac{30}{h+9}$$

Our table is now completely filled in:

	Rate ×	Time =	Distance
Hiking up	h	$\frac{30}{h}$	30
Skiing down	$h + 9$	$\frac{30}{h+9}$	30

Now it's time for an equation. The only given information we haven't used yet is that Maria spent a <u>total of 7 hours</u> on the trail — part of the 7 hours hiking up and the other part skiing down. In other words, the hiking time plus the skiing time equals 7 hours; we have our equation:

$$\frac{30}{h} + \frac{30}{h+9} = 7$$

This is a fractional equation; the LCD is $h(h + 9)$, so we multiply each side of the equation by the LCD:

$$\frac{30}{\cancel{h}}\left[\cancel{h}(h+9)\right] + \frac{30}{\cancel{h+9}}\left[h(\cancel{h+9})\right] = 7\left[h(h+9)\right]$$

A little cross-canceling simplifies things to

$$30(h + 9) + 30(h) = 7(h^2 + 9h)$$

which distributes out to become the quadratic equation

$$30h + 270 + 30h = 7h^2 + 63h$$

Bringing all the terms to the right to achieve standard quadratic form gives

$$0 = 7h^2 + 3h - 270$$

From here we factor and set each factor to 0:

$$0 = (7h + 45)(h - 6)$$

$$\Rightarrow \quad 7h + 45 = 0 \quad \text{or} \quad h - 6 = 0$$

$$\Rightarrow \quad h = -\frac{45}{7} \quad \text{or} \quad h = 6$$

A negative rate doesn't make sense in this problem, so we'll discard it and keep the value of 6 for h, the hiking speed, making the skiing speed $h + 9 = 6 + 9 = 15$. We're finally done:

Her hiking speed was **6 mph**
and her skiing speed was **15 mph**.

EXAMPLE 4: **Jason can go 15 miles upstream in his motorboat in the same time he can go 63 miles downstream. If the speed of his boat is 13 mph in still water, what is the rate of the current?**

Solution: We again create a table, with our distance formula in the top row, and "Upstream" in the second row and "Downstream" in the third row. Let's put some data into our

table right now, noting that the upstream distance is 15 mi while the downstream distance is 63 mi.

	Rate × Time = Distance		
Upstream (against the current)			15
Downstream (with the current)			63

Next, we analyze the rate (speed) column of the table. First, we see that the problem tells us that Jason's motorboat can go 13 mph in still water, without any current affecting the speed.

Second, notice that the problem asks for the rate of the current. It's time for the variable; we'll let c = the rate of the current.

Third, we need to carefully think about the meaning of the terms "upstream" and "downstream." Going upstream means that the boat is going in the opposite direction of the current, which hinders the boat's normal speed. Thus, the upstream rate is the boat's normal speed minus the rate of the current:

$$\text{upstream rate} = 13 - c$$

Going downstream means that the boat is going in the same direction as the current, which boosts the boat's normal speed.

So the downstream rate is the boat's normal speed plus the rate of the current:

$$\text{downstream rate} = 13 + c$$

We can now fill in the Rate column of the table:

	Rate ×	Time =	Distance
Upstream (against the current)	$13 - c$		15
Downstream (with the current)	$13 + c$		63

The expressions that go into the Time column are based on the time formula we wrote in the previous example: $t = \frac{d}{r}$

	Rate ×	Time =	Distance
Upstream (against the current)	$13 - c$	$\frac{15}{13-c}$	15
Downstream (with the current)	$13 + c$	$\frac{63}{13+c}$	63

It's now time to come up with an equation. Notice the phrase "*in the same time*" in the problem. This means that even though Jason traveled different distances at different speeds as he motorboated upstream and downstream, it took the same amount of time for each part of the trip. Hence, our equation is found by setting the two Times equal to each other:

$$\frac{15}{13-c} = \frac{63}{13+c}$$

Multiply each side of the equation by the LCD, $(13 - c)(13 + c)$:

$$\left[(13-c)(13+c)\right]\frac{15}{13-c} = \frac{63}{13+c}\left[(13-c)(13+c)\right]$$

Cross-cancel the common factors:

$$\left[\cancel{(13-c)}(13+c)\right]\frac{15}{\cancel{13-c}} = \frac{63}{\cancel{13+c}}\left[(13-c)\cancel{(13+c)}\right]$$

This leaves

$$15(13 + c) = 63(13 - c)$$

$\Rightarrow \quad 195 + 15c = 819 - 63c$

$\Rightarrow \quad 78c = 624$

$\Rightarrow \quad \underline{c = 8}$

We conclude that

The rate of the current is 8 mph.

Homework

5. Phil hikes up a 32-mile mountain trail. He then rents a pair of skis and skis back down the trail. His skiing speed is 8 mph faster than his hiking speed. If Phil spent a total of 6 hours on the trail, what were his hiking speed and his skiing speed?

6. Raya can go 9 miles upstream in her motorboat in the same time she can go 33 miles downstream. If the speed of her boat is 7 mph in still water, what is the rate of the current?

7. Ann hikes up a 30-mile mountain trail. She then rents a pair of skis and skis back down the trail. Her skiing speed is 4 mph faster than her hiking speed. If Ann spent a total of 8 hours on the trail, what were her hiking speed and her skiing speed?

8. Marcus can go 7 miles upstream in his motorboat in the same time he can go 147 miles downstream. If the speed of his boat is 11 mph in still water, what is the rate of the current?

Review Problems

9. Millie hikes up a 30-mile mountain trail. She then rents a pair of skis and skis back down the trail. Her skiing speed is 10 mph faster than her hiking speed. If Millie spent a total of 8 hours on the trail, what were her hiking speed and her skiing speed?

10. Joseph can go 35 miles upstream in his motorboat in the same time he can go 175 miles downstream. If the speed of his boat is 15 mph in still water, what is the rate of the current?

Solutions

1. $R_T \approx 0.192\,\Omega$

2. $40\,\Omega$

3. $R_T = \dfrac{R_1 R_2}{R_1 + R_2}$

4. $R_2 = \dfrac{R_1 R_T}{R_1 - R_T}$

5. 8 mph and 16 mph

6. 4 mph

7. 6 mph and 10 mph

8. 10 mph

9. 5 mph and 15 mph

10. 10 mph

"It is the Law that any difficulties that can come to you at any time, no matter what they are, must be exactly what you need most at the moment, to enable you to take the next step forward by overcoming them. The only real misfortune, the only real tragedy, comes when we suffer without learning the lesson."

– *Emmet Fox*

ASSET ALLOCATION – INVESTING IN THE STOCK MARKET



The time to prepare for your financial future is NOW. But knowing where to invest your money – taking into account both profit and safety – is a complex and confusing dilemma.

The term ***asset*** in the chapter title refers to the money you will invest (or the things you buy with your money); the word ***allocation*** describes how you will distribute the money to the various investments. In short, asset allocation is how you divvy up your dough. To prepare for the upcoming applications, let's first brush up on translating English expressions to algebra.

Homework

1. Translate each of the following English phrases into an algebraic expression:

   a. 12 more than x
   b. 8 more than y
   c. twice n
   d. 3 times Q
   e. 1 more than z
   f. 5 times m
   g. twice $2a$
   h. 3 times $4x$
   i. 7 less than t
   j. 13 less than B
   k. 4 more than twice x
   l. 20 less than three times a

m. 9 less than T
n. 8 less than 7 times R
o. the difference of X and Y
p. 8 times $3x$
q. 12 times $9a$
r. 20 times $8w$
s. 4 times the product of 3 and x.
t. 3 times the product of 13 and b.
u. 9 times the product of 5 and T.
v. twice the product of π and x.

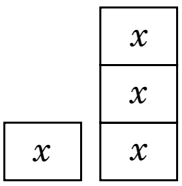
❑ *EXAMPLES*

EXAMPLE 1: **Ms. Zimmerman has $37,000 to invest in Coke and Microsoft stocks. She wants the amount invested in Microsoft to be 3 times the amount invested in Coke. How should the asset of $37,000 be allocated to the two investments? That is, how much money should she invest in each company?**

When you buy stock in a company, you own a piece of that company. If the company does well, and if the price of the stock goes up, and if you sell your shares at the right time, you can make a profit. If the reverse happens, you can lose money.

Solution: Since the Microsoft amount is based on the Coke amount, we'll let x represent the amount of money invested in Coke, which implies that $3x$ represents the amount invested in Microsoft:

x = Coke investment
$3x$ = Microsoft investment



One investment is 3 times the other.

The sum of these amounts must equal the total investment of $37,000. This fact leads to the equation:

$$x + 3x = 37{,}000$$
$$\Rightarrow \quad 4x = 37{,}000 \qquad \text{(combine like terms)}$$
$$\Rightarrow \quad x = 9250 \qquad \text{(divide each side by 4)}$$

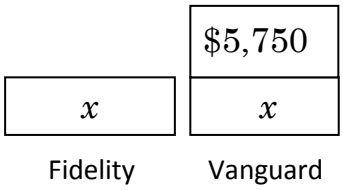
Since x = Coke investment, Ms. Zimmerman should invest \$9,250 in Coke. The Microsoft investment is $3x = 3(\mathbf{9250}) = 27{,}750$. Her plan is therefore to

Invest \$9,250 in Coke and \$27,750 in Microsoft

EXAMPLE 2: **\$42,750 will be invested in two mutual funds, Vanguard and Fidelity. The Vanguard fund will have \$5,750 more than the Fidelity fund (maybe because it's a little safer). How much will be invested in each mutual fund?**

A *mutual fund* is a basket of stocks. When you buy a mutual fund you're diversifying your investment across dozens or even thousands of companies. This way, if one company tanks big time, it will hurt your investment just a little bit.

Solution: Since the Vanguard fund amount is more than the Fidelity fund amount, we'll let x represent the smaller Fidelity amount. This implies that $x + 5750$ will be the larger Vanguard amount. Since the total investment is \$42,750, the relevant equation is



The Vanguard investment is \$5750 more than the Fidelity investment.

$$x + (x + 5750) = 42{,}750 \qquad \text{(the 2 funds total \$42,750)}$$
$$\Rightarrow \quad x + x + 5750 = 42{,}750 \qquad \text{(drop the parentheses)}$$
$$\Rightarrow \quad 2x + 5750 = 42{,}750 \qquad \text{(combine like terms)}$$

$\Rightarrow \quad 2x = 37{,}000$ (subtract 5750)

$\Rightarrow \quad x = 18{,}500$ (divide by 2)

Therefore, the Fidelity amount should be \$18,500. And the Vanguard amount would be

$$x + 5750 = \mathbf{18{,}500} + 5750 = 24{,}250$$

Thus, the \$42,750 should be split between the two mutual funds as follows:

Fidelity: \$18,500
Vanguard: \$24,250

EXAMPLE 3: **The amount invested in stock A is to be \$3,600 less than the amount invested in stock B. If the total investment is \$17,300, determine the asset allocation.**



Solution: Since the amount for stock A is based on the amount for stock B, we'll let b represent the investment in stock B. The wording of the problem then implies that $b - 3600$ is the amount for stock A (because it's "\$3,600 less than . . . stock B"). In short,

b = amount invested in stock B

$b - 3600$ = amount invested in stock A

Since the total investment is \$17,300, we add the two investments to get the total:

$b + (b - 3600) = 17{,}300$ (total investment: \$17,300)

$\Rightarrow \quad b + b - 3600 = 17{,}300$ (drop the parentheses)

$\Rightarrow \quad 2b - 3600 = 17{,}300$ (combine like terms)

$\Rightarrow \quad 2b = 20{,}900$ (add 3600 to each side)

$\Rightarrow \quad b = 10{,}450$ (divide each side by 2)

That's the amount for stock B. To calculate the amount for stock A, we compute $b - 3600 =$ **10,450** $- 3600 = 6850$.

The asset allocation should therefore be

stock A: \$6850 stock B: \$10,450

EXAMPLE 4: **Marvyn has \$22,400 to invest in his three favorite companies: Reebok, MTV, and Dell Computers. He wants to allocate his assets so that the amount invested in MTV is three times the amount invested in Reebok, and he also desires the amount invested in Dell to be four times the amount invested in MTV. How much should Marvyn invest in each company?**

Solution: Letting R = Reebok, M = MTV, and D = Dell, we translate the info in the problem like this:

"the amount invested in MTV is three times the amount invested in Reebok" becomes the equation

$$M = 3R$$

"the amount invested in Dell [is] to be four times the amount invested in MTV" becomes the equation

$$D = 4M$$

Since the total investment, \$22,400, is to be allocated to the three investments, the sum of the three investments must be \$22,400:

$$R + M + D = 22{,}400$$

$\Rightarrow$ $R + 3R + 4M = 22{,}400$ (since $M = 3R$ and $D = 4M$)

$\Rightarrow$ $R + 3R + 4(3R) = 22{,}400$ (since $M = 3R$)

$\Rightarrow$ $R + 3R + 12R = 22{,}400$ (associate the 4 and the 3)

$\Rightarrow$ $16R = 22{,}400$ (combine like terms)

$\Rightarrow$ $R = 1400$ (Reebok)

$\Rightarrow$ $M = 3R = 3(\mathbf{1400}) = 4200$ (MTV)

$\Rightarrow$ $D = 4M = 4(\mathbf{4200}) = 16{,}800$ (Dell)

The investments should be allocated as follows:

$1400 to Reebok
$4200 to MTV
$16,800 to Dell

Notice that the total investment is $22,400.

EXAMPLE 5: **You are in charge of investing $50,000 according to the following allocation plan: equal amounts of money in each of three stocks, a fixed amount of $5,000 for a CD, and an amount in a mutual fund that is twice the amount invested in any of the three stocks. How much will you invest in each of the five investments?**

Whereas the stock market fluctuates (wildly in the past 10 years), and you can make (or lose) a fortune, a CD (*certificate of deposit*) is more like a savings account. It has a set interest rate (say, 5%) for a fixed term (say, 1 year). At the end of the year you are guaranteed a 5% return on your investment – no more, but no less.

Solution: Since the three stocks all get the same amount, we'll let x represent that amount, i.e.,

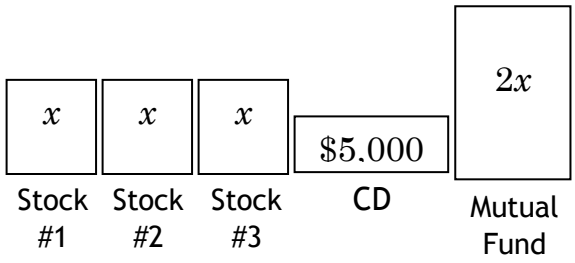
x = the amount invested in Stock #1
x = the amount invested in Stock #2
x = the amount invested in Stock #3

The CD investment is exactly \$5,000, so

5000 = the amount invested in the CD

The mutual fund gets twice the amount as any of the three stocks:

$2x$ = the amount invested in the mutual fund



Adding up all these amounts gives the equation

$$x + x + x + 5{,}000 + 2x = 50{,}000$$
$$\Rightarrow \quad 5x + 5{,}000 = 50{,}000 \quad \text{(combine like terms)}$$
$$\Rightarrow \quad 5x = 45{,}000 \quad \text{(subtract 5000 from each side)}$$
$$\Rightarrow \quad x = 9{,}000 \quad \text{(divide each side by 5)}$$

Therefore, the \$50,000 should be allocated as follows:

\$9,000	= amount invested in Stock #1
\$9,000	= amount invested in Stock #2
\$9,000	= amount invested in Stock #3
\$5,000	= amount invested in the CD
\$18,000	= amount invested in the mutual fund

Homework



2. One part of a $41,500 investment is designed to be four times the other part of the investment. How will the money be allocated? That is, find the two parts of the investment.

3. Sarah wants to invest a total of $22,500. Some will be in a stock and some will be in a municipal bond. The amount in the bond will be $10,500 more than the amount in the stock. Find the amount of each investment.

4. An investment of $9,500 will be split among two stocks, each with equal amounts, a CD containing $2,000, and an index fund containing three times the amount in either of the two stocks. Determine the asset allocation.

5. Joseph has $5,950 to invest in his three favorite companies, Coke, Starbucks, and Nike. He wants to allocate his assets so that the amount invested in Starbucks is twice the amount invested in Coke, and he also desires the amount invested in Nike to be twice the amount invested in Starbucks. How much should Joseph invest in each company?

6. One part of an $88,025 investment is designed to be six times the other part of the investment. Find the asset allocation.

7. Mark wants to invest a total of $40,000. Some will be in a stock and some will be in a municipal bond. The amount in the bond will be $7,500 less than the amount in the stock. Find the amount of each investment.

8. An investment of $22,500 is to be split among three stocks, each with equal amounts, a CD containing $5,000, and an index fund containing four times the amount in any of the three stocks. Determine the asset allocation.

9. Melanie has $39,200 to invest in her three favorite companies, Coke, Starbucks, and Nike. She wants to allocate her assets so that the amount invested in Starbucks is $2000 more than the amount invested in Coke. She also desires the amount invested in Nike to be equal to the total invested in Coke and Starbucks. How much should Melanie invest in each company?



10. The amount invested in stock A is to be $494 less than the investment in stock B. If the total investment is $7,528, how much should be invested in each stock?

11. The amount invested in stock A is to be $2,183 less than the investment in stock B. If the total investment is $4,517, how much should be invested in each stock?

12. The amount invested in stock A is to be $274 more than the investment in stock B. If the total investment is $14,452, how much should be invested in each stock?

13. The total investment in three stocks is $27,568. The amount spent on stock A must be 5 times the amount spent on stock C, while the amount spent on stock B is to be 2 times the amount spent on stock C. Determine the asset allocation.

14. The total investment in three stocks is $19,716. The amount spent on stock A must be 2 times the amount spent on stock C, while the amount spent on stock B is to be 3 times the amount spent on stock C. Determine the asset allocation.

15. The total investment in three stocks is $16,856. The amount spent on stock A must be 2 times the amount spent on stock C, while the amount spent on stock B is to be 4 times the amount spent on stock C. Determine the asset allocation.

16. The total investment in three stocks is $45,695. The amount spent on stock C must be 3 times the amount spent on stock B, while the amount spent on stock B is to be 3 times the amount spent on stock A. How much should be invested in each stock?



17. The total investment in three stocks is $52,949. The amount spent on stock C must be 2 times the amount spent on stock B, while the amount spent on stock B is to be 4 times the amount spent on stock A. How much should be invested in each stock?

Solutions

1.

a.	$x + 12$	b.	$y + 8$	c.	$2n$	d.	$3Q$
e.	$z + 1$	f.	$5m$	g.	$2(2a)$, or $4a$	h.	$3(4x)$, or $12x$
i.	$t - 7$	j.	$B - 13$	k.	$2x + 4$	l.	$3a - 20$
m.	$T - 9$	n.	$7R - 8$	o.	$X - Y$	p.	$8(3x) = 24x$
q.	$108a$	r.	$160w$	s.	$12x$	t.	$39b$
u.	$45T$	v.	$2\pi x$				

2. $8,300 and $33,200

3. $6,000 in the stock and $16,500 in the bond

4. $1500 in each of the two stocks, $2000 in the CD, and $4500 in the index fund

5. $850 in Coke, $1700 in Starbucks, and $3400 in Nike

6. $12,575 and $75,450

7. $23,750 in the stock and $16,250 in the bond

8. $2,500 in each of the three stocks, $5,000 in the CD, and $10,000 in the index fund.

9. $8,800 in Coke, $10,800 in Starbucks, and $19,600 in Nike

10. $3,517 in stock A and $4,011 in stock B

11. $1,167 in stock A and $3,350 in stock B

12. $7,363 in stock A and $7,089 in stock B

13. $17,230 in stock A; $6,892 in stock B; $3,446 in Stock C

14. $6,572 in stock A; $9,858 in stock B; $3,286 in Stock C

15. $4,816 in stock A; $9,632 in stock B; $2,408 in Stock C

16. $3,515 in stock A; $10,545 in stock B; $31,635 in Stock C

17. $4,073 in stock A; $16,292 in stock B; $32,584 in Stock C

"Whether you
think that you can
– or that you can't –
you are usually right."



Henry Ford
(1863-1947)

LINEAR MODELING USING GIVEN INFORMATION

❑ *INTRODUCTION*

Sometimes it's not too hard to create a formula from words. For example the phrase *density is the quotient of mass and volume* translates to the formula $d = \frac{m}{V}$. But this method of formula creation is useful only if we comprehend the words that need to be translated into Algebra.



More often, we're given a *situation* that needs translation. For example, if we buy 12 lbs of cashews at a unit price of \$14/lb, and we wish to determine the total cost, we have to understand that multiplication does the trick; no one's going to tell you that — you have to reason it out yourself. So the total cost is

$$12 \text{ lbs} \times \$14/\text{lb} = \underline{\$168}$$

That's fine for one or a few calculations. But if you need to program a calculator, or perhaps a spreadsheet, or maybe a programming language, you need to teach the machine what to do, and that can only be done with *variables*. So in the cashew problem above, we could let Q stand for the quantity (# of pounds) of cashews, let P be the unit price (the price of 1 pound) of the cashews, and let C be the total cost. Again, if we understand that the key to the formula will be multiplication, we can write a formula that can be entered into a computer (using *, the computer symbol for multiplication):



$$C = P * Q \qquad \text{[Cost = Unit Price times Quantity]}$$

❑ *JOINING THE CLUB*

EXAMPLE : **To join the Model Railroad Club, a member must pay an up-front fee of $25 to join, and then pay $10 per month for each month in the club.**



a. **Find the total cost for someone to be a club member for 8 months.**

b. **Find a formula for someone to be a club member for *m* months.**

c. **Use your formula to calculate the total to be in the club for 36 months.**

Solution: Let's begin with a simple table that shows the cost of membership for various months:

Months	0	1	2	3	4	5	6	7	8
Cost	$25	$35	$45	$55	$65	$75	$85	$95	$105

a. The cost, C, is $25 plus 8 months at $10 per month:

$$C = 25 + 10(8) = 25 + 80 = 105$$

Therefore, the cost of 8 months is **$105**.

b. Just change the 8 in part a. to the variable m:

$$C = 25 + 10m, \text{ or } \boldsymbol{C = 10m + 25}$$

c. Using the formula in part b., we get a cost of

$$C = 10m + 25 = 10(\mathbf{36}) + 25 = 360 + 25 = \mathbf{\$385}$$

Homework



1. a. It costs \$2 per mile to take a taxi. If m represents the total miles traveled, write a formula for the total cost, C.

   b. Use your formula to calculate the total cost of a 20-mile trip.

2. Sprint offers a monthly phone plan that has a base fee of \$20, plus \$0.25 for each minute of use. Write a formula for the monthly cost, C, if you used the phone for m minutes in a month.

3. a. It costs \$3 for the first mile of a taxi ride, and \$2 per mile for each additional mile. Calculate the total cost of a 21-mile taxi ride.

   b. It costs \$3 for the first mile of a taxi ride, and \$2 per mile for each additional mile. If m represents the total miles traveled, write a formula for the total cost, C.

   c. Use your new formula from part b. to calculate the total cost of a 15-mile trip.

4. It costs \$7 for the first mile of a taxi ride, and \$4 per mile for each additional mile. If m represents the total miles traveled, write a formula for the total cost, C.

5. The ultimate question on the taxi ride:

   Let C = the total cost of the taxi ride

   f = the cost of the first mile

   a = the cost of each mile after the first

   m = the total miles traveled

   Write a formula for the total cost of a taxi ride.

6. a. QRS, Incorporated sold 5000 widgets last month at a unit selling price of \$45/widget. Calculate the revenue obtained. Hint: Revenue is the total amount of money obtained.

   b. STU, Inc. sold w widgets last month at a unit selling price of \$$p$/widget. Create a formula for the revenue, R, obtained.

7. a. WXY, Inc. produced 600 widgets last quarter, and it costs \$50 to produce each widget. (\$50/widget is called the *unit cost*). In addition, fixed costs (rent, utilities, salaries, etc.) totaled \$2,400 last quarter. Calculate the total cost of producing the 600 widgets last quarter.

   b. Let w = the number of widgets produced
   c = unit cost to produce one widget
   f = fixed costs (rent, utilities, salaries, etc.)
   E = total expense to produce the w widgets
   Create a formula for the total expense, E, of producing w widgets.

8. Ernie earns a base salary of \$2000/month and a commission of \$50 for each iPad he sells. Find Ernie's total salary during a month in which he sold 75 iPads. Construct a formula which would give Ernie's total salary, S, during a month if he sold i iPads.



9. Andrew wants to place a classified ad in the Chronicle to sell his old iPad .

   a. The first 12 words cost \$7.50, and each additional word costs \$0.30. Find the total cost of an ad containing 20 words.

   b. The first 15 words cost \$9.75, and each additional word costs \$0.40. Construct a formula to calculate C, the total cost of an ad containing w words. [Assume that w is at least 15.]

10. A tole painter buys a piece of wood for $\$w$, paints it, and then sets the selling price at 4 times the cost of the wood. If the painter gets 75% of the selling price (the rest goes to the salesperson), find the tole painter's NET profit (money earned less expenses).

Review Problems

11. a. Suppose that the Foothill Middle School library currently has 7000 books, and is buying 150 more books per year. How many books will the library have 4 years from now?



b. Write a formula that will give the number of books, B, that the library will have y years from now.

12. You must pay \$75 up front to join the Painting Club, and then pay \$35 per month for each month that you're a member of the club. Create a formula that will give the total cost, C, to be in the club for m months.

13. It costs \$7.50 for the first mile of a taxi ride, and \$3.25 per mile for each additional mile. If m represents the total miles traveled, write a formula for the total cost, C.

14. The Speedy Car Rental Co. charges \$35/day and \$0.15/mile to rent a car. Let d represent the number of days you rent the car, and let m represent the number of miles driven. Find a formula for C, the total rental cost.

❑ *TO ∞ AND BEYOND*

Placing a Classified Ad

The first n words cost \$$I$, while each additional word costs \$$A$. Find the total cost, C, of an ad containing w words. You may assume that $w \geq n$.

Solutions

1. a. $C = 2m$ b. $C = 2m = 2(20) =$ **\$40**

2. $C = 20 + 0.25m$ [or, $C = 0.25m + 20$]

3. a. $3 + 2(20) = 3 + 40 =$ **\$43**
b. $C = 3 + 2(m - 1)$, or $\boldsymbol{C = 2(m - 1) + 3}$
c. $C = 2(m - 1) + 3 = 2(15 - 1) + 3 = 2(14) + 3 = 28 + 3 =$ **\$31**

4. $C = 7 + 4(m - 1)$ or $C = 4(m - 1) + 7$

5. $C = f + a(m - 1)$ or $C = a(m - 1) + f$

6. a. \$225,000 b. $R = pw$

7. a. $600(50) + 2{,}400 =$ **\$32,400** b. $E = cw + f$

8. \$2000 + \$50(75) = **\$5,750**
In general, $S = 2000 + 50(i)$, or $\boldsymbol{S = 50i + 2000}$

9. a. \$9.90 b. $C = 9.75 + 0.40(w - 15)$

10. $0.75(4w) - w$

11. a. $7000 + 4(150) =$ 7600 books
b. $B = 150y + 7000$ or $B = 7000 + 150y$

12. The cost would be \$75 plus \$35 for each month in the club.
That's \$75 + \$35 × the number of months in the club.
That's $C = 75 + 35m$, which could be written $\underline{C = 35m + 75}$

13. $C = 7.50 + 3.25(m - 1)$

14. $C = 35d + 0.15m$

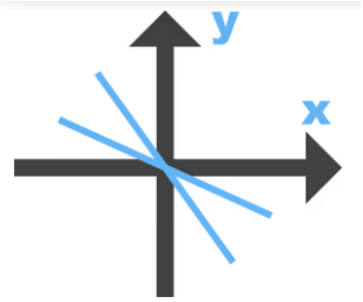
"Never doubt that a small group of thoughtful, committed citizens can change the world; indeed it is the only thing that ever has."

Margaret Mead, Anthropologist

LINEAR MODELING USING THE EQUATION OF A LINE

For the problems in this chapter you may use either of the two forms of a line we've studied: the slope-intercept form,

$$y = mx + b$$



or the point-slope form,

$$y - y_1 = m(x - x_1).$$

I believe you'll find the point-slope form easier because problems in this chapter will provide us with a pair of points on the line, not the *y*-intercept. In addition, this form is the preferred method in calculus.

EXAMPLE 1: **On day 1 the amount, *A*, of digoxin in the body was 3.2 milligrams (mg). On day 7 the amount was 17 mg. Assume that the amount of digoxin is a linear function of the day, *d*.**

a. Find the linear function that expresses the amount, *A*, as a function of the day, *d*.

b. What does the formula predict for the amount of digoxin on day 10?

c. On what day will *A* be 51.5 mg?

d. What does the *slope* of the line represent?

e. Solve the formula for *d*.

Solution: Don't be overwhelmed — almost all of the work is in part **a.** We start by organizing the given data into ordered pairs of the form (d, A), where d is the day and A is the amount of digoxin:

$$(1, 3.2) \quad (7, 17)$$

Note: d is basically the x, and A is basically the y.

a. The best method we have for finding the equation of the line through the two points is the point-slope method, whose formula is

$$A - A_1 = m(d - d_1)$$

where m is the slope and (d_1, A_1) is any point on the line.

So the only calculation we need is the slope:

$$m = \frac{\Delta A}{\Delta d} \qquad \left[\text{just like } \frac{\Delta y}{\Delta x}\right]$$

$$= \frac{17 - 3.2}{7 - 1}$$

$$= \frac{13.8}{6}$$

$$= \mathbf{2.3}$$

We can now choose either point; let's choose (7, 17):

$$A - \underline{17} = \mathbf{2.3}(d - \underline{7})$$

This a fine way to write the linear function, but it's actually better to rewrite it in slope-intercept form — this means solve for A and simplify:

$$A - 17 = 2.3(d - 7) \qquad \text{(the point-slope form)}$$

$$\Rightarrow \quad A - 17 = 2.3d - 16.1 \qquad \text{(distribute)}$$

$\Rightarrow \quad A = 2.3d - 16.1 + 17$ (add 17 to each side)

$\Rightarrow \quad \boxed{A = 2.3d + 0.9}$ (the slope-intercept form)

b. Just let $d = 10$ in the formula:

$$A = 2.3d + 0.9 = 2.3(\mathbf{10}) + 0.9 = 23 + 0.9 = \mathbf{23.09\ mg}$$

c. Now let $A = 51.5$ in the formula:

$$\mathbf{51.5} = 2.3d + 0.9 \Rightarrow 50.6 = 2.3d \Rightarrow d = 22 \Rightarrow \mathbf{Day\ 22}$$

d. The slope, 2.3, represents the average daily increase in the amount of digoxin.

e. $A = 2.3d + 0.9$

$\Rightarrow \quad A - 0.9 = 2.3d$

$\Rightarrow \quad \boldsymbol{d = \frac{A - 0.9}{2.3}}$

Homework

1. On day 5 the height of a plant is measured to be 38.1 cm. On day 9 the height has increased to 63.7 cm.

   a. Find the linear formula that expresses the height h of a plant as a function of the day, d.

   b. Use your formula to predict the height of the tree on day 11.

   c. On what day is the plant 127.7 cm tall?

   d. What does the slope of your line represent?

   e. Solve the formula for d.

2. During a recent snowstorm, there were 12 inches of snow on the ground at 7 p.m., and there were 19 inches at 11 p.m. If you were to graph this data (with time on the horizontal axis and snow on the vertical axis), calculate the slope, and then describe what the slope of the line represents in the context of this problem.

3. A scientist measured the pressure at a depth of 2 ft and determined that it was 15.59 lbs/in^2 (pounds per square inch). At a depth of 10 ft the pressure was 19.15 lbs/in^2.



   a. Find the linear formula that expresses the pressure P as a function of the depth d. [Round to 3 digits.]

   b. Use your formula to predict the pressure at a depth of 100 ft.

   c. At what depth would we find a pressure of 85.9 lbs/in^2?

   d. What does the 0.445 in your formula represent?

   e. According to the formula, what would be the pressure at sea level ($d = 0$)? Does this make any sense?

   f. Solve the formula for d.

Solutions

1. a. $h = 6.4d + 6.1$ b. 76.5 cm c. day 19

   d. It's the amount the plant grows each day (on average)

   e. $d = \dfrac{h - 6.1}{6.4}$

2. $m = \frac{7}{4}$ in/hr

The slope represents the average hourly rate of change in the snow level. That is, it's a measure of how fast it's snowing.

3. a. $P = 0.445d + 14.7$

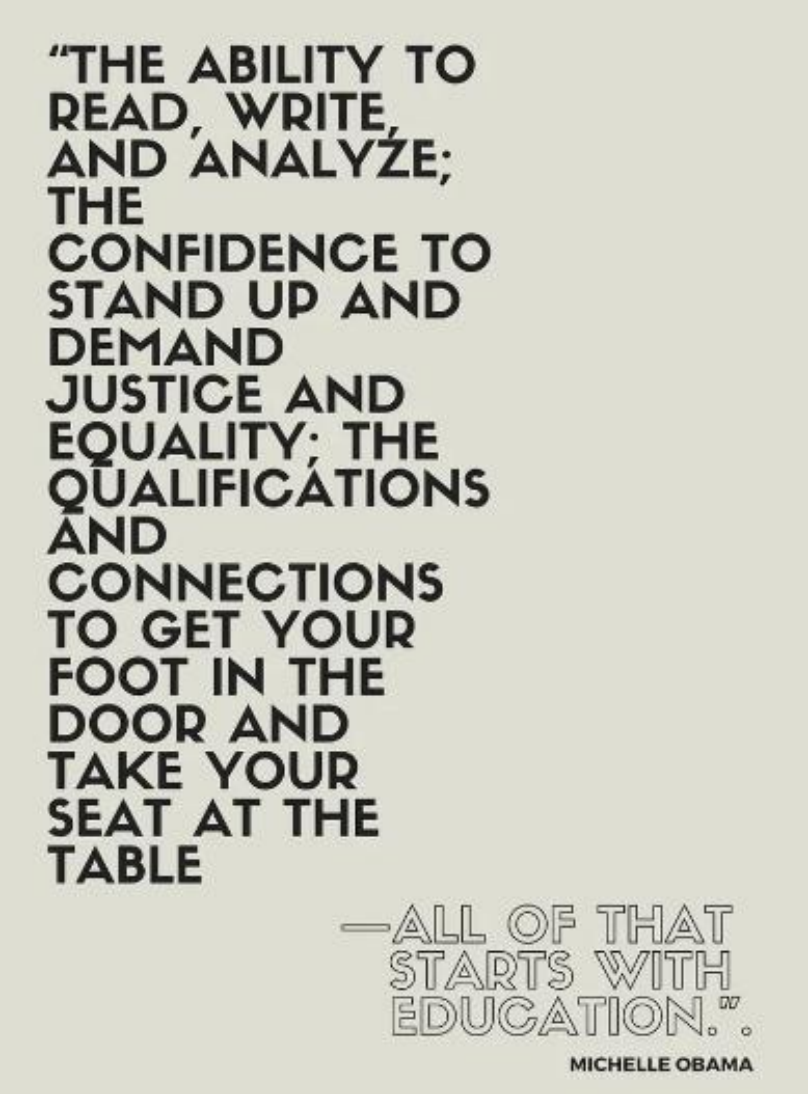
b. 59.2 lbs/in^2

c. 160 ft

d. The increase in pressure for each additional foot below sea level.

e. The pressure would be 14.7 lbs/in^2. It's not an error. Can you account for the pressure? Hint: It's not water pressure.

f. $d = \frac{P - 14.7}{0.445}$


"THE ABILITY TO READ, WRITE, AND ANALYZE; THE CONFIDENCE TO STAND UP AND DEMAND JUSTICE AND EQUALITY; THE QUALIFICATIONS AND CONNECTIONS TO GET YOUR FOOT IN THE DOOR AND TAKE YOUR SEAT AT THE TABLE
—ALL OF THAT STARTS WITH EDUCATION.".
MICHELLE OBAMA

MIXTURE PROBLEMS – COINS

❑ *INTRODUCTION*

Milo has 7 dimes. This means he has \$0.70 worth of dimes. We'll call the **7** the *number* of coins that Milo has, call \$0.10 (or 10¢) the *unit value*, and call the **\$0.70** (or 70¢) the *total value* of Milo's coins.



If Sierra has 5 quarters, then the *number* of coins is 5, the *unit value* is \$0.25 (or 25¢), and the *total value* is \$1.25 (or 125¢).

In both examples, how was the total value of the coins calculated? By *multiplying* the number of coins by their unit value. For example, using Sierra's quarters:

5 quarters × \$0.25 per quarter = \$1.25

In general,

Number of Coins × Unit Value = Total Value

❑ *COIN PROBLEMS*

EXAMPLE 1: **Brennan has 3 times as many nickels as dimes. (That is, the number of nickels is 3 times the number of dimes.) If the total value of the coins is \$2.75, how many of each coin does Brennan have?**

Solution: The easiest way to solve these kinds of problems (to avoid decimals) is to think of all money amounts in pennies (cents). For example, instead of saying that a quarter has a unit value of \$0.25, we'll say that its unit value is 25¢. And the total value of \$2.75 in this problem will be represented by the number 275¢. So, here we go.

First, we notice that we are asked for two different things: the number of nickels and the number of dimes. That's two variables we'll have to create -- let's declare ***n*** to represent the number of nickels, and let ***d*** equal the number of dimes. As you'll see, this means that we'll need two equations to find the values of the two variables.

Let's set up a chart to solve this problem. The central idea of the chart is that, for any given kind of coin, if we multiply the number of coins by its unit value, we get the total value of that coin. Study this chart carefully.

	Number of Coins × Unit Value = Total Value		
nickels	***n***	5	$5n$
dimes	***d***	10	$10d$
mixture	n/a	n/a	275

Since it's given that the number of nickels is 3 times the number of dimes, we get our first equation:

$$n = 3d$$

Also, the total value of the nickels ($5n$) plus the total value of the dimes ($10d$) should equal the total value of the mixture; that is,

$$5n + 10d = 275$$

Sure enough, we have two equations with two variables, just as we had predicted. And now we solve the set of two equations by rewriting the second equation, but using the fact that $n = 3d$:

$$5(\mathbf{3}\boldsymbol{d}) + 10d = 275 \qquad \text{(substitute } 3d \text{ for } n\text{)}$$

$\Rightarrow \quad 15d + 10d = 275$

$\Rightarrow \quad 25d = 275$

$\Rightarrow \quad d = 11$, so Brennan has 11 dimes

Since $d = 11$, it's pretty easy to calculate n:

$n = 3d = 3(\mathbf{11}) = 33$, so Brennan also has 33 nickels.

In summary, Brennan has

11 dimes and 33 nickels

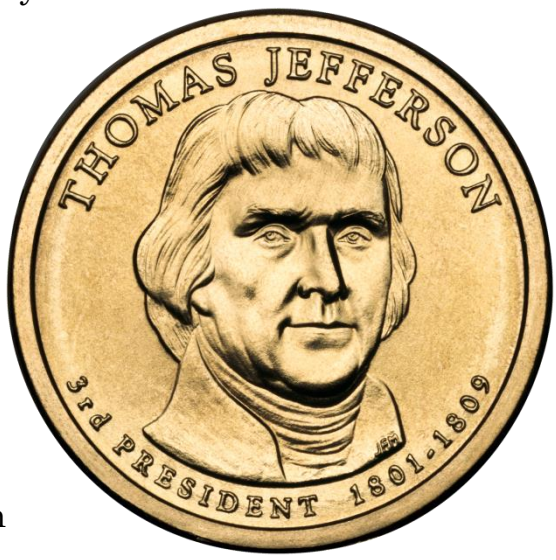
Homework

1. Lisa has 7 times as many dimes as nickels. If the total value of the coins is \$3.75, how many of each coin does Lisa have?

2. Orion has 5 times as many quarters as nickels. If the total value of the coins is \$3.90, how many of each coin does Orion have?

3. Lauren has 4 times as many pennies as dimes. If the total value of the coins is \$0.98, how many of each coin does Lauren have?

4. Cal has 10 times as many nickels as quarters. If the total value of the coins is \$6.75, how many of each coin does Cal have?



EXAMPLE 2: **Alana has 3 more quarters than pennies. If the total value of the coins is $5.17, how many of each coin does Alana have?**

Solution: Letting q represent the number of quarters and p the number of pennies, here's the chart:

	Number of Coins × Unit Value = Total Value		
quarters	q	25	$25q$
pennies	p	1	$1p$
mixture	n/a	n/a	517

The first sentence of the problem, the relationship between the number of quarters and the number of pennies, can also be expressed this way: *The number of quarters is 3 more than the number of pennies.* This leads to the equation:

$$q = p + 3$$

For our second equation, we use the same logic as Example 1. The value of the quarters ($25q$) plus the value of the pennies ($1p$) should equal the total value of the coin mixture; that is,

$$25q + 1p = 517$$

Since $q = p + 3$ from the first equation, we now change the q in the second equation to $p + 3$:

$$25(\mathbf{p + 3}) + 1p = 517 \qquad \text{(since } q = p + 3\text{)}$$

$$\Rightarrow \quad 25p + 75 + p = 517 \qquad \text{(distribute)}$$

$$\Rightarrow \quad 26p + 75 = 517 \qquad \text{(combine like terms)}$$

$$\Rightarrow \quad 26p = 442 \qquad \text{(subtract 75 from each side)}$$

$$\Rightarrow \quad p = 17 \qquad \text{(divide each side by 26)}$$

So Alana has 17 pennies.

Thus, the number of quarters is

$$q = p + 3 = \mathbf{17} + 3 = 20$$

To conclude, Alana has

20 quarters and 17 pennies

Homework

5. Alana has 5 more quarters than nickels. If the total value of the coins is \$4.25, how many of each coin does Alana have?

6. Ben has 7 more dimes than pennies. If the total value of the coins is \$1.47, how many of each coin does Ben have?

7. Andrew has 2 more nickels than quarters. If the total value of the coins is \$5.20, how many of each coin does Andrew have?

8. Justus has 10 more pennies than dimes. If the total value of the coins is \$3.73, how many of each coin does Justus have?

EXAMPLE 3: **Amir has a total of 14 coins, some quarters and some dimes. If the total value of the coins is \$2.75, how many of each coin does Amir have?**

Solution: Since Amir has some unknown number of quarters and dimes, let's let ***q*** stand for the number of quarters and let ***d*** be the number of dimes. The following chart is very similar to the previous ones:

	Number of Coins ×	Unit Value =	Total Value
quarters	q	25	$25q$
dimes	d	10	$10d$
mixture	14	n/a	275

We're ready for our first equation:

$q + d = 14$ (14 coins altogether)

What else do we know? As before, the total value of the quarters plus the total value of the dimes must equal the total value of the coin mixture, which is 275¢:

$25q + 10d = 275$

Let's summarize what we have, and what we're looking for. We're now trying to solve two equations with two variables:

$q + d = 14$
$25q + 10d = 275$ (we want to find both q and d)

How about we solve for q in the first equation, $q + d = 14$. By subtracting d from each side of the equation, our system of equations becomes

$q = 14 - d$
$25q + 10d = 275$

Now change the q in the second equation to $14 - d$ from the first equation:

$25(14 - d) + 10d = 275$ (since $q = 14 - d$)

What has this maneuver accomplished? We now have an equation with just <u>one</u> variable in it -- this we can solve:

$\Rightarrow \quad 350 - 25d + 10d = 275$ (distribute)

$\Rightarrow \quad 350 - 15d = 275$ (combine like terms)

$\Rightarrow \quad -15d = -75$ (subtract 350 from each side)

$\Rightarrow \quad d = 5$ (divide each side by –15)

This means that Amir has 5 dimes, and since $q = 14 - d$, we calculate that the number of quarters is $14 - 5 = 9$. In summary, Amir has

9 quarters and 5 dimes

Homework

9. The total number of coins is 20, some nickels and some dimes. If the total value of the coins is $1.40, how many of each coin are there?

10. The total number of coins is 29, some quarters and some pennies. If the total value of the coins is $3.89, how many of each coin are there?

11. The total number of coins is 40, some nickels and some quarters. If the total value of the coins is $5.00, how many of each coin are there?

12. The total number of coins is 87, some dimes and some pennies. If the total value of the coins is $6.09, how many of each coin are there?

Solutions

1. 5 nickels and 35 dimes

2. 15 quarters and 3 nickels

3. 28 pennies and 7 dimes

4. 90 nickels and 9 quarters

5. 15 quarters and 10 nickels

6. 14 dimes and 7 pennies

7. 19 nickels and 17 quarters

8. 43 pennies and 33 dimes

9. 12 nickels and 8 dimes

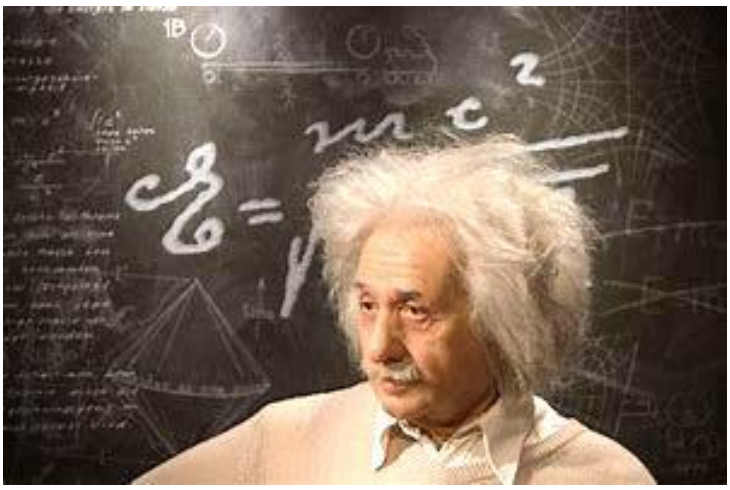
10. 15 quarters and 14 pennies

11. 25 nickels and 15 quarters

12. 58 dimes and 29 pennies

"Any fool can know. The point is to understand."

– Albert Einstein



PEANUTS, ETC.

❑ *THE BASICS*



Consider the following problem: You bought 12 lbs of peanuts that were priced at $4 per pound, written "$4/lb" (and called the *unit price*). What is the total cost (or total value) of the peanuts? It should be clear that we could multiply 12 lbs by $4/lb and end up with **$48**.

Similarly, if we bought N pounds of cashews at $9/lb, the total value would be $9 \times N$, which of course can be written $\mathbf{9}\boldsymbol{N}$.

And for our third example, if we had $(x + 5)$ lbs of walnuts which were worth $7/lb, the total value of those walnuts would be $\mathbf{7}\boldsymbol{(x + 5)}$.

In general,

Pounds of Nuts × Unit Price = **Total Value**

Here's another idea we need for this chapter. If you had 7 lbs of cashews, and you mixed them with 13 lbs of peanuts, you would have a total of **20** pounds of nuts — that should be obvious. But since this is an algebra class, we also need to deal with the following: If you had 10 lbs of almonds and "H" pounds of hazelnuts, what is the total weight of the nuts? It would be $10 + H$, or $\boldsymbol{H + 10}$.

We need to understand one more concept before we solve our nutty examples. Suppose you had a bucket of macadamia nuts whose total value was $25, and there's a second bucket containing almonds worth a total value of $35. Now we pour the two buckets of nuts into one huge

bin. The question is, What is the total value of the mixed nuts in the bin? Well, the correct answer is **$60**, easily found by adding the total value of the macadamia nuts ($25) and the total value of the almonds ($35). Make sense?

Homework

1. If you mix 17 pounds of almonds with 23 pounds of cashews, what is the total weight of the nuts?

2. If you mix A pounds of almonds with 25 pounds of peanuts, what is the total weight of the nuts?

3. If you mix P pounds of pistachios with W pounds of walnuts, what is the total weight of the nuts?

4. One bin contains $48 worth of peanuts; another bin contains $52 worth of cashews. If the two bins are combined into one bucket, what is the total value of nuts in the bucket?

5. A bucket of almonds is worth $210. A bucket of peanuts is worth 30P$. Now dump both buckets of nuts into one bin. What is the total value of the nuts in the bin?



6. Find the total value of 10 lbs of pecans if the unit price is $13/lb.

7. Find the total value of P lbs of pecans if the unit price is $25/lb?

8. If peanuts sell for $9/lb, find the total value of $(x + 5)$ pounds.

❑ *THE EXAMPLES*

EXAMPLE 1: **The Nutte Shoppe has 14 lbs of walnuts which are priced at \$15/lb. How many pounds of almonds which are priced at \$30/lb should be mixed with the walnuts to get a mixture that will sell for \$24/lb?**

Solution: We've got walnuts, 14 lbs at \$15/lb. We've got almonds — we don't know how many pounds — but we know they're worth \$30/lb. And when we mix them together, we get some number of pounds whose unit price is \$24/lb.

Do you see that the problem is asking for the number of pounds of almonds we need to mix with the walnuts? Since this is unknown, how about we let "$\boldsymbol{a}$" represent the number of pounds of almonds. Let's try to organize this information into a chart to make the solution clearer.

	Pounds of Nuts × Unit Price = Total Value		
walnuts	14	15	210
almonds	$\boldsymbol{a}$	30	$30\boldsymbol{a}$
mixture	$\boldsymbol{a} + 14$	24	$24(\boldsymbol{a} + 14)$

As mentioned before the example, adding the dollar values of the two kinds of nuts must equal the dollar value of the mixture; that is,

$210 + 30a = 24(a + 14)$ (dollar values must add up)

$\Rightarrow$ $210 + 30a = 24a + 336$ (distribute)

$\Rightarrow$ $210 + 6a = 336$ (subtract $24a$)

$\Rightarrow$ $6a = 126$ (subtract 210)

$\Rightarrow$ $a = 21$ (divide by 6)

We now know how many pounds of almonds are needed to mix with the walnuts to produce a mixture worth \$24/lb:

21 lbs of almonds

EXAMPLE 2: **Safeway has 25 lbs of peanuts which are priced at \$8/lb, and mixes them with 15 lbs of cashews which are priced at \$24/lb. Find the *unit price* of the mixture.**

Solution: First note that the chart below uses the general term *quantity* instead of *pounds of nuts*. Letting u represent the unit price of the mixture, we get the following chart:

	Quantity	× Unit Price	= Total Value
peanuts	25	8	200
cashews	15	24	360
mixture	40	u	$40u$

The 40 was obtained by adding the pounds of the two kinds of nuts that comprise the mixture, and each box in the Total Value column was obtained by multiplying the two boxes to its left. As before, we note that the total value of the peanuts plus the total value of the cashews must equal the total value of the mixture:

$$200 + 360 = 40u$$

$$\Rightarrow \quad 560 = 40u$$

$$\Rightarrow \quad u = 14$$

And thus the unit price of the mixture is | \$14 per pound |

EXAMPLE 3: **Hadley's sells walnuts which are priced at \$22/lb and cashews which are priced at \$26/lb. How many pounds of each type of nut should Hadley's mix together to get 28 pounds of nuts which would sell for \$25/lb?**

Solution: We're letting w represent the Quantity (number of pounds) of walnuts, and similarly for c. All the Unit Prices are given to us in the problem, and each Total Value is the product of the two entries to its left:

	Quantity	× Unit Price	= Total Value
walnuts	w	22	$22w$
cashews	c	26	$26c$
mixture	28	25	700

Since the quantities must total 28 lbs, we get the equation

$$w + c = 28$$

Since the total values must add up, we also get

$$22w + 26c = 700$$

We'll solve this pair of equations by Substitution, by solving the first equation for c:

$$c = 28 - w$$

Plugging this new equation into the second equation gives:

$$22w + 26(28 - w) = 700 \quad \text{(just one variable!)}$$

$$\Rightarrow \quad 22w + 728 - 26w = 700$$

$$\Rightarrow \quad -4w = -28$$

$$\Rightarrow \quad w = 7, \text{ so we have 7 lbs of walnuts}$$

Then $c = 28 - w = 28 - \mathbf{7} = 21$

7 lbs of walnuts
21 lbs of cashews

Homework

9. The Nut House has 16 lbs of peanuts which are priced at $13/lb. How many pounds of cashews which are priced at $18/lb should be mixed with the peanuts to get a mixture that will sell for $14/lb?

10. Hadley's has 4 lbs of peanuts which are priced at $12/lb, and mixes them with 11 lbs of almonds which are priced at $27/lb. Find the *unit price* of the mixture.



11. Walnut Heaven sells cashews which are priced at $4/lb and almonds which are priced at $7/lb. How many pounds of each type of nut should Walnut Heaven mix together to get 18 pounds of nuts which would sell for $6/lb?

12. Trader Joe's has 4 lbs of walnuts which are priced at $24/lb. How many pounds of almonds which are priced at $10/lb should be mixed with the walnuts to get a mixture that will sell for $18/lb?

13. Peanut Palace has 28 lbs of walnuts which are priced at $19/lb, and mixes them with 12 lbs of almonds which are priced at $29/lb. Find the *unit price* of the mixture.

14. Nutty Nuts sells cashews which are priced at $2/lb and almonds which are priced at $26/lb. How many pounds of each type of nut should Nutty Nuts mix together to get 32 pounds of nuts which would sell for $17/lb?

15. Safeway has 6 lbs of cashews which are priced at \$4/lb, and mixes them with 12 lbs of almonds which are priced at \$7/lb. Find the *unit price* of the mixture.

16. The Nutte Shoppe has 4 lbs of cashews which are priced at \$22/lb. How many pounds of almonds which are priced at \$13/lb should be mixed with the cashews to get a mixture that will sell for \$17/lb?



17. Hadley's sells walnuts which are priced at \$22/lb and cashews which are priced at \$26/lb. How many pounds of each type of nut should Hadley's mix together to get 28 pounds of nuts which would sell for \$25/lb?

Solutions

1. 40 lbs

2. $A + 25$ lbs

3. $P + W$ lbs

4. \$100

5. \$210 + \$30P

6. \$130

7. \$25P

8. \$9(x + 5)

9. 4 lbs of cashews

10. \$23/lb

11. 6 lbs of cashews and 12 lbs of almonds

12. 3 lbs of almonds

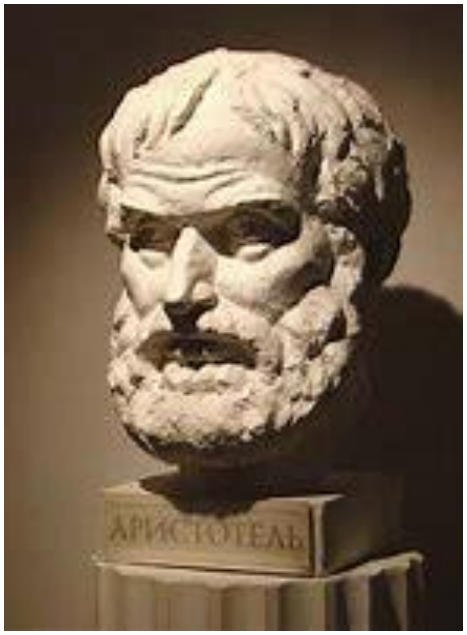
13. \$22/lb

14. 12 lbs of cashews and 20 lbs of almonds

15. \$6/lb

16. 5 lbs of almonds

17. 7 lbs walnuts and 21 lbs cashews



"It is the mark of an educated mind to entertain a thought without accepting it."

– ARISTOTLE –

PERCENT MIXTURE PROBLEMS – PART I

❑ *INTRODUCTION*



NaOCl
sodium hypochlorite

Suppose you have 12 quarts of a bleach solution which is at a 25% concentration. This means that 25% of the solution is bleach, while the other 75% is some neutral substance (usually water). How much of the 12 quarts is actually bleach? Since the solution is 25% bleach, we take 25% of 12, which is $12 \times 0.25 = 3$ quarts of bleach.

The 12 quarts of solution is called the total **quantity**.

The 25% is called the **concentration**.

The 3 quarts of bleach is called the actual **amount**.

The calculation above shows us the formula which ties all these ideas together, and is the basis for this chapter and the next one:

$$\text{Quantity} \times \%\ \text{Concentration} = \text{Amount}$$

Important Note:

To convert 63% to a decimal, move the decimal point (it's after the 3) two places to the left and drop the percent sign, giving 0.63. To convert the decimal 0.08 to a percent, move the decimal point two places to the right and add a percent sign to get 8%.

Homework

1. A 40-quart solution of sulfuric acid is at a 30% concentration. How many quarts of the solution are sulfuric acid? How many quarts are water?
2. A 25-kg solution of salt water is 20% NaCl (salt). How many kg of the solution consist of NaCl? How many kg are water?

❑ *EXAMPLES*

EXAMPLE 1: **A restaurant mixes 4 mL of one 19% sugar beverage with an unknown amount of an 84% sugar beverage. If the final mixture is 64% sugar, how many mL of the 84% sugar beverage were used?**

Solution: Let's organize all the known quantities and the unknown quantity in a chart for easy reference. Notice that the items being mixed (the ingredients) are in the first two rows, the final solution (the mixture) is in the bottom row, and the formula we developed above constitutes the column headings of columns 2, 3, and 4. Carefully re-read the problem and note how all of its information has been placed in the chart:

	Quantity	x Concentration	= Amount
19% sugar	4	19%	4(0.19)
84% sugar	x	84%	$0.84x$
final mixture	$x + 4$	64%	$0.64(x + 4)$

We're letting x represent the quantity of the 84% sugar beverage. Note that if we mix 4 mL with x mL, we'll get a total quantity of $x + 4$ mL.

The Quantity column provides us with a potential equation: $4 + x = x + 4$. Unfortunately, this equation falls apart and tells us nothing about x. So we hop over to the Amount column and use the following reasoning: The actual amount of sugar in the final mixture must equal the sum of the actual amounts of sugar in the ingredients:

$$4(0.19) + 0.84x = 0.64(x + 4)$$

$\Rightarrow$ $0.76 + 0.84x = 0.64x + 2.56$ (distribute)

$\Rightarrow$ $0.2x + 0.76 = 2.56$ (subtract $0.64x$)

$\Rightarrow$ $0.2x = 1.8$ (subtract 0.76)

$\Rightarrow$ $x = 9$ (divide by 0.2)

Therefore, 9 mL of the 84% sugar beverage were used.

EXAMPLE 2: **A 72% acid mixture is created by mixing 12 pints of a 66% acid solution with 8 pints of another acid solution. Find the percent concentration of the other acid solution.**

Solution: Since we are now experts on this subject, we'll let C stand for the unknown concentration and get right to the following chart:

	Quantity x Concentration = Amount		
66% acid	12	66%	12(0.66)
unknown acid	8	C	$8C$
final mixture	20	72%	20(0.72)

We could write an equation stating that the Quantities must add up, giving us $12 + 8 = 20$ — which is true, but useless. As in the previous example, the Amount column gives us our equation with the variable in it:

$$12(0.66) + 8C = 20(0.72)$$

$$\Rightarrow \quad 7.92 + 8C = 14.4 \qquad \text{(arithmetic)}$$

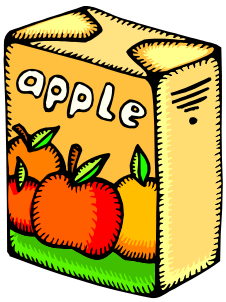
$$\Rightarrow \quad 8C = 6.48 \qquad \text{(subtract 7.92)}$$

$$\Rightarrow \quad C = 0.81 \qquad \text{(divide by 8)}$$

Convert the decimal 0.81 to a percent by moving the decimal point two places to the right.

Thus, the concentration of the other acid is $\boxed{81\%}$

EXAMPLE 3: **A grocer mixes 2 L of a 34% apple juice beverage with 4 L of a 19% apple juice beverage. What is the percent concentration of apple juice in the resulting mixture?**



Solution: This example is essentially the same as the previous one, except that this problem asks for the concentration of the final mixture, not one of the ingredients.

	Quantity	x Concentration	= Amount
34% juice	2	34%	2(0.34)
19% juice	4	19%	4(0.19)
final mixture	6	C	$6C$

Although the Quantity column gives us a true equation, $2 + 4 = 6$, this equation is useless in trying to solve for C. So let's jump to the Amount column and use the concept that the

actual amounts of apple juice in the 34% and the 19% juices must add up to the actual amount of apple juice in the mixture:

$$2(0.34) + 4(0.19) = 6C$$

$$\Rightarrow \quad 0.68 + 0.76 = 6C$$

$$\Rightarrow \quad 1.44 = 6C$$

$$\Rightarrow \quad C = 0.24$$

Hence, the concentration of the mixture is $\boxed{24\%}$

Homework

3. A chemist mixes 8 mL of a 37% glycerin solution with some 57% glycerin solution. If the final mixture is 49% glycerin, how many mL of the 57% glycerin solution were used?

4. A 63% acid solution is created by mixing 3 liters of an 87% acid solution with 12 liters of another acid solution. Find the concentration of the other acid solution.

5. An allergist wants to mix some pure water with 2 ounces of an 18% pollen serum to produce a serum that is 12% pollen. How many ounces of pure water should he use?

6. A restaurant mixes 5 liters of a 13% sugar liquid with some 21% sugar liquid. If the final mixture is 19% sugar, how many liters of the 21% sugar liquid were used?



7. An 18-gallon solution that is 45% vinegar is mixed with 9 gallons of another solution. The resulting solution is 58% vinegar. Find the percentage of vinegar in the other solution.

8. A researcher has a 4-L mixture that is 40% saline. How many liters of pure water should she add to make a mixture that is 16% saline?

9. A company mixes 14 L of pure sulfate with 16 L of 10% sulfate shampoo. What percentage sulfate is the resulting mixture?

10. What is the percent concentration of sand in a mixture a mason made by mixing 12 pounds of pure sand with 2 pounds of 72% sand cement?

11. How many mL of pure water should a detective mix with 2 mL of a 54% cyanide poison to produce a poison that is 12% cyanide?

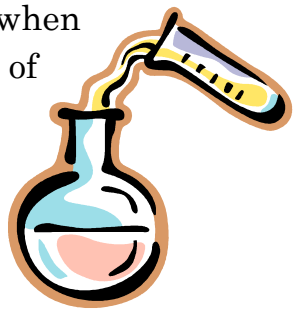
12. A bakery mixes 8 pounds of a 64% butter frosting with 4 pounds of a 73% butter frosting. What is the percentage of butter in the resulting mixture?

13. A 52% mixture is created by mixing 16 liters of a 97% solution with 24 liters of another solution. Find the concentration of the other solution.

14. What is the concentration of the mixture formed when 57 liters of a 30% solution are mixed with 3 liters of a 50% solution?

15. A 22% mixture is created by mixing 22 liters of a 5% solution with 17 liters of another solution. Find the concentration of the other solution.

16. What is the concentration of the mixture formed when 4 liters of a 43% solution are mixed with 60 liters of an 11% solution?



17. A 71% mixture is created by mixing 56 liters of a 98% solution with 54 liters of another solution. Find the concentration of the other solution.

18. What is the concentration of the mixture formed when 28 liters of a 90% solution are mixed with 36 liters of a 42% solution?

19. How many liters of a 62% solution must be added to 60 liters of a 77% solution to get a mixture whose concentration is 72%?

20. How many liters of a 99% solution must be added to 17 liters of a 96% solution to get a mixture whose concentration is 98%?

Solutions

1. 12 qts acid; 28 qts H_2O

2. 5 kg NaCl; 20 kg H_2O

3. 12 mL **4**. 57% acid **5**. 1 oz

6. 15 L **7**. 84% vinegar **8**. 6 L

9. 52% sulfate **10**. 96% sand **11**. 7 mL

12. 67% butter **13**. 22% **14**. 31%

15. 44% **16**. 13% **17**. 43%

18. 63% **19**. 30 L **20**. 34 L



"Few will have the greatness to bend history itself; but each of us can work to change a small portion of events, and in the total of all those acts will be written the history of this generation." Robert F. Kennedy

PERCENT MIXTURE PROBLEMS – PART II

❑ *INTRODUCTION*



We continue our discussion of percent mixture problems. There are two differences between this chapter's problems and the problems in Part I: First, we are going to be asked to find the amounts of the <u>two</u> quantities that must be mixed together to achieve the desired outcome. And second, we need to understand the following facts:

Pure bleach would have a **100%** bleach concentration, while **pure water** would have a **0%** bleach concentration. Be sure this makes sense to you.

Homework

1. What is the percent concentration of sodium in pure sodium?
2. What is the percent concentration of nitric acid in pure water?
3. Consider a gallon of pure antifreeze. What is the percent concentration of antifreeze? What is the percent concentration of water?
4. Consider a liter of pure water. What is the percent concentration of hydrochloric acid? What is the percent concentration of water?

❑ *EXAMPLES*

EXAMPLE 1: **How many quarts each of a 62% poison solution and a 6% poison solution must a detective mix to get 14 quarts of a mixture that is 30% poison?**

Solution: Let x represent the quarts of the 62% poison.
Let y represent the quarts of the 6% poison.

	Quantity	x Concentration	= Amount
62% poison	x	62%	$0.62x$
6% poison	y	6%	$0.06y$
final mixture	14	30%	14×0.30

Looking at the Quantity column, we're mixing x quarts with y quarts to get a total of 14 quarts in the final mixture. It makes sense to say that the sum of x and y must be 14:

$$x + y = 14 \qquad \textbf{[Equation \#1]}$$



Consider the Concentration column. Does adding the concentrations together make any sense? Of course not: $62\% + 6\% \neq 30\%$. In fact, our intuition tells us that the concentration of the final solution ought to be somewhere between the concentrations of the ingredients being mixed. But we need another equation -- since we have two variables x and y, we'll need two equations in order to find x and y.

Now look at the Amount column. Each ingredient being mixed together contains poison and water. Does it make sense that the actual amount of poison in the final solution must be the sum of the actual amounts of poison in the ingredients? This leads to the second equation:

$$0.62x + 0.06y = 14 \times 0.30$$

or, $$0.62x + 0.06y = 4.2 \qquad \textbf{[Equation \#2]}$$

Equations 1 and 2 constitute a system of two equations in two variables, which we will solve using the Addition method.

$$\begin{array}{lll} x + y = 14 & \text{(times } -0.62\text{)} \Rightarrow & -0.62x - 0.62y = -8.68 \\ 0.62x + 0.06y = 4.2 & \text{(leave alone)} \Rightarrow & 0.62x + 0.06y = 4.2 \\ & & \hline \\ & \text{Adding} \Rightarrow & 0 - 0.56y = -4.48 \\ & \Rightarrow & y = 8 \end{array}$$

Since y stood for the number of quarts of the 6% solution, we see that the detective needs 8 quarts of the 6% solution. Moreover, since $x + y = 14$, we see that $x + 8 = 14 \Rightarrow x = 6$, which means that the detective also needs 6 quarts of the 62% solution.

6 quarts of the 62% poison, &
8 quarts of the 6% poison

EXAMPLE 2: **A druggist wants to create 4 liters of a 94% anti-malaria medicine. How many liters each of pure anti-malaria medicine and a 92% anti-malaria medicine must she mix together?**

Solution:

Let x represent the liters of pure anti-malaria medicine.
Let y represent the liters of the 92% anti-malaria medicine.

Remembering that pure anti-malaria medicine has an anti-malaria concentration of 100%, we put all our information in the chart:

	Quantity x	Concentration =	Amount
pure medicine	x	100%	$1.00x$
92% medicine	y	92%	$0.92y$
final mixture	4	94%	4×0.94

One of our equations comes from the fact that the quantities x and y must add up to 4:

$$x + y = 4$$

The second equation comes from the fact that the amount of anti-malaria medicine in the 100% medicine plus the amount of anti-malaria medicine in the 92% medicine must equal the amount of anti-malaria medicine in the final mixture:

$$1.00x + 0.92y = 0.94 \times 4,$$

$$\text{or,} \quad x + 0.92y = 3.76$$

Let's solve this system of equations via the elimination method.

$$\begin{array}{lll} x + y = 4 & \text{(leave alone)} & \Rightarrow \quad x + y = 4 \\ x + 0.92y = 3.76 & \text{(times } -1\text{)} & \Rightarrow \quad -x - 0.92y = -3.76 \\ & & \hline \\ & \text{Adding} & \Rightarrow \quad 0 + 0.08y = 0.24 \\ & & \Rightarrow \quad y = 3 \end{array}$$

Since $y = 3$, and $x + y = 4$, it follows that $x = 1$. We now know how many liters of each medicine she must mix together.

3 liters of the 92% medicine, & 1 liter of the pure medicine

EXAMPLE 3: **How many fluid ounces each of pure water and a 10% albuterol inhalant must an allergist mix to get 5 fluid ounces of an inhalant that is 6% albuterol?**

Solution: This is the same scenario as the two previous examples, so we can get right to it; note, however, that pure

water contains 0% albuterol (that is, there is no albuterol in pure water).

	Quantity	x Concentration	= Amount
pure water	x	0%	$0x$
10% albuterol	y	10%	$0.10y$
final mixture	5	6%	5×0.06

The two equations are $x + y = 5$ and $0x + 0.10y = 5 \times 0.06$, or $0.10y = 0.3$. Setting up the system of equations neatly gives

$$x + y = 5$$
$$0.10y = 0.3$$

The second equation can be easily solved for y, so we don't need the Elimination method. Dividing each side of the second equation by 0.10, we get $y = 3$. Using $x + y = 5$, we find that $x = 2$. Thus,

2 ounces of water, &
3 ounces of the 10%

Homework

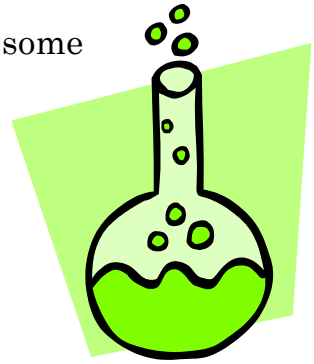
5. How many pounds each of a 97% poison solution and a 67% poison solution must a detective mix to get 15 pounds of a solution that is 91% poison?

6. A druggist wants to create 24 ounces of a 92% alcohol medicine. How many ounces each of pure alcohol medicine and a 76% alcohol medicine must he mix together?

7. How many ounces each of pure water and an 80% albuterol inhalant must an allergist mix to get 20 ounces of an inhalant that is 64% albuterol?

8. A druggist wants to mix some 59% alcohol medicine with some 21% alcohol medicine. How many mL of each substance must she use to get a 19-mL mixture that is 57% alcohol?

9. A detective wants to mix some pure poison with some 20% poison solution. How many pounds of each substance must she use to get a 25-pound mixture that is 52% poison?



10. How many liters each of a 99% solution and an 88% solution must be mixed together to get 22 liters of a mixture whose concentration is 97%?

11. How many liters each of a pure solution and a 12% solution must be mixed together to get 88 liters of a mixture whose concentration is 42%?

12. How many liters each of a 24% solution and a 68% solution must be mixed together to get 22 liters of a mixture whose concentration is 44%?

13. How many liters each of pure water and a 57% solution must be mixed together to get 38 liters of a mixture whose concentration is 42%?

14. How many liters each of a 79% solution and a 70% solution must be mixed together to get 36 liters of a mixture whose concentration is 76%?

15. How many liters each of pure water and a 48% solution must be mixed together to get 72 liters of a mixture whose concentration is 14%?

16. How many liters each of a 9% solution and a 94% solution must be mixed together to get 68 liters of a mixture whose concentration is 69%?

17. How many liters each of pure water and a 20% solution must be mixed together to get 100 liters of a mixture whose concentration is 9%?

18. How many liters each of a 73% solution and a 97% solution must be mixed together to get 12 liters of a mixture whose concentration is 93%?

19. How many liters each of a pure solution and a 58% solution must be mixed together to get 48 liters of a mixture whose concentration is 93%?

Solutions

1. 100%

2. 0%

3. 100% antifreeze; 0% water

4. 0% hydrochloric acid; 100% water

5. 12 pounds of the 97% solution and 3 pounds of the 67% solution

6. 16 ounces of pure alcohol and 8 ounces of the 76% alcohol medicine

7. 4 ounces of water and 16 ounces of the 80% albuterol inhalant

8. 18 mL of the 59% alcohol medicine and 1 mL of the 21% alcohol medicine

9. 10 pounds of pure poison and 15 pounds of the 20% poison solution

10. 18 L of the 99% solution and 4 L of the 88% solution

11. 30 L of the pure solution and 58 L of the 12% solution

12. 12 L of the 24% solution and 10 L of the 68% solution

13. 10 L of pure water and 28 L of the 57% solution

14. 24 L of the 79% solution and 12 L of the 70% solution

15. 51 L of pure water and 21 L of the 48% solution

16. 20 L of the 9% solution and 48 L of the 94% solution

17. 55 L of pure water and 45 L of the 20% solution

18. 2 L of the 73% solution and 10 L of the 97% solution

19. 40 L of the pure solution and 8 L of the 58% solution



"When you do the common things in life in an uncommon way, you will command the attention of the world."

- George Washington Carver (1864-1943)



PRODUCING WIDGETS

❑ *INTRODUCTION*



Suppose we have 10 apples and 13 bananas, which of course means that we have 23 pieces of fruit in total. This can be written with the simple equation: $\mathbf{10 + 13 = 23}$.

Assume we have a apples and b bananas, and also pretend that there are 12 pieces of fruit in total. Then we can write these facts as an equation: $\boldsymbol{a + b = 12}$.

Suppose that yellow widgets cost \$25 apiece to produce (called the *unit cost*), and that we will produce 40 yellow widgets. Then the total cost of production is given by the expression $\mathbf{\$25 \times 40}$ (or $40 \times \$25$).

Assume that the unit cost of maroon widgets is \$14, and that we will produce m of the maroon widgets. Then the total cost of production is the product of \$14 and m: $\mathbf{\$14}\boldsymbol{m}$.

If we make x widgets at \$9 each and y widgets at \$11 each, then

1) the total number of widgets made is $\boldsymbol{x + y}$, and
2) the total production cost is $\mathbf{\$9}\boldsymbol{x} \mathbf{+ \$11}\boldsymbol{y}$

If we make w widgets at \$12 each and q widgets at \$26 each, for a total cost of \$500, we can express this information in the equation $\$12w + \$26q = \mathbf{\$500}$.

If y is designated to be three times whatever x is, then we can write $\boldsymbol{y = 3x}$.

The phrase "The number of green widgets should be two times the number of blue widgets" can be translated into the equation $\boldsymbol{g = 2b}$.

Homework

1. There were 20 vegetables: c carrots and t tomatoes. Write an equation to express this fact.

2. There were r red widgets and p purple widgets. If there were 150 widgets in all, write the equation which expresses this fact.

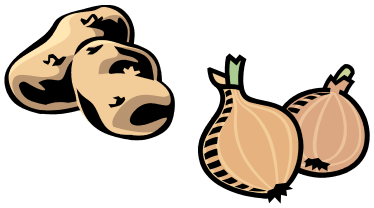
3. At a unit cost of \$49.75 per widget, what is the total cost to produce 100 widgets?

4. At \$3.75 per pound, what is the total cost to buy s pounds of top sirloin.

5. If the unit cost of a widget is \$500, and you produce w of them, find the total cost.

6. If 17 red widgets at a unit production cost of \$5 and 80 brown widgets at a unit production cost of \$1.75 are produced, calculate the total cost.

7. If you bought p pounds of potatoes at \$2 per pound and n pounds of onions at \$4 per pound, write an expression that represents the total cost.



8. In Problem 7, suppose the total cost was \$26. Write an equation which describes this situation. Dollar signs are generally omitted when writing an equation.

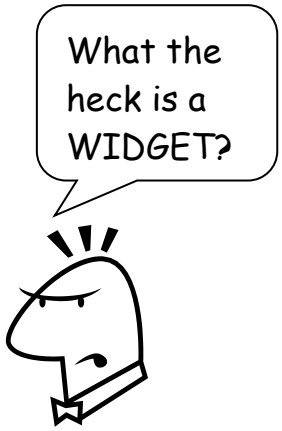
9. The factory spent a total of \$250 producing f flagons at \$7 apiece and d dragons at \$12 apiece. Write an equation which represents this situation.

10. Write an equation which expresses each statement:

    a. x is 4 more than y.
    b. T is 3 less than Z.
    c. u is twice as big as v.
    d. g is three times as big as h.
    e. w is four times as large as s.
    f. h is one-half as large as k.
    g. E is one-third the size of m.
    h. A is 12 less than B.
    i. W is 1 less than Q.

❑ *THE RIGHT MIXTURE*

I suppose if we had an infinite amount of money, we could produce all the widgets in all the colors we would ever need, but of course this is absurd. We have constraints (limitations) to worry about; the first will concern the number of widgets to be produced, and the second will be the amount of money we can spend producing them. Each of these constraints will be represented by an equation. Thus, each application we solve will have <u>two</u> equations that we must solve at the same time, for two different variables.



In this first application, we are told the total number of widgets to produce (some of one color and some of another) and the total amount of money we're allowed to spend.

<u>EXAMPLE 1:</u> **The unit cost of red widgets is $8 and the unit cost of green widgets is $10. Each production run can produce a total of 20 widgets, but we control how many of each color. The production budget is $168. How many widgets of each color should we manufacture?**

<u>Solution:</u> We're seeking two things in this problem: the number of red widgets and the number of green widgets to produce. Let's let

r represent the number of red widgets, and
g represent the number of green widgets.

We're given the unit costs of production: $8 for each red widget and $10 for each green widget. This means that it will cost $8r$ to produce the red widgets, and $10g$ to produce the green widgets.

Since we're seeking the values of two different variables, we expect to have two equations at our disposal. The first equation

represents the fact that we will produce 20 widgets altogether. Since r and g represent the number of widgets of each color, we get the equation

$$r + g = 20 \qquad \text{(20 widgets altogether)}$$

The total cost of production is the sum of the costs of the two colors of widgets -- this yields the equation

$$8r + 10g = 168 \qquad \text{(the total cost of \$168)}$$

We now have a system of two equations in two unknowns:

$$\begin{aligned} r + g &= 20 \\ 8r + 10g &= 168 \end{aligned}$$

$$\begin{aligned} r + g = 20 &\xrightarrow{\text{times } -8} -8r - 8g = -160 \\ 8r + 10g = 168 &\xrightarrow{\text{leave alone}} 8r + 10g = 168 \\ \text{Add the equations:} \quad & 0r + 2g = 8 \\ & g = 4 \end{aligned}$$

This value of g tells us that we need to produce 4 green widgets. Since we need 20 widgets total, it's clear that we need to make 16 of the red widgets. More formally,

$$r + g = 20 \;\Rightarrow\; r + 4 = 20 \;\Rightarrow\; r = 16$$

To summarize, the production schedule should be

16 red widgets & 4 green widgets

Check:

16 + 4 = 20 widgets ✓

16 @ \$8 = \$128

4 @ \$10 = \$ 40

\$168 ✓

In our second application, market research has determined that black widgets sell about twice as well as white widgets. To satisfy this demand, the production department has to solve the following problem:

EXAMPLE 2: **Assume that black widgets cost \$3 apiece and white widgets cost \$7 apiece to produce. Our goal is that the number of black widgets should be twice the number of white widgets, while our production budget is \$260. How many widgets of each color should we produce to satisfy all the stated conditions?**

Solution: We'll let

b = the number of black widgets to produce
w = the number of white widgets to produce

One of the conditions stipulates that the number of black widgets must be twice the number of white widgets. This yields the equation

$$b = 2w \qquad \text{(\# of blacks is twice \# of whites)}$$

Since we're producing b black widgets at a unit cost of \$3, the cost of producing the black widgets is $3b$. Similarly, the cost of the white widgets is $7w$. The sum of these two costs must be \$260, the given budget. This fact gives us the equation

$$3b + 7w = 260 \qquad \text{(the total cost is \$260)}$$

We thus have a system of two equations in two unknowns:

$$\begin{aligned} b &= 2w \\ 3b + 7w &= 260 \end{aligned}$$

To solve this system by the addition method, a little rearranging is in order. In the first equation subtract $2w$ from each side, and leave the second equation alone:

$$\begin{aligned} b - 2w &= 0 \\ 3b + 7w &= 260 \end{aligned}$$

Now the variables are lined up.

Now we'll multiply the equations by appropriate numbers so that when we add the equations we'll eliminate the b:

$$\begin{array}{lcl} b - 2w = 0 & \xrightarrow{\text{times } -3} & -3b + 6w = 0 \\ 3b + 7w = 260 & \xrightarrow{\text{leave alone}} & 3b + 7w = 260 \end{array}$$

Add the equations: $13w = 260$

$\Rightarrow \quad w = 20$

Since $b = 2w$, it follows that $b = 2(\mathbf{20}) = 40$. In conclusion, we instruct the factory to produce

20 white widgets & 40 black widgets

Check: 40 black is twice 20 white ✓

20 @ \$7 = \$140

40 @ \$3 = \$120

\$260 ✓

ALTERNATE METHOD: We can solve the system of equations in the previous example by the *Substitution Method*. Here is the system again:

$$\begin{array}{r} b = 2w \\ 3b + 7w = 260 \end{array}$$

Since $b = 2w$, we can substitute $2w$ for b in the second equation:

$$\begin{array}{ll} & 3(2w) + 7w = 260 \\ \Rightarrow & 6w + 7w = 260 \\ \Rightarrow & 13w = 260 \\ \Rightarrow & w = \mathbf{20} \\ \Rightarrow & b = 2w = 2(20) = \mathbf{40} \end{array}$$

And we get the same values for w and b.

EXAMPLE 3: **Assume that green widgets cost \$5 apiece and yellow widgets cost \$9 apiece to produce. Our goal is that we produce 8 more yellow widgets than green widgets. If we have a total of \$352 to spend, how many widgets of each color should we manufacture?**

Solution: We'll set g and y represent the number of green and yellow widgets. Since we need "8 more yellow widgets than green widgets," we get

$$y = g + 8$$

The total cost to produce all the widgets gives us

$$5g + 9y = 352$$

Substitute the first equation into the second:

$$5g + 9(g + 8) = 352$$

$$\Rightarrow \quad 5g + 9g + 72 = 352$$

$$\Rightarrow \quad 14g = 280$$

$$\Rightarrow \quad g = 20$$

And using the first equation: $y = g + 8 = \mathbf{20} + 8 = 28$

20 green widgets & 28 yellow widgets

Notes: You should check this solution to make sure we have the right numbers of widgets. Also, we could have used the Addition Method to solve the system of equations; you might try this for practice.

Homework

11. The unit cost of red widgets is $6 and the unit cost of white widgets is $8. If the goal is to make 25 widgets for a total of $170, how many widgets of each color should we make?

12. The unit cost of maroon widgets is $5 and the unit cost of white widgets is $9. If the goal is that the number of white widgets should be 8 times the number of maroon widgets, and if the total budget is $462, how many widgets of each color should we make?

13. The unit cost of red widgets is $7 and the unit cost of yellow widgets is $6. If the goal is that the number of yellow widgets should be 75 more than the number of red widgets, and if the total budget is $645, how many widgets of each color should we make?

14. The unit cost of red widgets is $7 and the unit cost of maroon widgets is $8. If the goal is to make 23 widgets for a total of $183, how many widgets of each color should we make?

15. The unit cost of white widgets is $9 and the unit cost of black widgets is $6. If the goal is that the number of black widgets should be 7 more than the number of white widgets, and if the total budget is $57, how many widgets of each color should we make?

16. The unit cost of yellow widgets is $9 and the unit cost of white widgets is $8. If the goal is to make 35 widgets for a total of $305, how many widgets of each color should we make?

17. The unit cost of black widgets is \$9 and the unit cost of white widgets is \$11. If the goal is that the number of white widgets should be 8 times the number of black widgets, and if the total budget is \$2425, how many widgets of each color should we make?

18. The unit cost of green widgets is \$9 and the unit cost of black widgets is \$11. If the goal is to make 9 widgets for a total of \$85, how many widgets of each color should we make?

❑ *A Graphical Approach to the Widget Production Problem*

Each of the widget production problems was solved by generating two equations in two variables from the data given in the problem, followed by solving the resulting system of equations. But in more advanced math, business, and science problems, the system of equations may be unsolvable using algebra; the best solution we may get is an approximation. One of the approximation methods used is ***graphing***. We create the equations from the stated problem, graph each equation, and then estimate where the graphs cross each other (intersect).

EXAMPLE 4: **The unit cost of red widgets is \$13 and the unit cost of green widgets is \$25. Each production run can produce a total of 50 widgets, but we control how many of each color. The production budget is \$758. By graphing, determine how many widgets of each color we should manufacture.**

Solution: This problem leads to the system of equations:

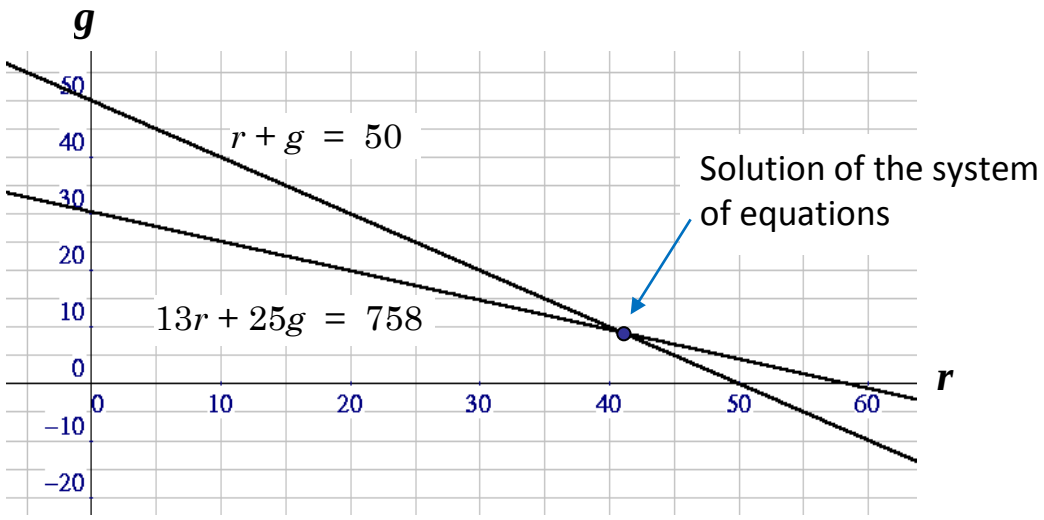
$$\begin{aligned} r + g &= 50 \\ 13r + 25g &= 758 \end{aligned}$$

To graph these two equations, we realize that each of them is a line, and we've learned that the simplest way to graph a line is to use the intercepts. For no particular reason, we will label the horizontal axis as r, and the vertical axis as g. This means that the ordered pairs will be written in the form (r, g).

$r + g = 50$: Setting $r = 0$ gives $g = 50$. Setting $g = 0$ gives $r = 50$. The intercepts are (0, 50) and (50, 0).

$13r + 25g = 758$: When $r = 0$, $g = 30.32$, and when $g = 0$, $r = 58.31$. The intercepts are therefore (0, 30.32) and (58.31, 0).

We now plot each line on the same grid, and then estimate the point of intersection of the two lines.



Focusing on the point of intersection, we estimate the r-coordinate to be about 41 and the g-coordinate to be about 9. That is, the point of intersection is about (41, 9). Our conclusion? We should make **41 red widgets and 9 green widgets**.

There's no homework for this section. All the problems in this chapter should be done using the *Elimination or Substitution Method*.

Review Problems

19. The unit cost of black widgets is \$11 and the unit cost of maroon widgets is \$5. If the goal is that the number of maroon widgets should be 8 times the number of black widgets, and if the total budget is \$306, how many widgets of each color should we make?



20. The unit cost of yellow widgets is \$10 and the unit cost of maroon widgets is \$6. If the goal is that the number of maroon widgets should be 135 more than the number of yellow widgets, and if the total budget is \$1242, how many widgets of each color should we make?

21. The unit cost of green widgets is \$8 and the unit cost of black widgets is \$9. If the goal is to make 45 widgets for a total of \$389, how many widgets of each color should we make?

Solutions

[Notice that we are NOT writing dollar signs in the answers, since we won't use them when we're solving equations.]

1. $c + t = 20$ **2.** $r + p = 150$ **3.** 4975

4. $3.75s$ **5.** $500w$ **6.** 225

7. $2p + 4n$ **8.** $2p + 4n = 26$ **9.** $7f + 12d = 250$

10. a. $x = y + 4$ b. $T = Z - 3$ c. $u = 2v$

d. $g = 3h$ e. $w = 4s$ f. $h = \frac{1}{2}k$

g. $E = \frac{1}{3}m$ h. $A = B - 12$ i. $W = Q - 1$

11. 15 red and 10 white

12. 6 maroon and 48 white

13. 15 red and 90 yellow

14. 1 red and 22 maroon

15. 1 white and 8 black

16. 25 yellow and 10 white

17. 25 black and 200 white

18. 7 green and 2 black

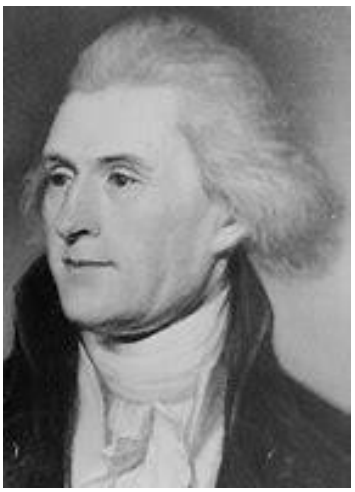
19. 6 black and 48 maroon

20. 27 yellow and 162 maroon

21. 16 green and 29 black

"I'm a great believer in luck, and I find the harder I work, the more I have of it."

– *Thomas Jefferson*



MIXTURE PROBLEMS – RADIATOR

❑ INTRODUCTION



Suppose a 12-liter radiator is filled with an antifreeze/water solution at a 25% concentration of antifreeze. In that case, 25% of the 12 liters is antifreeze, while the rest (75%) is water. Multiplying 12 liters by 25% shows that 3 liters of the solution are antifreeze, and the other 9 liters (12 liters × 75%) are water. Thus, the key formula we need for the percent mixture problems in this chapter is

Quantity × Concentration = Actual Amount

❑ EXAMPLE

A 10-liter radiator is filled with 42% antifreeze solution. How much of the antifreeze solution must be drained and replaced with pure antifreeze to bring the concentration of the final solution up to 58%?

Solution: We'll set up all the info in a chart, utilizing the above formula:

	Quantity (in liters)	× Concentration (of antifreeze)	= Actual Amount (of antifreeze)
Original	10	42%	4.2
Drain	x	42%	$0.42x$
Replace	x	100%	x
Final	10	58%	5.8

Notice that the four rows in our chart represent each of the problem's four phases.

In the statement of the problem, we are given that it's a 10-liter radiator filled with 42% antifreeze. The product of these two numbers is 4.2, as per the formula given above.

The question asks us how much to remove -- we don't know, so this is where the variable x comes in. What is the concentration of the x liters we're to remove? It's the same as the concentration of the entire 10 liters: 42%. And, of course, the product of x with 42% is $0.42x$.

Now for the 3rd row of the chart. The number of liters of pure antifreeze needed to refill the radiator must equal the number of liters drained off, namely x. And the refill fluid is pure antifreeze, which has a concentration of 100%; hence, the last entry in the row is $x \times 100\%$, or simply x.

The bottom row: The radiator has been refilled, so it contains 10 liters of solution, just as in the original row. The concentration of this final mixture is stipulated to be 58%, and the product is clearly 5.8.

Thus, looking at the right-hand column of the chart (Actual Amount), we can say that we started with 4.2 liters of antifreeze, subtracted $0.42x$ liters, added x liters, and ended up with 5.8 liters of antifreeze. Now we translate these thoughts into an equation:

$$4.2 - 0.42x + x = 5.8$$

$$\Rightarrow \quad 0.58x = 1.6$$

$$\Rightarrow \quad x = 2.76, \text{ and thus}$$

2.76 liters should be drained out

Note: If the antifreeze solution needs to be diluted, then we drain a certain amount of the solution and replace it with pure water, which has an antifreeze concentration of 0%.

Homework

1. A 9-liter radiator is filled with 67% antifreeze solution. How much of the antifreeze solution must be drained and replaced with pure antifreeze to bring the concentration of the final mixture up to 95% antifreeze?

2. A 5-liter radiator is filled with 58% antifreeze solution. How much of the antifreeze solution must be drained and replaced with pure water to bring the concentration of the final mixture down to 25% antifreeze?

3. A 4-liter radiator is filled with 32% antifreeze solution. How much of the antifreeze solution must be drained and replaced with pure antifreeze to bring the concentration of the final mixture up to 75% antifreeze?

4. A 20-liter radiator is filled with 75% antifreeze solution. How much of the antifreeze solution must be drained and replaced with pure water to bring the concentration of the final mixture down to 23% antifreeze?

Solutions

1. 7.64 liters

2. Note: The concentration of antifreeze in pure water is 0%.
$5(.58) - .58x + 0x = 5(.25) \Rightarrow$ 2.84 liters

3. 2.53 liters

4. 13.87 liters

Motion Problems – An Introduction

Whether it's the police pursuing a bank robber, or a physicist determining the velocity of a proton in a linear accelerator, the concepts of time, distance, and speed are at the heart of all science and technology.

If you travel from San Francisco to L.A., 400 miles away, and you travel for 8 hours at an average speed of 50 miles per hour, then



you traveled a **DISTANCE** of 400 miles

at a **RATE** (speed) of 50 mph

during a **TIME** interval of 8 hours.

Notice that in this example, if you multiply the rate by the time (50 mph × 8 hrs), you get the distance (400 mi). This idea always holds:

$$\text{Rate} \times \text{Time} = \text{Distance}$$

Motion problems will involve two variables, for which you might want to construct two equations. Two chapters to review so that you can solve the systems of equations are *Systems of Equations – Substitution* and *Systems of Equations – Elimination*.

Homework



1. a. Moe traveled at a rate of 120 km/hr for 12 hours. Find Moe's distance.

   b. Larry flew a distance of 3000 miles in 6 hours. What was Larry's rate?

   c. Curly jogged 12 miles at a rate of 3 mph. How long was Curly jogging?

2. Which is the proper formula for distance?

   a. $d = rt$ b. $d = \frac{r}{t}$ c. $d = \frac{t}{r}$

3. Two skaters leave the skate park and skate in opposite directions, one at 10 mph and the other at 8 mph. After some time, they are 18 miles apart. If d_1 is the distance traveled by the first skater, and if d_2 is the distance traveled by the second skater, write an appropriate equation.

4. A woodpecker traveled from the maple tree to the oak tree at 13 mph, and then made a return trip at 19 mph. If d_1 is the distance it traveled to the oak tree, and if d_2 is the distance from the oak back to the maple, write an appropriate equation.

5. A 1096-mi trip took a total of 16 hours. The speed was 71 mph for the first part of the trip, and then decreased to 67 mph for the rest of the trip. If d_1 is the distance traveled on the first part of the trip, and if d_2 is the distance traveled on the second part of the trip, write an appropriate equation.

6. A jet ski leaves the beach. Nine hours later a motorboat begins to pursue the jet ski and finally catches up with it. If d_1 is the distance the jet ski travels, and if d_2 is the distance the motorboat travels, write an appropriate equation.

7. Mutt and Jeff leave the mall at the same time and head in the same direction. Jeff's speed is 9 mph more than 6 times Mutt's speed. Four hours later Jeff is 1036 miles ahead of Mutt. If d_1 is the distance Mutt traveled, and if d_2 is the distance Jeff traveled, write an appropriate equation.

Solutions

1. a. 1440 km b. 500 mi/hr c. 4 hrs

2. a. $d = rt$ or $rt = d$ **3**. $d_1 + d_2 = 18$ **4**. $d_1 = d_2$

5. $d_1 + d_2 = 1096$ **6**. $d_1 = d_2$

7. $d_1 + 1036 = d_2$ OR $d_2 - d_1 = 1036$ OR $d_2 - 1036 = d_1$

"Formal education is but an incident in the lifetime of an individual. Most of us who have given the subject any study have come to realize that education is a continuous process ending only when ambition comes to a halt."

– *R.I. Rees*

MOTION PROBLEMS – OPPOSITE DIRECTIONS

Mike and Sarah start from the burger stand and skate in opposite directions. Mike's speed is 5 less than 3 times Sarah's speed. In 10 hours they are 70 miles apart. Find the speed of both skaters.



Solution: Let's organize all the information in a table using the basic $rt = d$ formula. Down the first column are the names of our two skaters. Across the first row are the three components of motion, the rate (speed), the time, and the distance. We've written the formula to help us remember the basic relationship among these three concepts.

	Rate × **Time** = **Distance**		
Mike			
Sarah			

Since each skater's speed is being asked for (they're the unknowns), we'll let M stand for Mike's speed and S stand for Sarah's speed, and so these variables go into the Rate column.

As for the Time column, the problem states that each skater skated for exactly 10 hours, so each of their travel times is 10.

Since Distance = Rate × Time, the Distance column is simply the product of the Rate and Time columns for both Mike and Sarah.

	Rate	× Time	= Distance
Mike	M	10	$10M$
Sarah	S	10	$10S$

Since there are two unknowns in this problem, it's likely we'll need two equations. Let's look at the rates first: From the phrase in the problem "Mike's speed is 5 less than 3 times Sarah's speed" we create the equation

$$M = 3S - 5 \qquad \text{[Equation 1]}$$

To determine the second equation, we have to picture where the skaters are going. They start in the same place and then proceed to skate in opposite directions and end up 70 miles from each other. Therefore, the sum of their individual distances must be 70. Well, Mike skated a distance of $10M$ miles while Sarah went $10S$ miles. So 70 must be the sum of $10M$ and $10S$:

$$10M + 10S = 70 \qquad \text{[Equation 2]}$$

Now substitute Equation 1 into Equation 2:

$10(\mathbf{3S - 5}) + 10S = 70$ (replaced M with $3S - 5$)

$\Rightarrow \quad 30S - 50 + 10S = 70$ (distribute)

$\Rightarrow \quad 40S - 50 = 70$ (combine like terms)

$\Rightarrow \quad 40S = 120$ (add 50 to each side)

$\Rightarrow \quad \underline{S = 3}$ (divide each side by 40)

Recall that S stood for Sarah's speed, so we know for sure that Sarah skated 3 mph. To find Mike's speed we use Equation 1 and Sarah's speed:

$$\begin{aligned} M &= 3S - 5 \\ &= 3(\mathbf{3}) - 5 \\ &= 9 - 5 \\ &= \underline{4} \end{aligned}$$

Note: We could have used the **Elimination Method** to solve the system of equations:

$$M = 3S - 5$$
$$10M + 10S = 70$$

This shows that Mike skated at a rate of 4 mph. We now have the complete answer to the question:

> Mike's speed was 4 mph and
>
> Sarah's speed was 3 mph.

Homework

1. Two pedestrians leave the same place and walk in opposite directions. The speed of one of the pedestrians is 5 mph less than 7 times the other. In 6 hours they are 354 miles apart. Find the speed of each pedestrian.
2. Two skaters leave the same place and skate in opposite directions. The speed of one of the skaters is 8 mph less than 10 times the other. In 9 hours they are 819 miles apart. Find the speed of each skater.
3. Two joggers leave the same place and jog in opposite directions. The speed of one of the joggers is 9 mph more than 5 times the other. In 7 hours they are 357 miles apart. Find the speed of each jogger.
4. Two pedestrians leave the same place and walk in opposite directions. The speed of one of the pedestrians is 7 mph less than 9 times the other. In 10 hours they are 930 miles apart. Find the speed of each pedestrian.
5. Two skaters leave the same place and skate in opposite directions. The speed of one of the skaters is 1 mph more than 7 times the other. In 9 hours they are 513 miles apart. Find the speed of each skater.

Solutions

1. 8 mph & 51 mph

2. 9 mph & 82 mph

3. 7 mph & 44 mph

4. 10 mph & 83 mph

5. 7 mph & 50 mph

MOTION PROBLEMS – PURSUIT

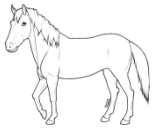
❑ *INTRODUCTION*

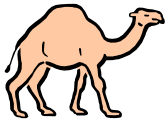
Homework

1. A jet ski leaves the beach. Nine hours later a motorboat begins to pursue the jet ski and finally catches up with it. If d_1 is the distance the jet ski travels, and if d_2 is the distance the motorboat travels, write an appropriate equation.

❑ *PURSUIT*

EXAMPLE 1: **A camel leaves the oasis traveling 5 mph. Eight hours later a horse begins to pursue the camel at a speed of 45 mph. How many hours after the horse leaves the oasis will it catch up with the camel?**





Solution: This problem gives us the rates of both animals, so those aren't an issue. In fact, the question is asking for the travel time of the horse. In addition, we don't know the travel time of the camel, either. So how about we let

c = the travel *time* of the camel, and

h = the travel *time* of the horse

These variables will go into the Time column of our table. And since the rates (speeds) of the animals are given, we'll simply place them in the Rate column. As in the previous example, the Distance is found by multiplying the Rate by the Time. Here's the table will all the known and unknown information in it:

	Rate	× **Time**	= **Distance**
camel	5	c	$5c$
horse	45	h	$45h$

Again, two variables will require two equations. We'll start with the Time column. Notice that the horse left after the camel (by 8 hours). This implies that the horse's travel time was 8 hours less than the camel's. This observation (which is not very obvious) leads to the first equation:

$$h = c - 8 \qquad \text{[Equation 1]}$$

To determine the second equation, we have to visualize where the animals are going. They start in the same place, leave one after the other, and then go in the same direction and end up in the same place. Thus, each of them went the same distance even though the camel left before the horse. This fact means that we can set the two distances in the table equal to each other:

$$5c = 45h \qquad \text{[Equation 2]}$$

Substituting Equation 1 into Equation 2:

$$5c = 45(\mathbf{c - 8}) \qquad \text{(since } h = c - 8\text{)}$$

$$\Rightarrow \quad 5c = 45c - 360 \qquad \text{(distribute)}$$

$$\Rightarrow \quad -40c = -360 \qquad \text{(subtract } 45c \text{ from each side)}$$

$$\Rightarrow \quad c = 9 \qquad \text{(divide each side by } -40\text{)}$$

This tells us that the camel traveled for 9 hours. But the question asked for the horse's travel time, so we use Equation 1 to find h:

$$h = c - 8 = 9 - 8 = 1$$

We conclude that

> It takes the horse 1 hour to catch up with the camel.

Homework

2. A camel leaves the oasis traveling 10 mph. Five hours later a dune buggy begins to pursue the camel at a speed of 15 mph. How many hours after the dune buggy leaves the oasis will it catch up with the camel?

3. A sailboat leaves the island traveling 14 mph. Five hours later a hydrofoil begins to pursue the sailboat at a speed of 24 mph. How many hours after the hydrofoil leaves the island will it catch up with the sailboat?

4. A robber leaves the bank traveling 15 mph. Four hours later a sheriff begins to pursue the robber at a speed of 35 mph. How many hours after the sheriff leaves the bank will he catch up with the robber?

5. A rowboat leaves the harbor traveling 26 mph. Seven hours later a speedboat begins to pursue the rowboat at a speed of 39 mph. How many hours after the speedboat leaves the harbor will it catch up with the rowboat?

6. A robber leaves the bank traveling 26 mph. Three hours later a sheriff begins to pursue the robber at a speed of 39 mph. How many hours after the sheriff leaves the bank will she catch up with the robber?

Solutions

1. $d_1 = d_2$

2. 10 hours

3. 7 hours

4. 3 hours

5. 14 hours

6. 6 hours

"Nothing in this world can take the place of *persistence*.

Talent can not; nothing is more common than unsuccessful men with talent.

Genius can not; unrewarded genius is almost a proverb.

Education can not; the world is full of educated derelicts.

Persistence and determination alone are omnipotent."

Calvin Coolidge, 30th President of the U.S.

MOTION PROBLEMS – ROUND TRIP

It takes a helicopter a total of 13 hours to travel from the mountain to the valley at a speed of 30 mph and return at a speed of 35 mph. How long does it take to get from the mountain to the valley?



Solution: We'll let x represent the time it takes to go from the mountain to the valley (since this is what's being asked for). Let's also choose y to stand for the time it takes to return from the valley to the mountain. The two rates (speeds) are given, and we know that each distance is the product of the rate and the time. So here's the table:

	Rate	× **Time**	= **Distance**
to valley	30	x	$30x$
to mtn	35	y	$35y$

Since the total travel time is 13 hours, we get our first equation:

$$x + y = 13 \qquad \text{[Equation 1]}$$

Now what about the distances, $30x$ and $35y$? Wouldn't you agree that the distance the mountain to the valley is the same as the distance the valley to the mountain? That is, $30x$ and $35y$ must be equal:

$$30x = 35y \qquad \text{[Equation 2]}$$

To solve this system of two equations in two unknowns, let's take Equation 1 and solve it for y:

$$x + y = 13 \qquad \text{(Equation 1)}$$

$\Rightarrow \quad y = 13 - x$ (subtract x each side)

We now replace the variable y in Equation 2 with the result just obtained:

$30x = 35(\mathbf{13 - x})$

$\Rightarrow \quad 30x = 455 - 35x$ (distribute)

$\Rightarrow \quad 65x = 455$ (add $35x$ to each side)

$\Rightarrow \quad \underline{x = 7}$ (divide each side by 65)

Now, what did x represent? Go back to the table and see that x represented the travel time the mountain to the valley. Since this is exactly what was being asked for in the problem, we're done.

It takes 7 hours to travel from the mountain to the valley.

Homework

1. A helicopter traveled the hospital to the battlefield at a speed of 22 mph and returned at a speed of 44 mph. If the entire trip took 18 hours, find the travel times to and the battlefield.

2. A hang glider traveled the oceanside to the mountaintop at a speed of 18 mph and returned at a speed of 21 mph. If the entire trip took 13 hours, find the travel times to and the mountaintop.

3. A helicopter traveled the hospital to the battlefield at a speed of 36 mph and returned at a speed of 24 mph. If the entire trip took 20 hours, find the travel times to and the battlefield.

4. A tractor traveled the wheat field to the chicken coop at a speed of 27 mph and returned at a speed of 36 mph. If the entire trip took 21 hours, find the travel times to and the chicken coop.

5. A hang glider traveled the oceanside to the mountaintop at a speed of 34 mph and returned at a speed of 51 mph. If the entire trip took 15 hours, find the travel times to and the mountaintop.

Solutions

1. 12 hrs & 6 hrs

2. 7 hrs & 6 hrs

3. 8 hrs & 12 hrs

4. 12 hrs & 9 hrs

5. 9 hrs & 6 hrs

MOTION PROBLEMS – SAME DIRECTION

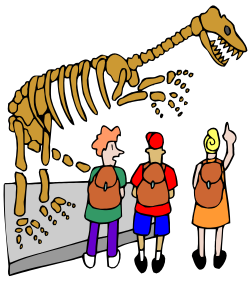
❑ *INTRODUCTION*

Homework

1. Mutt and Jeff leave the mall at the same time and head in the same direction. Jeff's speed is 9 mph more than 6 times Mutt's speed. Four hours later Jeff is 1036 miles ahead of Mutt. If d_1 is the distance Mutt traveled, and if d_2 is the distance Jeff traveled, write an appropriate equation.

❑ *SAME DIRECTION*

EXAMPLE 1: **Six hours after Mary and Moe leave the museum at the same time and head in the same direction, Moe is 252 miles ahead of Mary. If Moe's speed is 8 mph less than 3 times Mary's speed, find the speeds of Mary and Moe.**



Solution: Mary and Moe left the museum at the same time and each traveled for 6 hours, so both times in the table must be 6. Since the rates are unknown, we let x represent Mary's speed and let y represent Moe's speed.

	Rate ×	Time =	Distance
Mary	x	6	$6x$
Moe	y	6	$6y$

First, we need an equation relating x and y. The phrase "Moe's speed is 8 mph less than 3 times Mary's speed" translates to

$y = 3x - 8$ [First equation]

Now, according to the table, Mary traveled $6x$ miles, while Moe traveled $6y$ miles. The problem says that at the end of the 6 hours, Moe is 252 miles ahead of Mary. This means that Moe's distance ($6y$) is 252 miles more than Mary's distance ($6x$), which translates into our second equation:

$6y = 6x + 252$ [Second equation]

Substituting the first equation into the second equation gives us

$$\begin{aligned} & 6(3x - 8) = 6x + 252 \\ \Rightarrow\ & 18x - 48 = 6x + 252 && \text{(distribute)} \\ \Rightarrow\ & 12x = 300 && \text{(subtract } 6x \text{ and add 48)} \\ \Rightarrow\ & \underline{x = 25} && \text{(divide by 12)} \end{aligned}$$

which implies that Moe's rate is $y = 3x - 8 = 3(\mathbf{25}) - 8 = 67$.

Mary's speed was 25 mph and
Moe's speed was 67 mph.

Homework

2. Moe and Curly leave the airport at the same time and head in the same direction. Curly's speed is 8 mph less than 3 times Moe's

speed. Five hours later Curly is 430 miles ahead of Moe. Find the speeds of Moe and Curly.

3. Sally and Maria leave the mall at the same time and head in the same direction. Maria's speed is 6 mph less than 2 times Sally's speed. Ten hours later Maria is 410 miles ahead of Sally. Find the speeds of Sally and Maria.



4. Lucy and Ethyl leave the mall at the same time and head in the same direction. Ethyl's speed is 4 mph more than 4 times Lucy's speed. Four hours later Ethyl is 604 miles ahead of Lucy. Find the speeds of Lucy and Ethyl.

5. George and Gracie leave the stadium at the same time and head in the same direction. Gracie's speed is 3 mph less than 4 times George's speed. Eight hours later Gracie is 1080 miles ahead of George. Find the speeds of George and Gracie.

6. Mutt and Jeff leave the stadium at the same time and head in the same direction. Jeff's speed is 8 mph less than 3 times Mutt's speed. Ten hours later Jeff is 660 miles ahead of Mutt. Find the speeds of Mutt and Jeff.

Solutions

1. $d_1 + 1036 = d_2$ OR $d_2 - d_1 = 1036$ OR $d_2 - 1036 = d_1$

2. 47 mph & 133 mph

3. 47 mph & 88 mph

4. 49 mph & 200 mph

5. 46 mph & 181 mph

6. 37 mph & 103 mph

MOTION PROBLEMS – TWO-PART JOURNEY

A limousine traveled at 29 mph for the first part of a 540-mile trip, and then increased its speed to 53 mph for the rest of the trip. How many hours were traveled at each rate if the total trip took 12 hours?



Solution: The rates for each part of the trip are given, so just put them in the table in the right places. Let x be the travel time for the first part of the trip and let y be the travel time for the second part of the trip. Finally, the Distance column is the product of the Rate and Time columns.

	Rate	× **Time**	= **Distance**
1st part	29	x	$29x$
2nd part	53	y	$53y$

The total travel is given to be 12 hours. Therefore,

$$x + y = 12$$

Since the total distance traveled was 540 miles, adding the distance of the 1st part of the trip plus the distance of the 2nd part of the trip should give a total of 540:

$$29x + 53y = 540$$

Solving the first equation for y gives $y = 12 - x$. Substituting $12 - x$ for y in the second equation gives:

$$
\begin{aligned}
& 29x + 53(\mathbf{12} - \boldsymbol{x}) = 540 \\
\Rightarrow\ & 29x + 636 - 53x = 540 && \text{(distribute)} \\
\Rightarrow\ & -24x + 636 = 540 && \text{(combine like terms)} \\
\Rightarrow\ & -24x = -96 && \text{(subtract 636)} \\
\Rightarrow\ & \underline{x = 4}
\end{aligned}
$$

This means that the first part of the limo trip took 4 hours. Using the equation $y = 12 - x$, we calculate the time for the rest of the trip as $y = 12 - x = 12 - \mathbf{4} = 8$. In short,

4 hours at 29 mph and 8 hours at 53 mph

Homework

1. A 1096-mi trip took a total of 16 hours. The speed was 71 mph for the first part of the trip, and then decreased to 67 mph for the rest of the trip. How many hours were traveled at each speed?

2. A 730-mi trip took a total of 11 hours. The speed was 68 mph for the first part of the trip, and then decreased to 59 mph for the rest of the trip. How many hours were traveled at each speed?

3. A 664-mi trip took a total of 12 hours. The speed was 30 mph for the first part of the trip, and then increased to 68 mph for the rest of the trip. How many hours were traveled at each speed?

4. A 489-mi trip took a total of 9 hours. The speed was 45 mph for the first part of the trip, and then increased to 59 mph for the rest of the trip. How many hours were traveled at each speed?

5. A 556-mi trip took a total of 14 hours. The speed was 38 mph for the first part of the trip, and then increased to 42 mph for the rest of the trip. How many hours were traveled at each speed?

Solutions

1. 6 hrs & 10 hrs

2. 9 hrs & 2 hrs

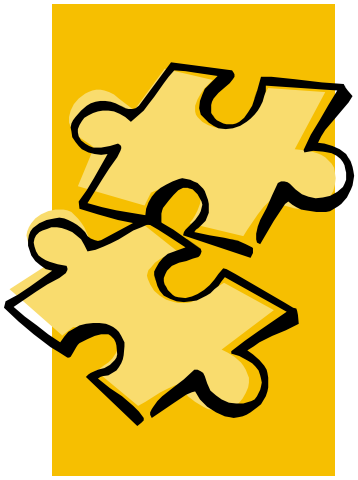
3. 4 hrs & 8 hrs

4. 3 hrs & 6 hrs

5. 8 hrs & 6 hrs

NUMBER WORD PROBLEMS

❑ *TERMINOLOGY*



To **double** a number means to multiply it by **2**.

The double of 12 is 2(12) = 24.

When n is doubled, it becomes $\boldsymbol{2n}$.

To **square** a number means to multiply it by itself.

The square of 12 is $12^2 = 12 \times 12 = 144$.

When n is squared, it becomes $\boldsymbol{n^2}$.

To **triple** a number means to multiply it by **3**.

The result of tripling x is $\boldsymbol{3x}$.

To **quadruple** a number means to multiply it by **4**.

To **quintuple** a number means to multiply it by **5**.

10 is added to n: $\boldsymbol{n + 10}$

N is subtracted from x: $\boldsymbol{x - N}$

13 is subtracted from twice m: $\boldsymbol{2m - 13}$

Phrases such as "the result is," "the final result is," "the answer is," and "is the same as" translate to the **equal sign** in the equation. In fact, the word "is" alone usually (but not always) represents the equal sign.

Homework

1. Beginners frequently confuse *doubling* and *squaring*. Let's clear this up right now.

   a. What is the double of 15?
   b. What is the square of 15?
   c. What's 9 squared?
   d. What's 9 doubled?
   e. 5 doubled gives ____.
   f. 5 squared gives ____.
   g. (20)(2) =
   h. 20^2 =
   i. Doubling 25 gives ____.
   j. Squaring 25 gives ____.
   k. The square of y is ____.
   l. The double of y is ____.

2. Now we add to the confusion. Consider the number 0. Notice that 0 doubled is $2(0) = \underline{0}$. Also, 0 squared is $0^2 = 0 \times 0 = \underline{0}$. In other words, 0 is a special number whose double is the same as its square. Try to find the <u>only</u> other number whose double is equal to its square.

❑ *I'M THINKING OF A NUMBER*

<u>EXAMPLE 1:</u> **I'm thinking of a number. If I double it and add 10, the result is 64. What is the number?**

<u>Solution:</u> Let n represent the unknown number. Doubling it produces the expression $2n$. Adding 10 to that gives the expression $2n + 10$. The phrase "the result is" becomes an equal sign, and so the equation we need to solve is



$2n + 10 = 64$ (English to Algebra)

$\Rightarrow \quad 2n + 10 \mathbf{- 10} = 64 \mathbf{- 10}$ (subtract 10 from each side)

$\Rightarrow \quad 2n = 54$ (simplify)

$\Rightarrow \quad \frac{2n}{\mathbf{2}} = \frac{54}{\mathbf{2}}$ (divide each side by 2)

$\Rightarrow \quad n = 27$ (simplify)

Since n stood for the number we were seeking, we can conclude that the number asked for in the question is

27

EXAMPLE 2: **I'm thinking of a number. If 7 is subtracted from the number, and then that result is multiplied by 10, the final result is 150. What is the number?**

Solution: For variety, we'll select x to represent the unknown number. First 7 is subtracted from x; this can be written $x - 7$ (not the other way around!). Next, that quantity is to be multiplied by 10. How shall we indicate that 10 must be multiplied by the quantity $x - 7$? We use parentheses: $10(x - 7)$. And, of course, the phrase "the final result is" produces our equal sign, yielding us the equation

	$10(x - 7) = 150$	(English to Algebra)
$\Rightarrow$	$10x - 70 = 150$	(distribute)
$\Rightarrow$	$10x - 70 \mathbf{+ 70} = 150 \mathbf{+ 70}$	(add 70 to each side)
$\Rightarrow$	$10x = 220$	(simplify)
$\Rightarrow$	$\frac{10x}{\mathbf{10}} = \frac{220}{\mathbf{10}}$	(divide each side by 10)
$\Rightarrow$	$x = 22$	(simplify)

We conclude that my number was | 22 |

EXAMPLE 3: **Solve each "I'm thinking of a number" problem:**

A. I'm thinking of a number. If I add 3 to the number, and then divide that sum by 2, the result is 7. What is the number?

Solution: Translating the words into algebra gives the equation

$$\frac{n+3}{2} = 7$$

$$\Rightarrow \quad \frac{n+3}{2}\left[2\right] = 7\left[2\right] \qquad \text{(multiply each side by 2)}$$

$$\Rightarrow \quad n + 3 = 14 \qquad \text{(simplify)}$$

$$\Rightarrow \quad \boldsymbol{n = 11} \qquad \text{(subtract 3 from each side)}$$

B. I'm thinking of a number. If I divide the number by 9, and then subtract 5 from that quotient, the answer is 3. Find the number.

Solution: In this problem we divide first, and then subtract. Our equation to solve is therefore

$$\frac{x}{9} - 5 = 3$$

$$\Rightarrow \quad \frac{x}{9} = 8 \qquad \text{(add 5 to each side)}$$

$$\Rightarrow \quad \frac{x}{9}\left[9\right] = 8\left[9\right] \qquad \text{(multiply each side by 9)}$$

$$\Rightarrow \quad \boldsymbol{x = 72} \qquad \text{(simplify)}$$

C. I'm thinking of a number. If I subtract 5 from the number, and then divide that difference by 3.1, the result is –2.7. What is the number?

Solution: Except for the ugliness of the numbers, this is just like part A above:

$$\frac{y-5}{3.1} = -2.7$$

$\Rightarrow$ $y-5 = (-2.7)(3.1)$ (multiply each side by 3.1)

$\Rightarrow$ $y-5 = -8.37$ (simplify)

$\Rightarrow$ $\boldsymbol{y = -3.37}$ (add 5 to each side)

Homework

Solve each of the following "I'm thinking of a number" problems by creating and solving an appropriate equation:

3. a. If I add 7 to the number, the result is 53.
   b. If I subtract 9 from the number, the result is 33.
   c. If I double the number, the result is 23.
   d. If I triple the number, the result is 72.
   e. If I multiply the number by 7, the result is 861.
   f. If I multiply the number by 8, the result is 100.
   g. If I divide the number by 19, the result is 17.
   h. If I divide the number by 3, the result is 287.

4. a. If I double it and add 17, the result is 50.
   b. If I triple it and subtract 9, the result is 0.
   c. If I multiply it by 7 and then add 50, the result is –3.
   d. If I add 3 to the number, and then multiply that result by 9, the final result is 34.
   e. If I subtract 5 from the number, and then multiply that result by 17, the final result is 20.

f. If 30 is added to the number, and then that result is multiplied by 3, the final result is 99.

g. If 17 is subtracted from the number, and then that result is multiplied by –5, the result is 30.

5. a. If I add 7 to the number, and then divide that sum by 3, the result is 20.

b. If I subtract 9 from the number, and then divide that difference by 5, the result is –10.

c. If I divide the number by 3, and then add 13 to the quotient, the result is 12.

d. If I divide the number by 12, and then subtract 50 from the quotient, the result is –23.

❑ *More Number Problems*

The phrase "7 more than x" represents the situation where 7 has been added to x, and therefore can be represented by the expression $x + 7$.

The phrase "9 less than w" represents the situation where 9 has been subtracted from w, and thus can be written as the expression $w - 9$.

Here are some more translations from English to Algebra:

the sum of A and B	$A + B$
the difference of x and y	$x - y$
the product of a and b	ab
the quotient of m and n	$\frac{m}{n}$ (or m/n or $m \div n$)
a number p is divided by 7	$\frac{p}{7}$
10 more than z	$z + 10$

8 less than a	$a - 8$
3 more than twice n	$2n + 3$
5 less than 11 times y	$11y - 5$
the sum of 4 times a and 4 times b	$4a + 4b$
4 times the sum of a and b	$4(a + b)$
the difference of twice q and 3	$2q - 3$
twice the difference of q and 3	$2(q - 3)$
a number n increased by 12	$n + 12$
a number w decreased by 7	$w - 7$
100 less than the square of n	$n^2 - 100$
23 less than the double of u	$2u - 23$
the square of the sum of x and y	$(x + y)^2$
the difference of the squares of A and B	$A^2 - B^2$

Homework

6. Translate from English to Algebra:

a. n doubled
b. x squared
c. a tripled
d. b quadrupled
e. t quintupled
f. 3 added to u
g. 9 subtracted from x.
h. twice Q
i. 7 added to 3 times n
j. 2 subtracted from 4 times z
k. 9 less than w
l. 4 more than x
m. 18 more than T
n. 27 less than P
o. 3 more than twice x
p. 3 less than 10 times y
q. 1 more than the triple of t.
r. 6 less than the quadruple of n

s. the quotient of x and 3

t. the product of a and b

u. 3 times a added to 4 times a

v. 9 times R

w. 6 times the sum of x and y

x. 9 times the difference of A and B

y. 12 times the sum of x and 20

z. 7 times the difference of n and 200

7. a. Translate "the square of the sum of u and w" to Algebra.

b. Translate "the sum of the squares of u and w" to Algebra.

c. Find a counterexample to the statement that the expressions in part a. and part b. are equal.



EXAMPLE 4: **Eight times a number is 6 less than twice the number. Find the number.**

Solution: Let x represent the number. We'll translate the English sentence into algebra like this:

Eight times a number | is | 6 less than twice the number

$8x$ | $=$ | $2x - 6$

In other words, the equation we must solve is

$$8x = 2x - 6$$

$\Rightarrow$ $8x - \mathbf{2x} = 2x - \mathbf{2x} - 6$ (subtract $2x$ from each side)

$\Rightarrow$ $6x = -6$ (simplify)

$\Rightarrow$ $\frac{6x}{\mathbf{6}} = \frac{-6}{\mathbf{6}}$ (divide each side by 6)

$\Rightarrow$ $x = -1$ (simplify)

Therefore, the number we're searching for is $\boxed{-1}$

EXAMPLE 5: **Find a number given that 3 times the difference of the number and 1 is 13 more than the number.**

Solution: If we let n represent the unknown number, we can translate as follows:

"Three times difference of the number and 1" $\Rightarrow$ $\mathbf{3(n-1)}$
"is" $\Rightarrow$ $\mathbf{=}$
"13 more than the number" $\Rightarrow$ $\mathbf{n+13}$

Putting it all together:

	$3(n-1) = n+13$	(the equation we need to solve)
$\Rightarrow$	$3n-3 = n+13$	(distribute)
$\Rightarrow$	$2n-3 = 13$	(subtract n from each side)
$\Rightarrow$	$2n = 16$	(add 3 to each side)
$\Rightarrow$	$n = 8$	(divide each side by 2)

The number we're seeking is $\boxed{8}$

Homework

8. Seven times a number is 2 more than 9 times the number. What is the number?

9. Nine more than 2 times a number is equal to 9 more than 6 times the number. Find the number.

10. Six times the difference of a number and 3 is 13 less than the number. Find the number.

11. Find a number given that 5 times the sum of the number and 4 is 32 more than the number.

12. If 3 times a number is added to 4 times the number, the result is 700. Find the number.

13. Eight times a number is 3 more than 5 times the number. What is the number?

14. Nine less than 5 times a number is equal to 7 more than 8 times the number. Find the number.

15. Five times the difference of a number and 4 is 18 less than the number. Find the number.

16. Find a number given that 6 times the sum of the number and 7 is 8 less than the number.

17. Find a number given that 7 times the sum of the number and 9 is the same as 201 more than the number.

18. Find a number given that 10 times the difference of the number and 10 is the same as 64 less than the number.

19. Find a number given that 6 times the difference of the number and 7 is the same as 43 more than the number.

20. Find a number given that 5 times the sum of the number and 3 is the same as 141 less than the number.

21. Find a number given that 9 times the sum of the number and 8 is the same as 224 more than the number.

22. Find a number given that 9 times the difference of the number and 7 is the same as 263 less than the number.

23. 47 less than 8 times a number is equal to 26 less than the number. What is the number?

24. 5 less than 3 times a number is equal to 11 more than the number. What is the number?

25. 43 more than 12 times a number is equal to 122 less than the number. What is the number?

26. Find a number given that 12 times the sum of the number and 4 is the same as 499 more than the number.

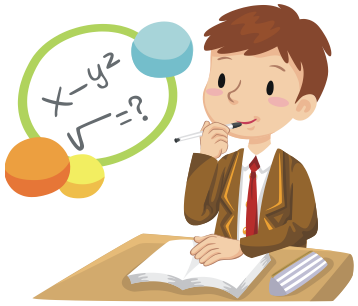
27. 5 more than 6 times a number is equal to 50 more than the number. What is the number?

28. 47 more than 14 times a number is equal to 229 more than the number. What is the number?

29. Find a number given that 8 times the difference of the number and 1 is the same as 162 less than the number.

Review Problems

30. I'm thinking of a number. If 10 is subtracted from the number, and then that result is multiplied by 11, the final answer is –165. What is the number?



31. I'm thinking of a number. If –2 is subtracted from the number, and then that result is multiplied by –11, the final answer is –165. What is the number?

32. I'm thinking of a number. If I divide the number by –5, and then add –6, the final result will be –17. What is the number?

33. I'm thinking of a number. If 114 is subtracted from the number, and then that result is divided by –11, the final answer is 10. What is the number?

34. I'm thinking of a number. If –46 is subtracted from the number, and then that result is divided by –8, the final answer is –5. What is the number?

35. Seventeen times a number is 2 less than 10 times the number. What is the number?

36. Nine less than 7 times a number is equal to 19 more than 4 times the number. Find the number.

37. Three times the difference of a number and 5 is 15 less than the number. Find the number.

38. Find a number given that 15 times the sum of the number and 4 is 32 less than the 6 times the number.

Solutions

1. a. 30 b. 225 c. 81 d. 18 e. 10 f. 25

g. 40 h. 400 i. 50 j. 625 k. y^2 l. $2y$

2. I'd rather not give it away just yet.

3. a. $n + 7 = 53$; $n = 46$ b. $n - 9 = 33$; $n = 42$

c. $2n = 23$; $n = \frac{23}{2}$ d. $3n = 72$; $n = 24$

e. $7n = 861$; $n = 123$ f. $8n = 100$; $n = \frac{25}{2}$

g. $\frac{n}{19} = 17$; $n = 323$ h. $\frac{n}{3} = 287$; $n = 861$

4. a. $2n + 17 = 50$; $n = \frac{33}{2}$ b. $3n - 9 = 0$; $n = 3$

c. $7n + 50 = -3$; $n = -\frac{53}{7}$ d. $9(n + 3) = 34$; $n = \frac{7}{9}$

e. $17(n - 5) = 20$; $n = \frac{105}{17}$ f. $3(n + 30) = 99$; $n = 3$

g. $-5(n - 17) = 30$; $n = 11$

5. a. $\frac{n+7}{3} = 20;\ n = 53$ b. $\frac{n-9}{5} = -10;\ n = -41$

c. $\frac{n}{3} + 13 = 12;\ n = -3$ d. $\frac{n}{12} - 50 = -23;\ n = 324$

6. –5 **7**. 13 **8**. –1 **9**. 4 **10**. –6

11. a. $2n$ b. x^2 c. $3a$ d. $4b$

e. $5t$ f. $u + 3$ g. $x - 9$ h. $2Q$

i. $3n + 7$ j. $4z - 2$ k. $w - 9$ l. $x + 4$

m. $T + 18$ n. $P - 27$ o. $2x + 3$ p. $10y - 3$

q. $3t + 1$ r. $4n - 6$ s. $\frac{x}{3}$ t. ab

u. $4a + 3a$ v. $9R$ w. $6(x + y)$ x. $9(A - B)$

y. $12(x + 20)$ z. $7(n - 200)$

12. a. $(u + w)^2$

b. $u^2 + w^2$

c. Pick a pair of numbers for u and w and show that you get a different answer for each expression.

13. $7x = 9x + 2;\ x = -1$ **14**. $2x + 9 = 6x + 9;\ x = 0$

15. $6(x - 3) = x - 13;\ x = 1$ **16**. $5(x + 4) = x + 32;\ x = 3$

17. 100 **18**. 1 **19**. $-\frac{16}{3}$ **20**. $\frac{1}{2}$ **21**. –10

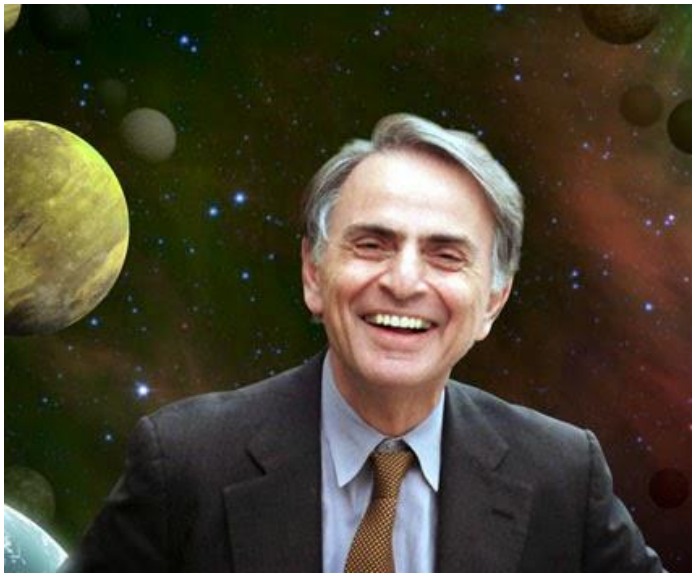
22. 23 **23**. 4 **24**. 17 **25**. –39 **26**. 19

27. –25 **28**. 3 **29**. 8 **30**. –15 **31**. 41

32. 9 **33**. 14 **34**. –22 **35**. $-\frac{2}{7}$ **36**. $\frac{28}{3}$

37. 0 **38**. $-\frac{92}{9}$

"It is far better to grasp the Universe as it really is than to persist in delusion, however satisfying and reassuring."



– Carl Sagan (1934 – 1996)

PEANUTS, ETC.

❑ *THE BASICS*



Consider the following problem: You bought 12 lbs of peanuts that were priced at $4 per pound, written "$4/lb" (and called the *unit price*). What is the total cost (or total value) of the peanuts? It should be clear that we could multiply 12 lbs by $4/lb and end up with **$48**.

Similarly, if we bought N pounds of cashews at $9/lb, the total value would be $9 \times N$, which of course can be written $\mathbf{9}\boldsymbol{N}$.

And for our third example, if we had $(x + 5)$ lbs of walnuts which were worth $7/lb, the total value of those walnuts would be $\mathbf{7}\boldsymbol{(x + 5)}$.

In general,

Pounds of Nuts × Unit Price = Total Value

Here's another idea we need for this chapter. If you had 7 lbs of cashews, and you mix them with 13 lbs of peanuts, you would have a total of **20** pounds of nuts. That should be obvious. But since this is an algebra class, we also need to deal with the following: If you had 10 lbs of almonds and "H" pounds of hazelnuts, what is the total weight of the nuts? It would be $10 + H$, or $\boldsymbol{H + 10}$.

We need to understand one more concept before our nutty examples. Suppose you had a bucket filled with macadamia nuts whose total value was $25, and there's a second bucket containing almonds worth a total value of $35. Now we pour the two buckets of nuts into one huge

bin. The question is, What is the total value of the mixed nuts in the bin? Well, the correct answer is **$60**, easily found by adding the total value of the macadamia nuts ($25) and the total value of the almonds ($35). Make sense?

Homework

1. If you mix 17 pounds of almonds with 23 pounds of cashews, what is the total weight of the nuts?

2. If you mix A pounds of almonds with 25 pounds of peanuts, what is the total weight of the nuts?

3. If you mix P pounds of pistachios with W pounds of walnuts, what is the total weight of the nuts?

4. One bin contains $48 worth of peanuts; another bin contains $52 worth of cashews. If the two bins are combined into one bucket, what is the total value of nuts in the bucket?

5. A bucket of almonds is worth $210. A bucket of peanuts is worth 30P$. Now dump both buckets of nuts into one bin. What is the total value of the nuts in the bin?



6. Find the total value of 10 lbs of pecans if the unit price is $13/lb.

7. Find the total value of P lbs of pecans if the unit price is $25/lb?

8. If peanuts sell for $9/lb, find the total value of $(x + 5)$ pounds.

❑ *THE EXAMPLES*

EXAMPLE 1: **The Nutte Shoppe has 14 lbs of walnuts which are priced at \$15/lb. How many pounds of almonds which are priced at \$30/lb should be mixed with the walnuts to get a mixture that will sell for \$24/lb?**

Solution: We've got walnuts, 14 lbs at \$15/lb. We've got almonds — we don't know how many pounds — but we know they're worth \$30/lb. And when we mix them together, we get some number of pounds worth \$24/lb.

Do you see that the problem is asking for the number of pounds of almonds we need to mix with the walnuts? Since this is unknown, how about we let "***a***" represent the number of pounds of almonds. Let's try to organize this information into a chart to make the solution clearer.

	Pounds of Nuts ×	Unit Price =	Total Value
walnuts	14	15	210
almonds	a	30	$30a$
mixture	$a + 14$	24	$24(a + 14)$

As mentioned before the example, adding the dollar values of the two kinds of nuts must equal the dollar value of the mixture; that is,

$210 + 30a = 24(a + 14)$ (dollar values must add up)

$\Rightarrow$ $210 + 30a = 24a + 336$ (distribute)

$\Rightarrow$ $210 + 6a = 336$ (subtract $24a$)

$\Rightarrow$ $6a = 126$ (subtract 210)

$\Rightarrow$ $a = 21$ (divide by 6)

We now know how many pounds of almonds are needed to mix with the walnuts to produce a mixture worth \$24/lb:

21 lbs of almonds

EXAMPLE 2: **Safeway has 25 lbs of peanuts which are priced at \$8/lb, and mixes them with 15 lbs of cashews which are priced at \$24/lb. Find the *unit price* of the mixture.**

Solution: First note that the chart below uses the general term *quantity* instead of *pounds of nuts*. Letting u represent the unit price of the mixture, we get the following chart:

	Quantity	× Unit Price	= Total Value
peanuts	25	8	200
cashews	15	24	360
mixture	40	u	$40u$

The 40 was obtained by adding the pounds of the two kinds of nuts that comprise the mixture, and each box in the Total Value column was obtained by multiplying the two boxes to its left. As before, we note that the total value of the peanuts plus the total value of the cashews must equal the total value of the mixture:

$$200 + 360 = 40u$$

$$\Rightarrow \quad 560 = 40u$$

$$\Rightarrow \quad u = 14$$

And thus the unit price of the mixture is | \$14 per pound |

EXAMPLE 3: **Hadley's sells walnuts which are priced at $22/lb and cashews which are priced at $26/lb. How many pounds of each type of nut should Hadley's mix together to get 28 pounds of nuts which would sell for $25/lb?**

Solution: We're letting w represent the Quantity (number of pounds) of walnuts, and similarly for c. All the Unit Prices are given as numbers, and each Total Value is the product of the two entries to its left:

	Quantity	× Unit Price	= Total Value
walnuts	w	22	$22w$
cashews	c	26	$26c$
mixture	28	25	700

Since the quantities must total 28 lbs, we get the equation

$$w + c = 28$$

Since the values must add up, we also get

$$22w + 26c = 700$$

We'll solve this pair of equations by Substitution, by solving the first equation for c:

$$c = 28 - w$$

Plugging this new equation into the second equation gives:

$$22w + 26(28 - w) = 700 \qquad \text{(just one variable!)}$$

$$\Rightarrow \quad 22w + 728 - 26w = 700$$

$$\Rightarrow \quad -4w = -28$$

$$\Rightarrow \quad w = 7, \text{ so we have 7 lbs of walnuts}$$

Then $c = 28 - w = 28 - \mathbf{7} = 21$

7 lbs of walnuts
21 lbs of cashews

Homework

9. The Nut House has 16 lbs of peanuts which are priced at $13/lb. How many pounds of cashews which are priced at $18/lb should be mixed with the peanuts to get a mixture that will sell for $14/lb?

10. Hadley's has 4 lbs of peanuts which are priced at $12/lb, and mixes them with 11 lbs of almonds which are priced at $27/lb. Find the *unit price* of the mixture.



11. Walnut Heaven sells cashews which are priced at $4/lb and almonds which are priced at $7/lb. How many pounds of each type of nut should Walnut Heaven mix together to get 18 pounds of nuts which would sell for $6/lb?

12. Trader Joe's has 4 lbs of walnuts which are priced at $24/lb. How many pounds of almonds which are priced at $10/lb should be mixed with the walnuts to get a mixture that will sell for $18/lb?

13. Peanut Palace has 28 lbs of walnuts which are priced at $19/lb, and mixes them with 12 lbs of almonds which are priced at $29/lb. Find the *unit price* of the mixture.

14. Nutty Nuts sells cashews which are priced at $2/lb and almonds which are priced at $26/lb. How many pounds of each type of nut should Nutty Nuts mix together to get 32 pounds of nuts which would sell for $17/lb?

15. Safeway has 6 lbs of cashews which are priced at \$4/lb, and mixes them with 12 lbs of almonds which are priced at \$7/lb. Find the *unit price* of the mixture.

16. The Nutte Shoppe has 4 lbs of cashews which are priced at \$22/lb. How many pounds of almonds which are priced at \$13/lb should be mixed with the cashews to get a mixture that will sell for \$17/lb?



17. Hadley's sells walnuts which are priced at \$22/lb and cashews which are priced at \$26/lb. How many pounds of each type of nut should Hadley's mix together to get 28 pounds of nuts which would sell for \$25/lb?

Solutions

1. 40 lbs

2. $A + 25$ lbs

3. $P + W$ lbs

4. \$100

5. \$210 + \$30P

6. \$130

7. \$25P

8. \$9(x + 5)

9. 4 lbs of cashews

10. \$23/lb

11. 6 lbs of cashews and 12 lbs of almonds

12. 3 lbs of almonds

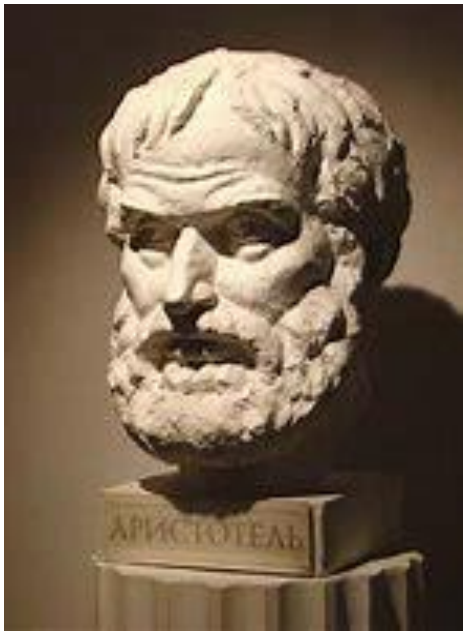
13. \$22/lb

14. 12 lbs of cashews and 20 lbs of almonds

15. \$6/lb

16. 5 lbs of almonds

17. 7 lbs walnuts and 21 lbs cashews



"It is the mark of an educated mind to be able to entertain a thought without accepting it."

– ARISTOTLE –

PRODUCING WIDGETS

❑ *INTRODUCTION*



Suppose we have 10 apples and 13 bananas, which of course means that we have 23 pieces of fruit in total. This can be written with the simple equation: $10 + 13 = 23$.

Assume we have a apples and b bananas, and also pretend that there are 12 pieces of fruit in total. Then we can write these facts as an equation: $a + b = 12$.

Suppose that yellow widgets cost \$25 apiece to produce (called the *unit cost*), and that we will produce 40 yellow widgets. Then the total cost of production is given by the expression 25×40 (or 40×25).

Assume that the unit cost of maroon widgets is \$14, and that we will produce m of the maroon widgets. Then the total cost of production is the product of 14 and m: $14m$.

If we make x widgets at \$9 each and y widgets at \$11 each, then

1) the total number of widgets made is $x + y$, and

2) the total production cost is $9x + 11y$

If we make w widgets at \$12 each and q widgets at \$26 each, for a total cost of \$500, we can express this information in the equation $12w + 26q = 500$.

If y is designated to be three times whatever x is, then we can write $y = 3x$.

The phrase "The number of green widgets should be two times the number of blue widgets" can be translated into the equation $g = 2b$.

Homework

1. There were 20 vegetables: c carrots and t tomatoes. Write an equation to express this fact.

2. There were r red widgets and p purple widgets. If there were 150 widgets in all, write the equation which expresses this fact.

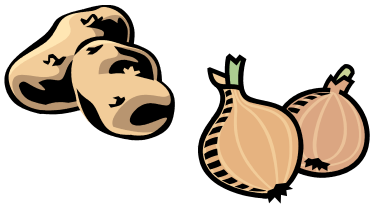
3. At a unit cost of \$49.75 per widget, what is the total cost to produce 100 widgets?

4. At \$3.75 per pound, what is the total cost to buy s pounds of top sirloin.

5. If the unit cost of a widget is \$500, and you produce w of them, find the total cost.

6. If 17 red widgets at a unit production cost of \$5 and 80 brown widgets at a unit production cost of \$1.75 are produced, calculate the total cost.

7. If you bought p pounds of potatoes at \$2 per pound and n pounds of onions at \$4 per pound, write an expression that represents the total cost.



8. In Problem 7, suppose the total cost was \$26. Write an equation which describes this situation. Dollar signs are generally omitted when writing an equation.

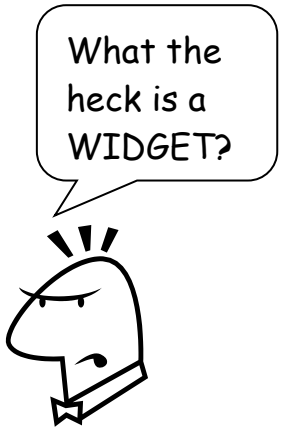
9. The factory spent a total of \$250 producing f flagons at \$7 apiece and d dragons at \$12 apiece. Write an equation which represents this situation.

10. Write an equation which expresses each statement:

    a. x is 4 more than y.
    b. T is 3 less than Z.
    c. u is twice as big as v.
    d. g is three times as big as h.
    e. w is four times as large as s.
    f. h is one-half as large as k.
    g. E is one-third the size of m.
    h. A is 12 less than B.
    i. W is 1 less than Q.

❑ *THE RIGHT MIXTURE*

I suppose if we had an infinite amount of money, we could produce all the widgets in all the colors we would ever need, but of course this is absurd. We have constraints (limitations) to worry about; the first will concern the number of widgets to be produced, and the second will be the amount of money we can spend producing them. Each of these constraints will be represented by an equation. Thus, each application we solve will have <u>two</u> equations that we must solve at the same time, for two different variables.



In this first application, we are told the total number of widgets to produce (some of one color and some of another) and the total amount of money we're allowed to spend.

<u>EXAMPLE 1:</u> **The unit cost of red widgets is $8 and the unit cost of green widgets is $10. Each production run can produce a total of 20 widgets, but we control how many of each color. The production budget is $168. How many widgets of each color should we manufacture?**

<u>Solution:</u> We're seeking two things in this problem: the number of red widgets and the number of green widgets to produce. Let's let

r represent the number of red widgets, and
g represent the number of green widgets.

We're given the unit costs of production: $8 for each red widget and $10 for each green widget. This means that it will cost $8r$ to produce the red widgets, and $10g$ to produce the green widgets.

Since we're seeking the values of two different variables, we expect to have two equations at our disposal. The first equation

represents the fact that we will produce 20 widgets altogether. Since r and g represent the number of widgets of each color, we get the equation

$$r + g = 20 \qquad \text{(20 widgets altogether)}$$

The total cost of production is the sum of the costs of the two colors of widgets — this yields the equation

$$8r + 10g = 168 \qquad \text{(the total cost of \$168)}$$

We now have a system of two equations in two unknowns:

$$\begin{aligned} r + g &= 20 \\ 8r + 10g &= 168 \end{aligned}$$

$$\begin{aligned} r + g = 20 &\xrightarrow{\text{times } -8} -8r - 8g = -160 \\ 8r + 10g = 168 &\xrightarrow{\text{leave alone}} 8r + 10g = 168 \\ \text{Add the equations:} \quad & 0r + 2g = 8 \\ & g = 4 \end{aligned}$$

This value of g tells us that we need to produce 4 green widgets. Since we need 20 widgets total, it's clear that we need to make 16 of the red widgets. More formally,

$$r + g = 20 \;\Rightarrow\; r + 4 = 20 \;\Rightarrow\; r = 16$$

To summarize, the production schedule should be

16 red widgets & 4 green widgets

Check:

16 + 4 = 20 widgets ✓

16 @ \$8 = \$128

4 @ \$10 = \$ 40

\$168 ✓

In our second application, market research has determined that black widgets sell about twice as well as white widgets. To satisfy this demand, the production department has to solve the following problem:

EXAMPLE 2: **Assume that black widgets cost $3 apiece and white widgets cost $7 apiece to produce. Our goal is that the number of black widgets should be twice the number of white widgets, while our production budget is $260. How many widgets of each color should we produce to satisfy all the stated conditions?**

Solution: We'll let

b = the number of black widgets to produce
w = the number of white widgets to produce

One of the conditions stipulates that the number of black widgets must be twice the number of white widgets. This yields the equation

$$b = 2w \qquad \text{(\# of blacks is twice \# of whites)}$$

Since we're producing b black widgets at a unit cost of $3, the cost of producing the black widgets is $3b$. Similarly, the cost of the white widgets is $7w$. The sum of these two costs must be $260, the given budget. This fact gives us the equation

$$3b + 7w = 260 \qquad \text{(the total cost is \$260)}$$

We thus have a system of two equations in two unknowns:

$$\begin{aligned} b &= 2w \\ 3b + 7w &= 260 \end{aligned}$$

To solve this system by the elimination method, a little rearranging is in order. In the first equation subtract $2w$ from each side, and leave the second equation alone:

$$\begin{aligned} b - 2w &= 0 \\ 3b + 7w &= 260 \end{aligned}$$

Now the variables are lined up.

Now we'll multiply the equations by appropriate numbers so that when we add the equations we'll eliminate the b:

$$\begin{aligned} b - 2w &= 0 \quad \xrightarrow{\text{times } -3} \quad -3b + 6w = 0 \\ 3b + 7w &= 260 \quad \xrightarrow{\text{leave alone}} \quad 3b + 7w = 260 \end{aligned}$$

Add the equations: $13w = 260$

$\Rightarrow \quad w = 20$

Since $b = 2w$, it follows that $b = 2(\mathbf{20}) = 40$. In conclusion, we instruct the factory to produce

20 white widgets & 40 black widgets

Check: 40 black is twice 20 white ✓

20 @ \$7 = \$140

40 @ \$3 = \$120

\$260 ✓

ALTERNATE METHOD: We can solve the system of equations in the previous example by the *Substitution Method*. Here is the system again:

$$\begin{aligned} b &= 2w \\ 3b + 7w &= 260 \end{aligned}$$

Since $b = 2w$, we can substitute $2w$ for b in the second equation:

$$\begin{aligned} & 3(2w) + 7w = 260 \\ \Rightarrow \quad & 6w + 7w = 260 \\ \Rightarrow \quad & 13w = 260 \\ \Rightarrow \quad & w = \mathbf{20} \\ \Rightarrow \quad & b = 2w = 2(20) = \mathbf{40} \end{aligned}$$

And we get the same values for w and b.

EXAMPLE 3: **Assume that green widgets cost \$5 apiece and yellow widgets cost \$9 apiece to produce. Our goal is that we produce 8 more yellow widgets than green widgets. If we have a total of \$352 to spend, how many widgets of each color should we manufacture?**

Solution: We'll let g and y represent the number of green and yellow widgets. Since we need "8 more yellow widgets than green widgets," we get

$$y = g + 8$$

The total cost to produce all the widgets gives us

$$5g + 9y = 352$$

Substitute the first equation into the second:

$$5g + 9(g + 8) = 352$$

$$\Rightarrow \quad 5g + 9g + 72 = 352$$

$$\Rightarrow \quad 14g = 280$$

$$\Rightarrow \quad g = 20$$

Last, using the first equation: $y = g + 8 = \mathbf{20} + 8 = 28$

20 green widgets & 28 yellow widgets

Notes: You should check this solution to make sure we have the right numbers of widgets. Also, we could have used the Elimination Method to solve the system of equations; you might try this for practice.

Homework

11. The unit cost of red widgets is $6 and the unit cost of white widgets is $8. If the goal is to make 25 widgets for a total of $170, how many widgets of each color should we make?

12. The unit cost of maroon widgets is $5 and the unit cost of white widgets is $9. If the goal is that the number of white widgets should be 8 times the number of maroon widgets, and if the total budget is $462, how many widgets of each color should we make?

13. The unit cost of red widgets is $7 and the unit cost of yellow widgets is $6. If the goal is that the number of yellow widgets should be 75 more than the number of red widgets, and if the total budget is $645, how many widgets of each color should we make?

14. The unit cost of red widgets is $7 and the unit cost of maroon widgets is $8. If the goal is to make 23 widgets for a total of $183, how many widgets of each color should we make?

15. The unit cost of white widgets is $9 and the unit cost of black widgets is $6. If the goal is that the number of black widgets should be 7 more than the number of white widgets, and if the total budget is $57, how many widgets of each color should we make?

16. The unit cost of yellow widgets is $9 and the unit cost of white widgets is $8. If the goal is to make 35 widgets for a total of $305, how many widgets of each color should we make?

17. The unit cost of black widgets is \$9 and the unit cost of white widgets is \$11. If the goal is that the number of white widgets should be 8 times the number of black widgets, and if the total budget is \$2425, how many widgets of each color should we make?

18. The unit cost of green widgets is \$9 and the unit cost of black widgets is \$11. If the goal is to make 9 widgets for a total of \$85, how many widgets of each color should we make?

❑ *A Graphical Approach to the Widget Production Problem*

Each of the widget production problems was solved by generating two equations in two variables from the data given in the problem, followed by solving the resulting system of equations. But in more advanced math, business, and science problems, the system of equations may be unsolvable using algebra; the best solution will come from an approximation. One of the approximation methods used is ***graphing***. We create the equations from the stated problem, graph each equation, and then estimate where the graphs cross each other (intersect).

EXAMPLE 4: **The unit cost of red widgets is \$13 and the unit cost of green widgets is \$25. Each production run can produce a total of 50 widgets, but we control how many of each color. The production budget is \$758. By graphing, determine how many widgets of each color we should manufacture.**

Solution: This problem leads to the system of equations:

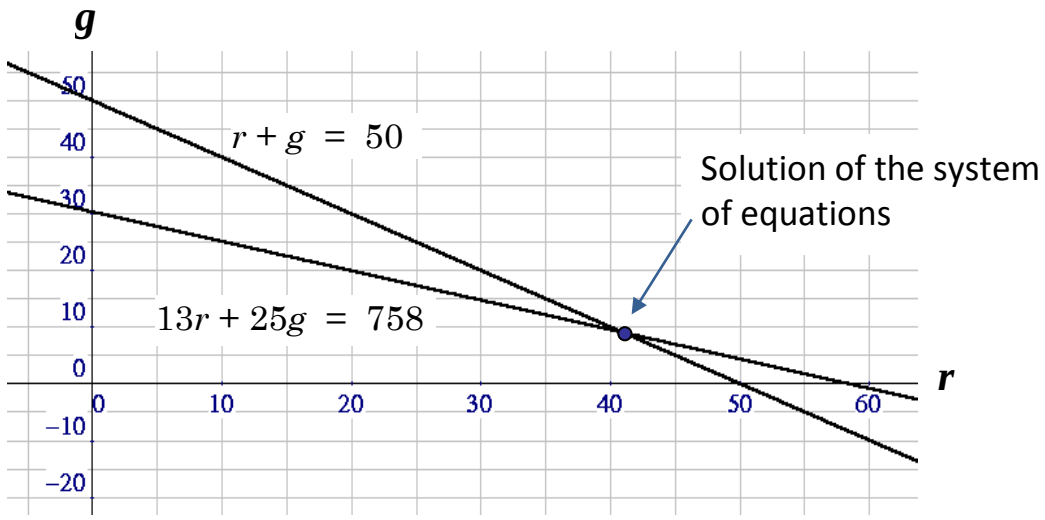
$$\begin{aligned} r + g &= 50 \\ 13r + 25g &= 758 \end{aligned}$$

To graph these two equations, we realize that each of them is a line, and we've learned that the simplest way to graph a line is to use the intercepts. For no particular reason, we will label the horizontal axis as r, and the vertical axis as g. This means that the ordered pairs will be written in the form (r, g).

$r + g = 50$: Setting $r = 0$ gives $g = 50$. Setting $g = 0$ gives $r = 50$. The intercepts are (0, 50) and (50, 0).

$13r + 25g = 758$: When $r = 0$, $g = 30.32$, and when $g = 0$, $r = 58.31$. The intercepts are therefore (0, 30.32) and (58.31, 0).

We now plot each line on the same grid, and then estimate the point of intersection of the two lines.



Focusing on the point of intersection, we estimate the r-coordinate to be about 41 and the g-coordinate to be about 9. That is, the point of intersection is about (41, 9). Our conclusion? We should make **41 red widgets and 9 green widgets**.

There's no homework for this section. All the problems in this chapter should be done using the *Elimination or Substitution Method*.

Review Problems

19. The unit cost of black widgets is \$11 and the unit cost of maroon widgets is \$5. If the goal is that the number of maroon widgets should be 8 times the number of black widgets, and if the total budget is \$306, how many widgets of each color should we make?



20. The unit cost of yellow widgets is \$10 and the unit cost of maroon widgets is \$6. If the goal is that the number of maroon widgets should be 135 more than the number of yellow widgets, and if the total budget is \$1242, how many widgets of each color should we make?

21. The unit cost of green widgets is \$8 and the unit cost of black widgets is \$9. If the goal is to make 45 widgets for a total of \$389, how many widgets of each color should we make?

Solutions

1. $c + t = 20$

2. $r + p = 150$

3. 4975

4. $3.75s$

5. $500w$

6. 225

7. $2p + 4n$

8. $2p + 4n = 26$

9. $7f + 12d = 250$

10. a. $x = y + 4$ b. $T = Z - 3$ c. $u = 2v$

d. $g = 3h$ e. $w = 4s$ f. $h = \frac{1}{2}k$

g. $E = \frac{1}{3}m$ h. $A = B - 12$ i. $W = Q - 1$

11. 15 red and 10 white

12. 6 maroon and 48 white

13. 15 red and 90 yellow

14. 1 red and 22 maroon

15. 1 white and 8 black

16. 25 yellow and 10 white

17. 25 black and 200 white

18. 7 green and 2 black

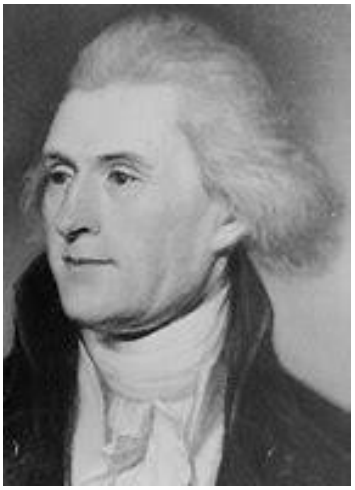
19. 6 black and 48 maroon

20. 27 yellow and 162 maroon

21. 16 green and 29 black

"I'm a great believer in luck, and I find the harder I work, the more I have of it."

– *Thomas Jefferson*



QUADRATIC MODELING – PART I



Whether you're trying to maximize your profits in a factory, or trying to design a cattle corral of maximum area with a given amount of fence, much of economics and engineering involves making the most of limited resources. The vertex of a parabola will allow us to solve such optimization problems.

❑ *SHORTCUT TO THE VERTEX*

Finding the vertex of a parabola by averaging the x-coordinates of the x-intercepts of the parabola is the same time-consuming process over and over again. Perhaps we can find the vertex of a generic parabola, yielding a simple formula that can find the vertex of any parabola quickly.

Let's start with a generic parabola (opening up or down) in standard form:

$$y = ax^2 + bx + c \qquad \text{[opens } up \text{ if } a > 0 \text{ and } down \text{ if } a < 0\text{]}$$

Our job now is to find the x-intercepts of this parabola. To find any x-intercepts, we set y to 0 and solve for x:

$$0 = ax^2 + bx + c$$

Let's first turn the equation around so it's in standard form:

$$ax^2 + bx + c = 0$$

How do we solve this equation for x? Actually, it's quite simple. The solutions of this equation are precisely the solutions provided by the Quadratic Formula. Thus, the two solutions of this equation are

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

These two "numbers" are the x-coordinates of the x-intercepts of the parabola. The next step is to find the average of these two numbers; this is done by adding them together and dividing by 2:

$$\frac{\dfrac{-b+\sqrt{b^2-4ac}}{2a} + \dfrac{-b-\sqrt{b^2-4ac}}{2a}}{2}$$

$$= \frac{\dfrac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}}{2} \qquad \text{(common denominator)}$$

$$= \frac{\dfrac{-2b}{2a}}{2} \qquad \text{(the radicals sum to 0)}$$

$$= \frac{\dfrac{-b}{a}}{2} \qquad \text{(reduce the top fraction)}$$

$$= \frac{-b}{a} \cdot \frac{1}{2} \qquad \text{(multiply by reciprocal)}$$

$$= \frac{-b}{2a}$$

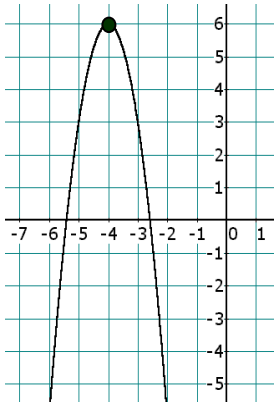
We've proved a theorem:

> The x-coordinate of the **vertex** of the parabola
>
> $$y = ax^2 + bx + c$$
>
> is given by the formula
>
> $$x = \frac{-b}{2a}$$

To find the y-coordinate of the vertex, just plug the x-coordinate of the vertex into the parabola formula.

For example, let's find the vertex of the parabola $y = -3x^2 - 24x - 42$. For this parabola, $a = -3$ and $b = -24$ (we don't need the value of c for this problem). So the x-coordinate of the vertex is

$$x = \frac{-b}{2a} = \frac{-(-24)}{2(-3)} = \frac{24}{-6} = -4$$



Now plug $x = -4$ into the parabola formula to find the y-coordinate of the vertex:

$$y = -3x^2 - 24x - 42 = -3(-4)^2 - 24(-4) - 42$$
$$= -3(16) - 24(-4) - 42 = -48 + 96 - 42 = \mathbf{6}$$

Conclusion: The vertex of the parabola

$y = -3x^2 - 24x - 42$ is **(–4, 6)**.

> Here's a good way to remember this vertex formula: Take a look at the Quadratic Formula, cross out the radical, and what remains is the x-coordinate of the vertex.

$$x = \frac{-b \pm \cancel{\sqrt{b^2 - 4ac}}}{2a}$$

Homework

1. Use the formula $x = \frac{-b}{2a}$ to find the **vertex** of each parabola:

   a. $y = x^2 + 2x - 48$

   b. $y = x^2 + 10x + 25$

   c. $y = 2x^2 + 8x + 5$

   d. $y = 3x^2 - 6x + 4$

   e. $y = -5x^2 + 40x - 20$

   f. $y = -3x^2 - 16$

2. Use the formula $x = \frac{-b}{2a}$ to find the **vertex** of each parabola:

a. $y = -2x^2 + 5x - 7$

b. $y = 7x^2 - x - 1$

c. $y = 3x^2 - 3x - 2$

d. $y = 7x^2 + 10$

e. $y = 4x^2 + 7x$

f. $y = -5x^2 - 14x + 10$

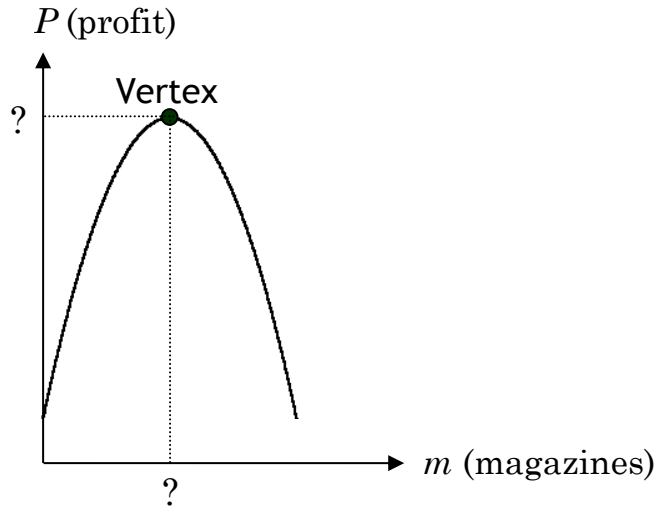
❑ *OPTIMIZATION PROBLEMS*

EXAMPLE 1: **A publishing company found that the profit they make on their latest magazine is given by the formula**

$$P = -2m^2 + 680m + 10200$$

where P is profit and m is the number of magazines sold. Find the number of magazines that the company needs to produce in order to maximize their profit. Also determine the maximum profit.

Solution: Wow, what a problem! It might help to realize that the variable m is playing the role of the variable x, while the P is playing the role of y. When viewed this way, the profit formula is simply a parabola that opens down (since $-2 < 0$). And where is the maximum point on a parabola that opens down? At its vertex, of course.



Once we find the vertex of the given parabola formula, we can find the number of magazines needed to maximize the profit, and then the dollar amount of the maximum profit itself.

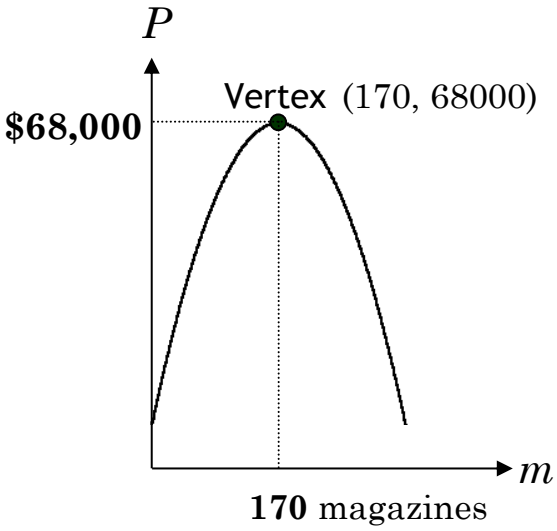
From the given parabola formula $P = -2m^2 + 680m + 10200$, we see that $a = -2$ and $b = 680$. Plugging these two values into our vertex formula gives us the m-coordinate of the vertex:

$$m = \frac{-b}{2a} = \frac{-680}{2(-2)} = \frac{-680}{-4} = \mathbf{170}$$

And since the m-axis is the horizontal axis (the magazine axis), we conclude that the number 170 represents magazines. That is, 170 magazines will produce the maximum profit. To calculate that profit, we merely let $m = 170$ in the profit formula:

$P = -2m^2 + 680m + 10200$	(the profit formula)
$= -2(\mathbf{170})^2 + 680(\mathbf{170}) + 10200$	($m = 170$)
$= -2(28900) + 680(170) + 10200$	(exponents first)
$= -57800 + 115600 + 10200$	(multiplications second)
$= \mathbf{68000}$	(additions last)

The vertex of the parabola is therefore the point (170, 68000).



In conclusion, producing **170** magazines will yield the company a maximum profit of **$68,000**.

EXAMPLE 2: **A company found that the cost for manufacturing a surfboard is given by the formula**

$$C = 3s^2 - 270s + 6800$$

where C is the cost in dollars and s is the number of surfboards produced. Find the number of boards the company needs to produce in order to minimize their cost. What is the minimum cost?

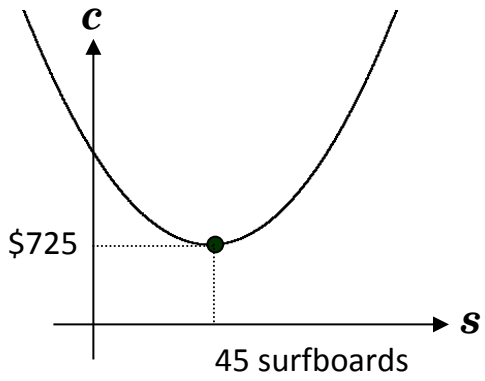
Solution: This problem is very similar to Example 1, except here we are to find the minimum of something, not the maximum. The graph of the cost formula is a parabola opening up (since $3 > 0$), so its vertex will be at the bottom of the parabola, and, as before, the vertex tells us everything we need to know.

Since $a = 3$ and $b = -270$, the s-coordinate of the vertex is

$$s = \frac{-b}{2a} = \frac{-(-270)}{2(3)} = \frac{270}{6} = \mathbf{45}$$

Plugging this value of s into the cost formula yields C = **725**. Therefore, the vertex of the parabola is **(45, 725)**.

What does all this mean? Geometrically, the point (45, 725) is at the bottom of the parabola. Therefore,



producing **45** surfboards will result in a minimum cost of **$725**.

Homework

3. A hard drive company found that the profit they make on their hard drive is given by the formula

$$P = -4h^2 + 200h + 1000,$$

where P is profit and h is the number of hard drives sold. Find the number of hard drives that the company needs to produce in order to maximize their profit. Also determine the maximum profit.

4. A company found that the cost for manufacturing a flash drive is given by the formula

$$C = 7d^2 - 140d + 745,$$

where C is the cost in dollars and d is the number of flash drives produced. Find the number of flash drives the company needs to produce in order to minimize their cost. What is the minimum cost?

5. The path of a baseball follows the parabola

$$y = -12x^2 + 456x + 100$$

At what value of x does the baseball reach its highest point? How high above the x-axis is the highest point?

6. The profit, P, is given by the formula

$$P = -10c^2 + 800c + 10000$$

Find the number of calculators, c, which would maximize the profit, and determine the maximum profit.

7. The cost, C, for manufacturing w widgets is given by

$$C = 7w^2 - 112w + 5000$$

How many widgets will minimize the cost, and what is the cost?

8. The path of a catapulted math teacher follows the parabola $y = -12x^2 + 312x + 250$. Find the point in the x-y plane where the teacher reaches his highest point.

9. A certain mountain has the approximate shape of the parabola $y = -13x^2 + 390x + 200$. Find the coordinates of the mountain peak.

10. A certain valley has the approximate shape of the parabola $y = 7.5x^2 - 1485x - 400$. Find the coordinates of the lowest point of the valley.

Review Problems

11. Using the short formula from this chapter, find the vertex of the parabola

$$y = -5x^2 + 70x - 90$$

12. The profit is given by the formula

$$P = -5b^2 + 750b + 10000$$

where P is profit and b is the number of books published. Find the number of books which would maximize the profit, and determine the maximum profit.

13. The cost, C, for manufacturing t trumpets is given by

$$C = 6t^2 - 276t + 5000$$

How many trumpets will minimize the cost, and what is the cost?

14. The path of a football follows the parabola $y = -7x^2 + 126x + 250$. Find the point in the x-y plane where the football reaches its highest point.

Solutions

1. a. $(-1, -49)$ b. $(-5, 0)$ c. $(-2, -3)$
d. $(1, 1)$ e. $(4, 60)$ f. $(0, -16)$

2. a. $\left(\frac{5}{4}, -\frac{31}{8}\right)$ b. $\left(\frac{1}{14}, -\frac{29}{8}\right)$ c. $\left(\frac{1}{2}, -\frac{11}{4}\right)$

d. (0, 10) e. $\left(-\frac{7}{8}, -\frac{49}{16}\right)$ f. $\left(-\frac{7}{5}, \frac{99}{5}\right)$

3. 25 hard drives; maximum profit = \$3,500

4. 10 flash drives; minimum cost = \$45

5. $x = 19$; height = 4,432

6. 40 calculators; maximum profit = \$26,000

7. 8 widgets; minimum cost = \$4552

8. (13, 2278)

9. (15, 3125)

10. (99, –73907.5)

11. V(7, 155)

12. 75 books; \$38,125

13. 23 trumpets; \$1826

14. (9, 817)

"The purpose of learning is growth, and our minds, unlike our bodies, can continue growing as we continue to live."



– Mortimer Adler

QUADRATIC MODELING – PART II

We continue our discussion of the uses of quadratic functions (parabolas) to solve business problems involving both revenue and expenses. We'll also solve problems involving the maximization of the area of a rectangular region using a given amount of fencing.

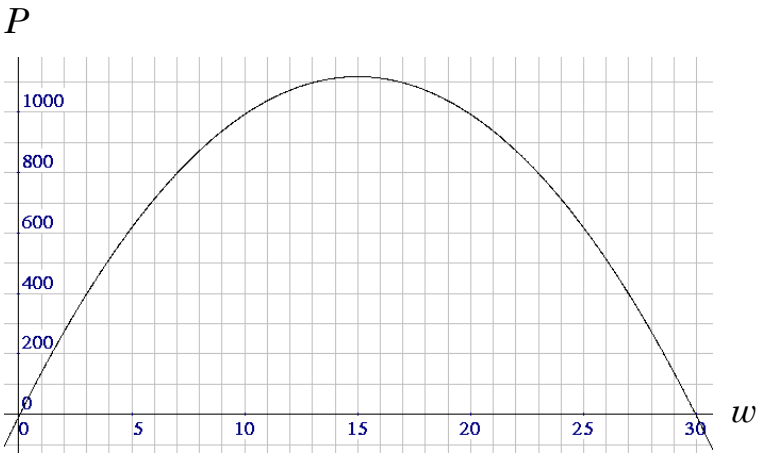


EXAMPLE 1: **The revenue function at Widget, Inc., where w is the number of widgets produced, is given by $R(w) = 2w^2 + 100w - 6$, while the expense function is $E(w) = 7w^2 - 50w + 1$. Find the production level of widgets that will yield a <u>maximum profit</u>, and then calculate the profit.**

Solution: Revenue is the money received from selling a product or service, and expenses are the costs incurred in providing that product or service to the customer. Profit is the difference between revenue and expenses, so we can write the formula

$$P(w) = R(w) - E(w)$$
$$\Rightarrow \quad P(w) = (2w^2 + 100w - 6) - (7w^2 - 50w + 1)$$
$$\Rightarrow \quad P(w) = 2w^2 + 100w - 6 - 7w^2 + 50w - 1$$
$$\Rightarrow \quad P(w) = -5w^2 + 150w - 7$$

This formula shows that the profit, P, is a quadratic function of the number of widgets, w. So its graph is a parabola opening down (since $-5 < 0$). Because we're trying to determine the maximum profit, we observe that the maximum value of P will be at the vertex (top) of the parabola. A rough sketch of the profit function (the parabola) should make this concept clear.



The w-coordinate of the vertex is given by the formula $\frac{-b}{2a}$, where $a = -5$ and $b = 150$. So, $w = \frac{-b}{2a} = \frac{-150}{2(-5)} = \frac{-150}{-10} = 15$.

Hence, the production level of widgets should be 15. This maximizes the profit. Fewer than 15 widgets would result in less profit; more than 15 widgets would also reduce profit. The optimum number of widgets is 15.

Now to calculate the profit on the 15 widgets:

$$P(w) = -5w^2 + 150w - 7$$

$$\Rightarrow \quad P(\mathbf{15}) = -5(\mathbf{15})^2 + 150(\mathbf{15}) - 7$$

$$\Rightarrow \quad P(15) = 1118$$

Producing **15** widgets gives a maximum profit of **$1,118**

This result is consistent with the graph of the parabola.

EXAMPLE 2: **Farmer Gertrude has 40 feet of fence to build a rectangular pigpen. What should the dimensions of the pigpen be so that it will enclose the maximum area? What will the area be with these dimensions?**

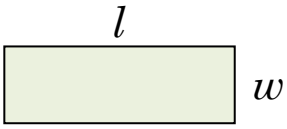


Solution: As a student, I remember thinking that the given 40 feet of fence would produce the same area regardless of the dimensions of the rectangle. However, consider the following rectangles:



Each rectangle has the same perimeter (the 40 feet of fence), but the areas (75 sq ft and 64 sq ft) are different — the first rectangle contains 11 additional sq ft, perhaps room for another pig.

The solution begins by labeling a rectangular pigpen:



Now, the **perimeter** of the rectangle is given to be 40 feet. Thus,

$$2l + 2w = 40$$

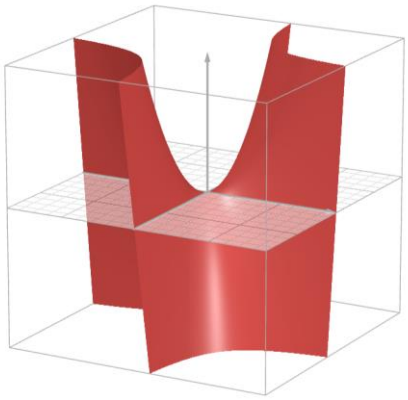
Also, the **area** of the rectangle is to be made as large as possible.

$$A = lw$$

The perimeter equation is called the **constraint** because it limits the possible dimensions of the rectangle.

The area formula is called the **objective function** because it represents the quantity we wish to maximize.

We'd like to graph this area function, $A = lw$, but the problem with this formula is that the area is a function of too many variables (we can't graph three variables in a two-dimensional coordinate system). But we can tie the perimeter formula to the area formula with a little substitution, thereby expressing the area, A, as a function of one variable instead of two:



This is the graph of $A = lw$, not helpful until Calculus.

Solve for l in the perimeter formula:

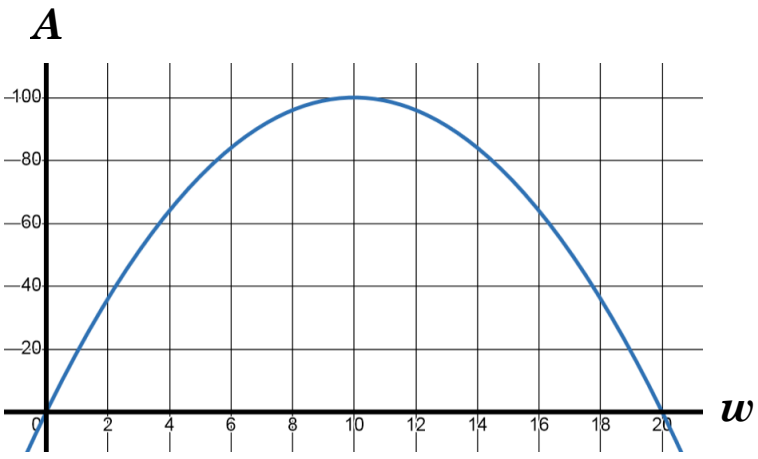
$$2l + 2w = 40 \quad \Rightarrow \quad 2l = 40 - 2w$$

$$\Rightarrow \quad l = \frac{40 - 2w}{2} \quad \Rightarrow \quad l = \frac{40}{2} - \frac{2w}{2} \quad \Rightarrow \quad \underline{l = 20 - w}$$

Now substitute $20 - w$ for l in the area formula:

	$A = lw$	(the area formula)
$\Rightarrow$	$A = (\underline{20 - w})w$	(letting $l = 20 - w$)
$\Rightarrow$	$A = 20w - w^2$	(distribute)
$\Rightarrow$	$A = -w^2 + 20w$	(better form)

We have written the area as a function of just one variable, w. We want to find the value of w that will maximize A. We can find this value of w because the equation we just wrote is a parabola:



Looking at the graph, we conjecture that the vertex — the maximum point on the curve — occurs at the point (10, 100). This is verified by using the formula for the vertex, where the w-coordinate is given by

$$w = \frac{-b}{2a} = \frac{-20}{2(-1)} = \frac{-20}{-2} = 10$$

Thus, the width is 10 feet. Using the formula $l = 20 - w$ we see that the length is also 10 feet. The rectangle of maximum area is therefore actually a

10 ft × 10 ft square

whose area is 100 sq ft.

Homework

1. The revenue function is $R(x) = 2x^2 - 5x - 25$, while the expense function is $E(x) = 7x^2 - 85x + 75$. Find the production level that will yield a maximum profit. What is the profit?
   Recall: Profit = Revenue – Expenses.

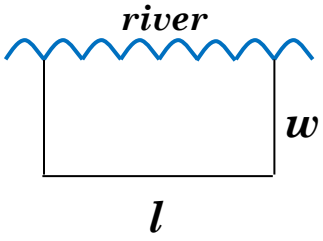
2. The revenue function is $R(x) = 2x^2 + 2000x + 200$, while the expense function is $E(x) = 11x^2 + 596x - 50$. Find the production level that will yield a maximum profit. What is the profit?

3. To protect his corn from Farmer Gertrude's pigs (Gertrude made the 10 ft × 10 ft pigpen alright, but she made it only 9 inches tall!), Uncle Eb needs to enclose his corn in a rectangular region, using only 84 feet of fence. He would like as much area as he can get with his fence. What should be the dimensions of Uncle Eb's cornfield?

4. Same as the previous problem, but assume 180 feet of fence.

5. Rancher Millie liked what Gertrude and Uncle Eb had done, so when she needed to build a cow corral, she consulted them about her problem. Gertrude and Eb told Millie that a square would definitely be the best shape, since a square gave both of them the most area for the given amount of fence. What Millie didn't think was relevant to tell her friends was that a river would be used as one side of the corral; that is, only three sides of the rectangle would have to be fenced:





If Rancher Millie has 1000 feet of fence to build the 3-sided rectangle, what dimensions would yield her the maximum area?

6. The same scenario as the previous problem, except Rancher Millie has 1200 feet of fence.

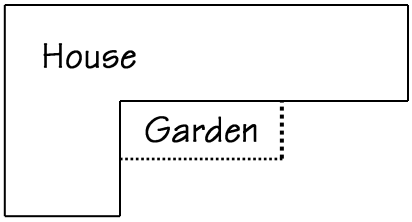
Review Problems

7. The revenue function for the widget factory is $R(w) = 23w^2 + 950w + 95$, and the expense function is $E(w) = 33w^2 + 350w - 5$. How many widgets should be produced to achieve the maximum profit?

8. What dimensions should a rectangular region be if the perimeter must be 100 ft and if the maximum area is desired?

9. A rectangular corral is to be built with 144 feet of fence, using the side of a barn as one of the sides of the rectangle. If the goal is to enclose the maximum number of square feet, what should the dimensions of the rectangle be?

10. Bruce has 60 feet of fence with which to enclose a rectangular region for his flower garden. But an inside corner of the yard will provide two of the four sides of the rectangle. Find the dimensions of the rectangle that will provide Bruce with the maximum possible area for his garden.



Solutions

1. 8 items; $220

2. 78 items; $55,006

3. 21 ft × 21 ft

4. 45 ft × 45 ft

5. Hint: It's not a square

6. 600 ft × 300 ft

7. 30 widgets

8. 25 ft × 25 ft

9. 72 ft × 36 ft

10. 30 ft × 30 ft

Education is not

a preparation for life;

education is life itself.

John Dewey

❑ *THE RADIATOR PROBLEM*



If a 12-liter radiator is filled with an antifreeze/water solution which is at a 25% concentration of antifreeze, then 25% of the 12 liters is antifreeze and the rest (75%) is water. Multiplying 12 liters by 25% gives us the fact that 3 of the liters of solution are antifreeze, and the other 9 liters (12 liters × 75%) are water. Thus, the key formula we need for the percent mixture problems in this section is

Quantity × Concentration = Actual Amount

EXAMPLE 1: **A 10-liter radiator is filled with 42% antifreeze solution. How much of the antifreeze solution must be drained and replaced with pure antifreeze to bring the concentration of the final solution up to 58%?**

Solution: We'll set up all the info in a chart, utilizing the above formula:

	Quantity (in liters)	× **Concentration** (of antifreeze)	= **Actual Amount** (of antifreeze)
Original	10	42%	4.2
Drain	x	42%	$0.42x$
Replace	x	100%	x
Final	10	58%	5.8

Notice that there are four rows in our chart, representing each of the four phases of the problem.

In the statement of the problem, we are given that it's a 10-liter radiator filled with 42% antifreeze. The product of these two numbers is 4.2, as per the formula given above.

The question asks us how much to remove -- we don't know, so this is where the variable x comes in. What is the concentration of the x liters we're to remove? It's the same as the concentration of the entire 10 liters: 42%. And, of course, the product of x with 42% is $0.42x$.

Now for the 3rd row of the chart. The number of liters of pure antifreeze needed to refill the radiator must be equal to the number of liters that were drained off, namely x. And the refill fluid is pure antifreeze, which has a concentration of 100%; hence, the last entry in the row is $x \times 100\%$, or simply x.

The bottom row: The radiator has been refilled, so it contains 10 liters of solution, just as in the original row. The concentration of this final mixture is stipulated to be 58%, and the product is clearly 5.8.

Thus, looking at the right-hand column of the chart (Actual Amount), we can say that we started with 4.2 liters of antifreeze, subtracted $0.42x$ liters, added x liters, and ended up with 5.8 liters of antifreeze. Now we translate these thoughts into an equation:

$$4.2 - 0.42x + x = 5.8$$

$$\Rightarrow \quad 0.58x = 1.6$$

$$\Rightarrow \quad x = 2.76, \text{ and thus}$$

2.76 liters should be drained out

Note: If the antifreeze solution needs to be diluted, then we drain a certain amount of the solution and replace it with pure water, which has an antifreeze concentration of 0%.

Homework

1. A 9-liter radiator is filled with 67% antifreeze solution. How much of the antifreeze solution must be drained and replaced with pure antifreeze to bring the concentration of the final mixture up to 95% antifreeze?

2. A 5-liter radiator is filled with 58% antifreeze solution. How much of the antifreeze solution must be drained and replaced with pure water to bring the concentration of the final mixture down to 25% antifreeze?

3. A 4-liter radiator is filled with 32% antifreeze solution. How much of the antifreeze solution must be drained and replaced with pure antifreeze to bring the concentration of the final mixture up to 75% antifreeze?

4. A 20-liter radiator is filled with 75% antifreeze solution. How much of the antifreeze solution must be drained and replaced with pure water to bring the concentration of the final mixture down to 23% antifreeze?

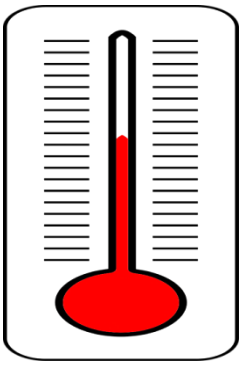
1. 7.64 liters

2. Note: The concentration of antifreeze in pure water is 0%.
$5(.58) - .58x + 0x = 5(.25) \Rightarrow 2.84$ liters

3. 2.53 liters

4. 13.87 liters

TEMPERATURE FORMULAS



The **Fahrenheit** temperature scale was designed so that water freezes at **32°F** and boils at **212°F**. Later, the **Celsius** (originally called centigrade) scale was created to make the numbers easier to work with. On this new scale, water freezes at **0°C** and boils at **100°C**. We conclude that

0°C = 32°F and

100°C = 212°F

❑ *CREATING THE FORMULA: F = 1.8C + 32*

Our goal here is to create, from scratch, a formula that converts from degrees Celsius to degrees Fahrenheit. All we need to accomplish this goal are two facts:

0°C = 32°F

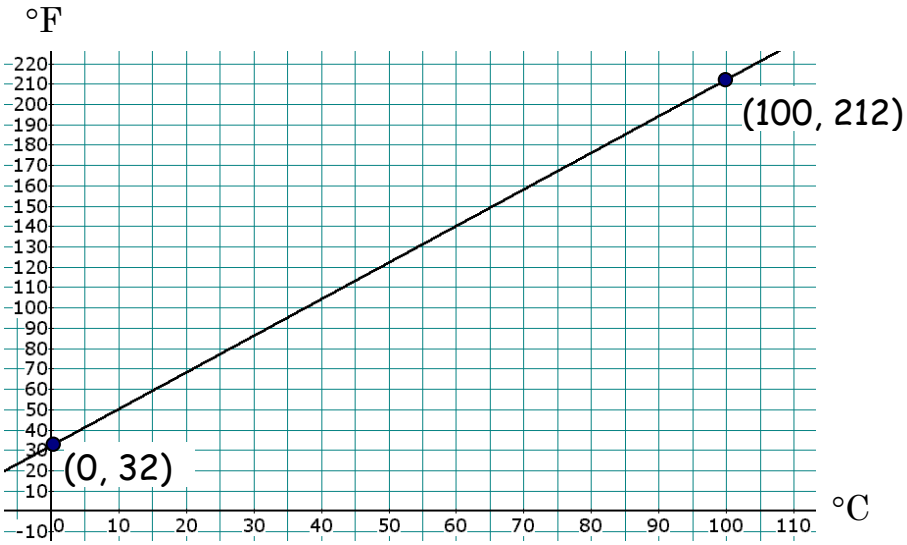
100°C = 212°F

and one assumption:

The relationship is **linear** (the graph is a straight line).

If we label the horizontal axis °C and the vertical axis °F, then the temperature facts above translate into the following two points on the line: (0, 32) and (100, 212).

As we proceed through the derivation, notice that °C is just a new name for x, and °F is a new name for y.



We begin with a linear formula relating °C to °F, where m represents the slope of the line, and b is the "F-intercept."

$$F = mC + b$$ (note that C is the x-value and F is the y-value)

First, we use the two points on the line to calculate the **slope** of the line:

$$m = \frac{\Delta F}{\Delta C} = \frac{212 - 32}{100 - 0} = \frac{180}{100} = 1.8$$

So now our equation is

$$F = 1.8C + b$$

Second, we use one of the given points — we'll use (0, 32) — to find b:

$$\mathbf{32} = 1.8(\mathbf{0}) + b \quad \Rightarrow \quad 32 = 0 + b \quad \Rightarrow \quad 32 = b$$

and we have our equation relating the two temperature scales:

$$\boxed{F = 1.8C + 32}$$

P.S. If using C and F in place of the usual x and y is too confusing, you're welcome to stick with x and y and work through the whole problem. Just be sure you write your final answer using the C and the F, because those are the variables contained in the formula.

❑ CREATING THE FORMULA: $C = \frac{F - 32}{1.8}$

What about the other way around?

> *What if we're given the Fahrenheit temperature and we wish to calculate the Celsius temperature?*

There are three ways we can answer this question.

Answer #1: Use the formula we derived above, $F = 1.8C + 32$, plug in the given Fahrenheit temperature, and solve for C with a little algebra. For example, to convert 50°F to Celsius, we can write

$F = 1.8C + 32$	(the formula we derived)
$\mathbf{50} = 1.8C + 32$	(substitute 50 for F)
$50 = 1.8C + 32$ $\mathbf{-32} \quad\quad \mathbf{-32}$	(subtract 32 from each side)
$18 = 1.8C$	(simplify each side)
$10 = C$	(divide each side by 1.8)

Conclusion: 50°F is equivalent to 10°C.

Answer #2: Just take the formula $F = 1.8C + 32$ and solve for C. After all, why not take advantage of our knowledge of solving formulas?

	$F = 1.8C + 32$	(the formula to find F)
$\Rightarrow$	$F - 32 = 1.8C$	(subtract 32 from each side)
$\Rightarrow$	$\frac{F-32}{\mathbf{1.8}} = \frac{1.8C}{\mathbf{1.8}}$	(divide each side by 1.8)
$\Rightarrow$	$\frac{F-32}{1.8} = C$	(simplify)

And so we now have our Celsius formula: $\boxed{C = \frac{F-32}{1.8}}$

So, to solve the problem stated in Answer #1 — to convert 50°F to °C — we just plug in 50 for F and calculate:

$$C = \frac{F-32}{1.8} = \frac{\mathbf{50}-32}{1.8} = \frac{18}{1.8} = 10$$

and again we conclude that <u>50°F is equivalent to 10°C</u>.

<u>Answer #3:</u> We could derive the formula for °C by mimicking the derivation of the formula $F = 1.8C + 32$ at the beginning of this chapter.

Homework

1. Derive the temperature formula $C = \frac{F-32}{1.8}$ in the same manner (the graphing) that the formula $F = 1.8C + 32$ was derived.

Solutions

1. Interchange the °C and °F axes, switch the coordinates of the two points on the graph, and proceed with the ideas of slope and intercept.

"It does not matter how slowly you go, as long as you do not stop."



– Confucius –

DIRECT AND INVERSE VARIATION

❑ *INTRODUCTION*



Let's start right off with an example:

Let D = number of hard ***D***rives sold
C = the ***C***apacity of the drive, in gigabytes (GB)
P = selling ***P***rice

Now let's make up a formula that will illustrate the ideas of this chapter:

$$D = \frac{5C}{P}$$

① What happens to drives sold, D, when the capacity, C, is increased? Let's assume that the drive capacity was 200 GB last month and the price was \$100. The number of drives sold was

$$D = \frac{5C}{P} = \frac{5 \cdot 200}{100} = \underline{10 \text{ drives}}$$

Now increase the capacity to 600 GB; the number of drives sold will be

$$D = \frac{5C}{P} = \frac{5 \cdot \mathbf{600}}{100} = \underline{30 \text{ drives}}$$

Make sense? Increasing the capacity (allowing the drives to hold more data) and keeping the price at \$100 should <u>increase</u> sales.

② Now let's see what happens if we increase the price and keep the capacity the same. Assume that last month the price, P, was \$150

when the capacity was 450 GB. We calculate the number of drives sold:

$$D = \frac{5C}{P} = \frac{5 \cdot 450}{150} = \underline{15 \text{ drives}}$$

Let's predict future sales if we increase the price from \$150 to \$225:

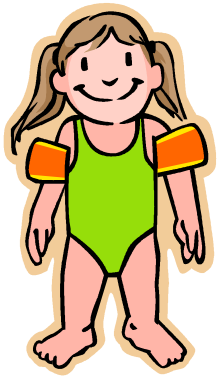
$$D = \frac{5C}{P} = \frac{5 \cdot 450}{225} = \underline{10 \text{ drives}}$$

Make sense? Increasing the price and keeping the capacity the same produced a <u>decrease</u> in sales.

Without going into the details, we could also demonstrate that if the capacity, C, goes down, so will D, the number of drives sold. And if the price, P, goes down, D will go up.

❑ *DIRECT AND INVERSE VARIATION*

Direct Variation What happens to the sale of bathing suits, B, when the temperature, T, goes up? As the temperature rises, so does the sale of bathing suits. We say that B is ***directly proportional*** to T, or that B ***varies directly*** as T. A formula of this type might be



$$B = 3T$$

For example, if the temperature is 80°, then 240 suits will be sold. But if the temperature rises to 100°, then 300 units will be sold. When it gets cold again, B will decrease. Whatever T does (increase or decrease), B does the **same**.

Inverse Variation What happens to the air pressure, P, as you increase your elevation, E? As your elevation goes up, the air pressure goes down. We say that P is ***inversely proportional*** to E, or that P ***varies inversely*** as E. An example might be the formula



$$P = \frac{2000}{E}$$

For instance, if $E = 200$, then $P = 10$. But if E is increased to 500, then the pressure P is reduced to 4. When you return to lower altitudes, the pressure goes back up. Whatever E does, P does the **reverse**.

See the numbers "3" and "2000" in our two formulas above? Each of these numbers is called a ***constant of proportionality*** (or ***constant of variation***).

In the following definitions, the letter ***k*** is the positive constant of proportionality (or constant of variation).

$y = kx$	is read "y is ***directly proportional to*** x," or "y ***varies directly as*** x," and means: If x increases, then y increases; and if x decreases, then y decreases. In other words, y does whatever x does.
$y = \frac{k}{x}$	is read "y is ***inversely proportional to*** x," or "y ***varies inversely as*** x," and means: If x increases, then y decreases; and if x decreases, then y increases. In other words, y does the reverse of what x does.

Homework

1. Consider the *direct variation* $D = 12Q$. We can pretend that Q is the quality of a car and D is the demand for that car.

   a. Find the value of D if Q is 40.
   b. Double the Q to 80 and recalculate D.
   c. When the quality increased, what happened to the demand?

d. Now reduce Q to 3 and recalculate D.
e. As the quality decreased, what happened to the demand?

2. Consider the *inverse variation* $P = \frac{200}{S}$. Let's assume that S is supply and P is price.

a. Find the value of P if $S = 10$.
b. Quadruple the S to 40 and recalculate P.
c. When the supply increased, what happened to the price?
d. Now reduce S to 5 and recalculate P.
e. As the supply decreased, what happened to the price?

❑ *EXTENSIONS OF THE BASIC VARIATION FORMULAS*

To extend the usefulness of problems in variation, we can add exponents and square roots to our direct and inverse variation formulas. For example, a fact of physics is that the kinetic energy of an object (energy of motion) varies directly as the square of its velocity, which can be written $E = kv^2$.

We can also combine direct and inverse variation into a single formula. For example, a chemistry principle states, "The volume of a gas is directly proportional to its temperature and inversely proportional to its pressure." This is summarized by the formula $V = \frac{kT}{P}$.

EXAMPLE 1: **Translate each variation statement into an algebraic formula, using k as the constant of variation:**

A. z varies directly as the cube of T. $z = kT^3$

B. R is inversely proportional to the square root of V. $R = \frac{k}{\sqrt{V}}$

C. P varies directly as the square of Q, and inversely as the square root of R. $P = \frac{kQ^2}{\sqrt{R}}$

D. B varies directly as the product of the square root of A and the cube of C. $B = k\sqrt{A}\,C^3$

E. y varies directly as the product of x and z, and inversely as the fourth power of w. $y = \frac{kxz}{w^4}$

Homework

3. Translate each variation statement into an algebraic formula, using ***k*** as the constant of proportionality:

a. b varies inversely as the 6th power of t.

b. n varies directly as the square root of b.

c. p is inversely proportional to the square of s.

d. v is directly proportional to c.

e. p varies inversely as the cube of x.

f. u is directly proportional to the square of t.

g. t varies directly as the square of h.

h. L is inversely proportional to the cube of s.

i. w varies directly as the 9th power of u.

j. R varies directly as the product of w and y.

k. g is directly proportional to the cube of c.

l. h varies directly as c and inversely as y.

m. x varies directly as the cube of t and inversely as the cube of h.

n. u is directly proportional to the product of c and the cube of r and inversely proportional to m.

o. y varies directly as the square of v and inversely as the cube of d.

p. n varies directly as the product of t and the square of d and inversely as b.

q. m is directly proportional to w and inversely proportional to z.

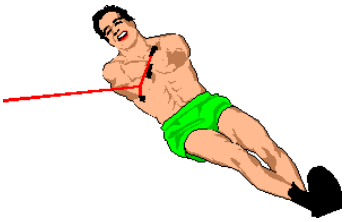
r. A varies directly as the product of u and the square of x and inversely as p.

s. n is directly proportional to r and inversely proportional to v.

t. c varies directly as the cube of u and inversely as the cube of z.

❑ *APPLICATIONS OF VARIATION*

EXAMPLE 2: **The number of bathing suits sold is directly proportional to the outside temperature and inversely proportional to the selling price. The number of suits sold is 1,600 when the temperature is 80° and the selling price is $50. Find the number of suits sold when the temperature rises to 95° and the price is reduced to $40.**





Solution: We'll start by letting

B = bathing suits sold

T = temperature

P = price

The first sentence of the problem, "The number of bathing suits sold is directly proportional to the outside temperature, and

inversely proportional to the selling price," gives us our variation formula:

$$B = \frac{kT}{P}$$

The second sentence, "The number of suits sold is 1,600 when the temperature is 80° and the selling price is \$50," allows us to find the value of k:

$$1{,}600 = \frac{k(80)}{50} \qquad \text{(substitute the given values)}$$

$$\Rightarrow \quad 80k = 80{,}000 \qquad \text{(multiply each side by 50)}$$

$$\Rightarrow \quad \boldsymbol{k = 1{,}000} \qquad \text{(divide each side by 80)}$$

The third sentence, "Find the number of suits sold when the temperature rises to 95° and the price is reduced to \$40," (with the value k = 1,000 we just calculated) gives us all the parts needed to compute the number of bathing suits sold under the new set of conditions:

$$B = \frac{kT}{P} \qquad \text{(our variation formula)}$$

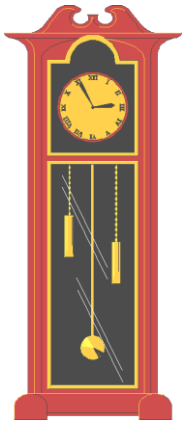
$$\Rightarrow \quad B = \frac{1{,}000T}{P} \qquad \text{(the constant } \boldsymbol{k} \text{ is 1,000)}$$

$$\Rightarrow \quad B = \frac{1{,}000(95)}{40} \qquad \text{(use the new values of } T \text{ and } P\text{)}$$

$$\Rightarrow \quad B = \frac{95{,}000}{40}$$

$$\Rightarrow \quad \boxed{B = 2{,}375 \text{ bathing suits}}$$

EXAMPLE 3: **The period of the pendulum (the time for one full swing) in a grandfather clock varies directly as the square root**



of its length. If the period is 50π when the length is 25, find the period when the length is 49.

Solution: The first sentence gives us our variation formula:

$$P = k\sqrt{L}$$

Substituting a period of 50π and a length of 25 gives:

$$50\pi = k\sqrt{25}$$

$\Rightarrow$ $50\pi = 5k$ (the positive square root of 25 is 5)

$\Rightarrow$ $\boldsymbol{k = 10\pi}$ (divide each side by 5)

Now we rewrite our variation formula using the k we just found:

$$P = k\sqrt{L} \Rightarrow P = 10\pi\sqrt{L}$$

Finally, find the period when the length is 49:

$$P = 10\pi\sqrt{49}$$

$\Rightarrow$ $P = 10\pi(7)$

$\Rightarrow$ $\boxed{P = 70\pi}$

Homework

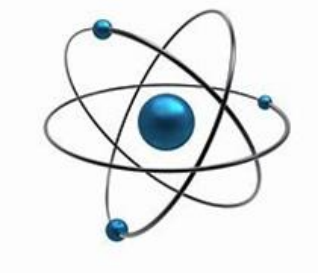


In a variation problem, DON'T forget "***k***", the *constant of variation*.

4. The area of a rectangle is directly proportional to its length. If the area is 247 when the length is 19, find the area when the length is 11.

5. The circumference of a circle is directly proportional to its radius. If the circumference is 40π when the radius is 20, then what is the circumference when the radius is 8?

6. The area of a circle varies directly as the square of its radius. If the area is 9π when the radius is 3, what is the area when the radius is 6?



7. The number of electrons varies directly as the square of the energy level. If the number of electrons is 32 when the energy level is 4, how many electrons are there when the energy level is 3?

8. The energy density is directly proportional to the fourth power of the temperature. If the energy is 512 when the temperature is 4, what is the energy when the temperature is 5?

9. The density of an object varies inversely as the object's volume. If the density is 22 when the volume is 18, what is the density when the volume is 6?

10. The acceleration of an object is inversely proportional to the object's mass. If the acceleration is 9 when the mass is 19, what is the acceleration when the mass is 1?

11. The force of gravity between two objects varies inversely as the square of the distance between the objects. If the force is 4 when the distance is 8, what is the force when the distance is 1?

12. The period of a pendulum varies directly as the square root of its length. If the period is 22π when the length is 121, what is the period when the length is 49?

13. The velocity of an object is directly proportional to the square root of its kinetic energy. If the velocity is 524 when the kinetic energy is 4, what is the velocity when the kinetic energy is 25?

14. The potential energy of an object varies directly as the product of its mass and height. If the potential energy is 80 when the mass is 4 and the height is 2, find the potential energy if the mass is 14 and the height is 13.

15. The fluid force on an object is directly proportional to the product of its area and its depth. If the fluid force is 1920 when the area is 10 and the depth is 24, find the fluid force if the area is 23 and the depth is 5.

16. The volume of a gas varies directly as its temperature, and inversely as its pressure. If the volume is 12 when the temperature is 9 and the pressure is 12, find the volume when the temperature is 5 and the pressure is 16.

17. The electric field is directly proportional to the charge, and inversely proportional to the area. If the electric field is 13 when the charge is 13 and the area is 9, find the electric field when the charge is 15 and the area is 3.



Review Problems

18. Translate the variation statement into an algebraic formula, using k as the constant of variation: "E varies directly as the product of x and the cube of y, and inversely as the square root of z."

19. The volume of a gas varies directly as its temperature, and inversely as its pressure. If the volume is 80 when the temperature is 40 and the pressure is 5, find the volume when the temperature is 30 and the pressure is 6.

20. The current in a circuit varies directly as the voltage. If the current is 336 when the voltage is 21, find the current when the voltage is 35.

21. The potential energy of an object varies directly as the product of its mass and its height. If the potential energy is 462 when the mass is 6 and the height is 7, find the potential energy if the mass is 24 and the height is 17.

22. The electric field is directly proportional to the charge, and inversely proportional to the area. If the electric field is 10 when

the charge is 15 and the area is 3, find the electric field when the charge is 12 and the area is 6.

23. The kinetic energy of a particle varies directly as the product of its mass and the square of its velocity. Assume that a particle of mass 10 and traveling at a velocity of 8 has a kinetic energy of 320. Find the kinetic energy of a particle with mass 7 and velocity 10.

Solutions

1. a. 480 b. 960 c. It increased d. 36 e. It decreased

2. a. 20 b. 5 c. It decreased d. 40 e. It increased

3. a. $b = \frac{k}{t^6}$ b. $n = k\sqrt{b}$ c. $p = \frac{k}{s^2}$ d. $v = kc$

e. $p = \frac{k}{x^3}$ f. $u = kt^2$ g. $t = kh^2$ h. $L = \frac{k}{s^3}$

i. $w = ku^9$ j. $R = kwy$ k. $g = kc^3$ l. $h = \frac{kc}{y}$

m. $x = \frac{kt^3}{h^3}$ n. $u = \frac{kcr^3}{m}$ o. $y = \frac{kv^2}{d^3}$ p. $n = \frac{ktd^2}{b}$

q. $m = \frac{kw}{z}$ r. $A = \frac{kux^2}{p}$ s. $n = \frac{kr}{v}$ t. $c = \frac{ku^3}{z^3}$

4. 143 **5**. 16π **6**. 36π

7. 18 **8**. 1,250 **9**. 66 **10**. 171

11. 256 **12**. 14π **13**. 1,310 **14**. 1,820

15. 920 **16**. 5 **17**. 45 **18**. $E = \frac{ky^3}{\sqrt{z}}$

19. 50

20. In electronics, current is denoted by the letter i.

$i = kV \Rightarrow 336 = k \cdot 21 \Rightarrow k = 16 \Rightarrow i = 16V$,
so when $V = 35$, $i = 16 \cdot 35 = 560$.

21. $E = kmh \Rightarrow 462 = k \cdot 6 \cdot 7 \Rightarrow k = 11 \Rightarrow E = 11mh$
$\Rightarrow E = 11(24)(17) = 4488$

22. $F = \frac{kC}{A}$; $k = 2$; $F = 4$

23. $E = kmv^2$ $E = 350$
[In physics, you'll learn that $k = \frac{1}{2}$.]

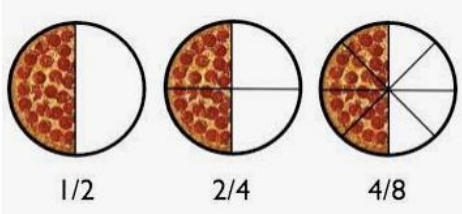
"It is never too late to be what you might have been."

– George Eliot

FRACTIONS AND DECIMALS

First, we need to understand what *equivalent fractions* are. Two fractions are ***equivalent*** if they represent the same number, even if they look different. For example, you know that we can reduce the fraction $\frac{3}{6}$ to $\frac{1}{2}$, meaning that the two fractions boil down to the same thing. The two fractions are therefore *equivalent.*

You can think of it this way: Whether you cut a pizza into 2 equal slices and eat 1 of them, or cut the pizza into 4 equal slices and eat 2 of them, or cut the pizza into 8 equal slices and eat 4 of them — you've eaten the same amount of pizza. Now we describe two ways to produce equivalent fractions.



3 Equivalent Fractions

❑ *THE FUNDAMENTAL PRINCIPLE OF FRACTIONS*

Dividing Top and Bottom by the Same Number

If you take any fraction, and divide both the numerator and denominator by the same non-zero number, the result is an ***equivalent fraction***.

For example, this is the rule we might use to reduce a fraction to lowest terms. If we start with the fraction $\frac{6}{10}$, and then divide the top and bottom by 2, we get the equivalent fraction $\frac{3}{5}$:

$$\frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \mathbf{\frac{3}{5}}$$

We can apply the same principle even if we have no intention of reducing a fraction:

$$\frac{140}{220} = \frac{140 \div 10}{220 \div 10} = \mathbf{\frac{14}{22}}$$

And the principle applies to Algebra as well:

$$\frac{aw}{bw} = \frac{aw \div w}{bw \div w} = \frac{a}{b}$$

The following section shows us how we can reverse this process.

Multiplying Top and Bottom by the Same Number

If you take any fraction, and then multiply both the numerator and denominator by the same non-zero number, the result is an equivalent fraction.

This is the process that allows us to "raise a fraction to higher terms." For example, if we want to convert the fraction $\frac{4}{7}$ into a fraction whose denominator is 21, we would multiply top and bottom by 3:

$$\frac{4}{7} = \frac{4 \times 3}{7 \times 3} = \frac{12}{21}$$

The **3** comes from asking, "What do I have to multiply the **7** by, to get **21**?"

The principle can even be used to convert a fraction containing decimals to one whose denominator is a whole number, a process we'll use with dividing decimals:

$$\frac{1.86745}{23.56} = \frac{1.86745\times100}{23.56\times100} = \frac{186.745}{2{,}356}$$ ← a whole number

In symbols, the Fundamental Principle of Fractions can be written as follows:

> For any non-zero number k, the fractions $\frac{a}{b}$ and $\frac{ak}{bk}$ are ***equivalent***; that is,
>
> $$\frac{a}{b} = \frac{ak}{bk}$$

This principle means that you may either multiply or divide the top and the bottom of a fraction by the same non-zero number and end up with an *equivalent fraction*.

❑ *ADDING AND SUBTRACTING FRACTIONS*

EXAMPLE 1:

A. Add: $2\frac{3}{4} + 5\frac{4}{5}$

The least common denominator (LCD) of 4 and 5 is 20:

$$\begin{aligned} 2\tfrac{3}{4} &= 2\tfrac{15}{20} \\ +\ 5\tfrac{4}{5} &= 5\tfrac{16}{20} \\ \hline 7\tfrac{31}{20} &= \mathbf{8\tfrac{11}{20}} \end{aligned}$$

B. Subtract: $18 - 5\frac{7}{10}$

Let's first line them up vertically:

$$\begin{array}{r} 18 \\ -\ 5\frac{7}{10} \\ \hline \end{array}$$

Since we can't subtract $\frac{7}{10}$ from nothing, we "borrow" one whole from the 18, and call it $\frac{10}{10}$. Then we can subtract:

$$\begin{array}{r} {}^{17}\cancel{18}\,\frac{10}{10} \\ -\ 5\frac{7}{10} \\ \hline \mathbf{12\frac{3}{10}} \end{array}$$

C. Subtract: $12\frac{5}{6} - 7\frac{8}{9}$

Let's first line them up vertically:

$$\begin{array}{r} 12\frac{5}{6} \\ -\ 7\frac{8}{9} \\ \hline \end{array}$$

The LCD (least common denominator) of 6 and 9 is 18:

$$\begin{array}{rcl} 12\frac{5}{6} & = & 12\frac{15}{18} \\ -\ 7\frac{8}{9} & = & 7\frac{16}{18} \\ \hline \end{array}$$

We can't subtract $\frac{16}{18}$ from $\frac{15}{18}$, so we borrow one whole from the 12, and call it $\frac{18}{18}$:

$$\begin{array}{rcl} 12\frac{15}{18} & = & {}^{11}\cancel{12}\,\frac{15}{18} + \frac{18}{18} \\ -\ 7\frac{16}{18} & & \\ \hline \end{array}$$

So we now have a subtraction problem we can finish:

$$\begin{array}{r} 11\frac{33}{18} \\ -\ 7\frac{16}{18} \\ \hline \mathbf{4\frac{17}{18}} \end{array}$$ (the result of adding $\frac{15}{18}$ and $\frac{18}{18}$)

❑ *Multiplying and Dividing Fractions*

A.

$$\frac{2}{3}\cdot\frac{5}{9} = \frac{2\cdot 5}{3\cdot 9} = \frac{10}{27}$$

B.

$$4\frac{1}{7}\times 14 = \frac{29}{7}\times\frac{14}{1} = \frac{29}{7}\times\frac{2\cdot 7}{1} = \frac{29}{\cancel{7}}\times\frac{2\cdot\cancel{7}}{1} = \frac{29\times 2}{1} = 58$$

C. $$\frac{4}{5}\div\frac{5}{7} = \frac{4}{5}\times\frac{7}{5} = \frac{28}{25} = 1\frac{3}{25}$$

D.

$$6\div 7\frac{4}{5} = \frac{6}{1}\div\frac{39}{5} = \frac{6}{1}\times\frac{5}{39} = \frac{2\cdot 3}{1}\times\frac{5}{3\cdot 13} = \frac{2\cdot\cancel{3}}{1}\times\frac{5}{\cancel{3}\cdot 13} = \frac{10}{13}$$

E.

$$\frac{2\frac{2}{3}}{11\frac{4}{9}} = 2\frac{2}{3}\div 11\frac{4}{9} = \frac{8}{3}\div\frac{103}{9} = \frac{8}{3}\cdot\frac{9}{103} = \frac{8}{\cancel{3}}\cdot\frac{\cancel{3}\times 3}{103} = \frac{24}{103}$$

❑ *DECIMAL OPERATIONS*

Adding: The key to adding decimals is threefold: First, realize that a whole number has its decimal point at the right of the number, as in:

$$234 = 234.$$

Second, line up the decimal points, and then add as usual. Third, bring the decimal point straight down into the sum.

$45 + 9.76 + 13.8 \longrightarrow$

```
 45.
  9.76
 13.8
```

or

```
 45.00
  9.76
 13.80
------
 68.56
```

Subtracting: Except for the unpleasant prospect of borrowing, the rules for subtracting decimals are identical to those for adding decimals. Be sure that the larger number is on top.

Multiplying: To multiply two decimals, line up the numbers flush on the right (just like whole-number multiplying -- putting the number with more digits on top) and multiply as usual. Now count up the total number of digits to the right of the decimals being multiplied, and make sure the answer has that many digits to the right of the point.

$2.3 \times 0.009 \longrightarrow$

```
 0.009   ← 3 digits to the right of the point
×  2.3   ← 1 digit to the right of the point
-------
   27
  180        3 + 1 = 4
-------
0.0207   ← 4 digits to the right of the point
```

Dividing: There are two cases to consider:

Case 1: The divisor (the outside number) is a whole number: Bring the decimal point in the dividend (the inside number) straight up, and divide as usual. Rounding is a separate issue and depends upon the original problem and your teacher's instructions.

```
   5.9
23|135.7
  115
  ---
   207
   207
   ---
     0
```

Case 2: The divisor (the outside number) is not a whole number:

Let's look at the division problem:

$$0.19\overline{)1.732}$$

Here's an approach consistent with the ideas presented in this chapter. Rewrite the problem as a fraction (inside number goes on the top);

$$\frac{1.732}{0.09}$$

Multiply top and bottom by whatever power of 10 turns the denominator into a whole number — in this case, that would be 100:

$$\frac{1.732}{0.09} = \frac{1.732\times 100}{0.09\times 100} = \frac{173.2}{9}$$

Now divide as explained in Case 1.

Here's the standard approach: Move the decimal point in the divisor to the right as many places as necessary to convert it to a whole number (that is, all the way to the right of the divisor). Now move the decimal point in the dividend to the right the same

number of places, and bring the decimal point straight up. Finally, divide as usual, and round according to the rules of the problem or the class. Check out the example:

$0.25\overline{)62.5}$	Divide 62.5 by 0.25
$0.25\overline{)62.50}$	Move each decimal point 2 places to the right
$25.\overline{)6250.}$ (quotient: 250.)	The final answer is **250**

❑ *Converting Between Fractions and Decimals*

Fraction → Decimal

To convert a **fraction to a decimal**, divide the numerator by the denominator. [Recall the Top-Inside-Left rule for division: The Top number of the fraction always goes Inside the division house.]

$\frac{13}{20} = 20\overline{)13} = 0.65$

$\frac{65}{50} = 50\overline{)65} = 1.3$

$\frac{2}{3} = 3\overline{)2} = 0.666666\ldots$ forever and ever,

which we might round to 0.67 or 0.667, or to whatever precision the problem requires.

Decimal → Fraction

To convert a **decimal to a fraction**, write the decimal number over the appropriate power of 10 (determined by the place value of the last digit), and reduce the fraction.

$0.9 = \frac{9}{10}$ (the 9 lies in the tenths place)

$0.125 = \frac{125}{1,000} = \frac{1}{8}$ (the 5 lies in the thousandths place)

$2.0005 = 2\frac{5}{10,000} = 2\frac{1}{2,000}$ (the 5 is in the ten-thousandths place)

Solutions

"Not with what we have,
but with what we enjoy,
do we find our true abundance."

Tibetan quote

NUMBER THEORY



The set of **Natural Numbers** is the set

$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$$

[The 3 dots mean 'go on forever and ever.'] Notice that there are infinitely many natural numbers. and although there is NO largest natural number, the smallest natural number is 1.

❑ *MULTIPLES*

If you were asked to "count by 7's, starting at 7, you would come up with the following set of numbers:

7, 14, 21, 28, 35, 42, . . .

We call these numbers the **multiples of 7**. Notice that 7 divides into each of its multiples with a *remainder* of 0. (The 7 "goes in evenly.")

If you count by 12's, you get the **multiples of 12**:

12, 24, 36, 48, 60, . . .

The **even numbers** can be defined as the **multiples of 2**:

2, 4, 6, 8, 10, 12, . . .

What would you get if you consider all the **multiples of 1**? You arrive back at the **natural numbers**:

1, 2, 3, 4, 5, . . .

If you'd like, you could add the number **0** to each list of multiples, since all the numbers 7, 12, 2, and 1 divide into 0 evenly (namely 0 times). But including 0 is not very useful, so we'll ignore 0 as a multiple.

Notes:

- Assuming that we ignore zero being a multiple, there is always a smallest multiple: the number itself. This implies that every multiple of a number is at least as big as the number itself.
- Every natural number has infinitely many multiples.

❑ *COMMON MULTIPLES AND THE LCM*

First Example: **Find the numbers that are multiples of both 3 and 5.**

That is, find the ***common multiples*** of 3 and 5.

Here are the multiples of **3**:

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45 . . .

And here are the multiples of **5**:

5, 10, 15, 20, 25, 30, 35, 40, 45, . . .

I'm now going to display the same two lists, the multiples of 3 and the multiples of 5 — but this time with boxes around the "common" multiples:

Multiples of **3**:

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, . . .

Multiples of **5**:

5, 10, 15, 20, 25, 30, 35, 40, 45, . . .

The *common multiples* are as follows:

15, 30, 45, . . . [These are the *common multiples* of 3 and 5.]

Sometimes the LCM of two numbers is simply their product.

Notice that this list of common multiples, 15, 30, 45, . . ., has a *smallest* number, namely the **15**. Since 15 is a common multiple of 3 and 5, and because it's the *smallest* such common multiple, we say that

15 is the ***Least Common Multiple* (*LCM*)** of 3 and 5.

We can abbreviate this statement by saying: LCM(3, 5) = 15

Second Example: **Find the LCM of 4 and 6**.

Multiples of **4**:

4, 8, 12, 16, 20, 24, 28, 32, 36, . . .

Multiples of **6**:

6, 12, 18, 24, 30, 36, . . .

Sometimes the LCM of two numbers is less than their product.

The **common multiples** have been boxed, and the smallest of these is **12**. Therefore,

LCM(4, 6) = 12

Third Example: **Find the LCM of 10 and 20.**

Multiples of **10**:

10, 20, 30, 40, 50, 60, . . .

Multiples of **20**:

20, 40, 60, 80, . . .

I hope it's clear that 20 is the least common multiple:

Sometimes the LCM of two numbers is equal to the larger of them.

LCM(10, 20) = 20

Fourth Example: **Find the LCM of the three numbers:**

6, 10, and 12.

Multiples of **6**:

6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72 . . .

Multiples of **10**:

10, 20, 30, 40, 50, 60, 70, 80, . . .

Multiples of **12**:

12, 24, 36, 48, 60, 72, 84, . . .

The smallest common multiple we see is 60. So,

LCM(6, 10, 12) = 60

And what it means is this: **60** is the smallest number that 6, 10, and 12 divide into evenly.

Homework

GCF
PRIMES

Solutions

PERCENTS

The Dow (Dow Jones Industrial Average) rose **250 points**. The S&P 500 (Standard & Poor) rose **75 points**. Did the Dow gain more than the S&P? Don't be so sure!





A cancer treatment increases the chance of a heart attack by 90%. Is the cancer treatment worth the risk?

❑ *UNDERSTANDING PERCENTS*

"**Per**" means "divided by" (like *miles per hour* means miles **divided** by hours). **"Cent"** refers to "hundred" (like 100 years in a century, or 100 cents in a dollar). Therefore,

PERCENT means "**divided by 100**"

For any quantity X in the whole universe,

$$X\% = \frac{X}{100}$$

Since percent means "out of **100**" or "divided by **100**," we can see what percents mean from the following examples:

1. $17\% = \frac{17}{\mathbf{100}}$
2. $9\% = \frac{9}{\mathbf{100}}$
3. $32\% = \frac{32}{\mathbf{100}} = \frac{8}{25}$
4. $6\% = \frac{6}{\mathbf{100}} = \frac{3}{50}$
5. $125\% = \frac{125}{\mathbf{100}} = 1\frac{25}{100} = 1\frac{1}{4}$
6. $800\% = \frac{800}{\mathbf{100}} = 8$

❑ *Converting Among Fractions, Decimals, and Percents*



Johnny sank 23 out of 32 free-throw attempts; Janie sank 29 out of 40 free-throw attempts. Who did better? Hard to tell — Janie made more baskets, but needed more attempts to do it. This section will show us how to determine who really did better. First, a couple of warnings:



1. There is to be NO rounding in this section; carry all divisions out until the remainder is 0.
2. All fractions must be reduced to lowest terms.

And notice that **100%** is just a fancy name for the number **1**. That is,

$$100\% = 1$$

Proof: $100\% = \frac{100}{\mathbf{100}} = 1$ ✓

Decimal → Percent

Method: 1) Move the decimal point two places to the right.

2) Attach a percent sign.

Example: **Convert 0.007 to a percent.**

1) Move the decimal point: **0.7**

2) Attach a percent sign:

0.7%

Example: **Convert 23.6 to a percent.**

Moving the decimal point and attaching a percent sign gives the final answer of

2,360%

Percent → Decimal

Method: 1) Move the decimal point two places to the left.

2) Remove the percent sign.

Example: **Convert 3.5% to a decimal.**

1) Move the decimal point: **0.035%**

2) Remove the percent sign:

0.035

Percent → Fraction

This procedure was covered in the Section Understanding Percents.

Fraction → Percent

Let's solve the basketball problem with Johnny and Janie. We begin by expressing each player's stats as fractions.

Johnny: $\frac{23}{32}$ Janie: $\frac{29}{40}$

To convert Johnny's score to a percent, first convert the fraction to a decimal, and then convert that decimal to a percent:

$$\frac{23}{32} = 0.71875 = \mathbf{71.875\%}$$

As for Janie's free-throw percentage:

$$\frac{29}{40} = 0.725 = \mathbf{72.5\%}$$

So we can conclude that Janie truly had the better stats.

Homework

1. Convert 56% to a reduced fraction.
2. Convert 5.9% to a decimal.
3. Convert .0045 to a percent.
4. Convert $\frac{4}{5}$ to a percent.
5. Convert 467% to a decimal.
6. Convert 3.89 to a percent.
7. Convert $\frac{7}{8}$ to a decimal.
8. Convert $\frac{33}{66}$ to a percent.

9. Convert 0.0077 to a percent.

10. Convert 0.025 to a reduced fraction.

❑ *PERCENT INCREASE AND DECREASE*

The price just went up $20! Is that a big deal? Let's see.

Suppose the $60,000 Lexus you ordered went up **$20**; would you cancel your order? Of course not; $20 out of $60,000 is trivial, a drop in the bucket relative to the original price.



But what if the monthly $25 fee for streaming Netflix increased by $**20** to a total of **$45**? Is that a big deal? Absolutely, and that's because $20 out of $25 is a huge increase, *relative to the original price*. Indeed, there's a really good chance that you might just cancel Netflix and get Disney+ instead.

How can the same $20 mean so little in one case and so much in another? Because we have to ask ourselves, "$20 compared to what?" Since a $20 increase alone is meaningless, we must calculate the $20 increase in relation to the starting price. The best way to do this is to calculate the ***percent increase***, and it's found by dividing the increase (the $20) by the original price (the $60,000 car or the $25 Netflix).

By the way, the Lexus increased in price by **0.03%**; the Netflix price increased by **80%**. The same $20 increase — drastically different percent increases!

EXAMPLE 1: **A $12 book has just risen to $15. Find the percent increase in the price of the book.**

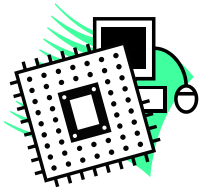


Solution: We're going to compute the increase in the price of the book, and then compare that increase to the original price:

$$\%\ \text{increase} = \frac{\text{increase}}{\text{original}} = \frac{\$3}{\$12} = 0.25 = \mathbf{25\%}$$

Notice that the increase was found by subtracting the old price from the new price.

EXAMPLE 2: **The price of an Intel i5 CPU chip has fallen from $297 to $103. Find the percent decrease.**



Solution: We use the same logic as before. We compare the change in price with the original price. The price change is found by subtracting $103 from $297, giving $194. Since the original price is $297, we're ready to calculate the percent decrease:

$$\%\ \text{decrease} = \frac{\text{decrease}}{\text{original}} = \frac{\$194}{\$297} \approx 0.653 \approx \mathbf{65\%}$$

EXAMPLE 3: **Show that a percent increase can exceed 100%, but a percent decrease can never exceed 100%.**



Solution: Consider that the price of gasoline increased from $0.50/gal in 1970 to $3.15/gal in 2020. We calculate the percent increase in the price of gasoline over this 50-year span like this:

$$\%\ \text{increase} = \frac{\text{increase}}{\text{original}} = \frac{\$2.65}{\$0.50} = 5.3 = \mathbf{530\%}$$

In fact, if something doubles, that's a 100% increase (you'll verify this fact in the homework), so any increase that's more than doubling will exceed a 100% increase. Therefore, there is no limit

on a percent increase; it could be a 6 billion percent increase and still be a valid calculation.

As for the highest possible percent decrease, consider this: A 100% decrease would be the removal of all of it. Since there's nothing left to remove, a decrease can never exceed 100%. For example, suppose that 200 frogs decreased in population by 100%. That means that every single one of them croaked. How can the frog population decrease any more than that? It can't, so a percent decrease is at most 100%.



Summary: In all three examples, we calculated the percent change (whether an increase or a decrease) by creating a fraction with the difference of the quantities on the top and the original quantity on the bottom. When that fraction was converted to a percent, we were done.

$$\text{Percent Change} = \frac{\text{Difference}}{\text{Original}}$$

Homework

Round your answers to the nearest whole percent:

11. Marie's wage increased from \$8/hr to \$12/hr. Find Marie's percent increase in wage.



12. The library increased its collection from 2500 books to 4200 books. Find the percent increase in the number of books.
13. Jim's weight went from 150 lb to 227 lb. Calculate the percent increase in weight.
14. The number of TB cases increased from 32 to 97. Calculate the percent increase in the number of TB cases.
15. The number of tumors increased from 49 to 88. Calculate the percent increase in the number of tumors.
16. The number of K rats increased from 12 to 42. Calculate the percent increase in the number of K rats.
17. The price of the Geo went from \$14,000 to \$19,000. The price of the Lexus went from \$52,000 to \$72,000. In terms of percent increase, which car increased in price more?
18. The price of top sirloin has doubled. Find the percent increase. [Hint: Make up your own numbers that represent a doubling of price.]
19. The number of swimmers has tripled. Find the percent increase.
20. The price of gasoline has quadrupled. Find the percent increase.



21. Thanks to Phoebe's "Smelly Cat" song, the number of cats has quintupled (yuk!). Find the percent increase.
22. a. Explain how it is possible that something could increase more than 100%.
    b. Explain why it is impossible for something to decrease more than 100%.

23. The number of hospitals decreased from 45 to 34. Calculate the percent decrease in the number of hospitals.
24. The number of K rats decreased from 22 to 10. Calculate the percent decrease in the number of K rats.
25. The number of prisons decreased from 52 to 36. Calculate the percent decrease in the number of prisons.
26. The price has dropped from $200 to $6. Find the percent decrease.
27. The number of dogs has fallen from 250 to 195. Find the percent decrease.
28. The number of heart attacks went from 760 before the drug was available to 490 after the drug was available. Calculate the percent decrease in heart attacks.
29. The number of dogs increased from 55 to 93. Calculate the percent increase in the number of dogs.
30. If the number of prisons increased by a factor of 6, make up your own numbers to find the percent increase. [An increase "by a factor of 6" means that the number of prisons is now six times what it was before.]



31. If the number of houses increased by a factor of 7, find the percent increase.
32. If Jesse's salary increased by a factor of 9, find the percent increase.
33. If the population increased by a factor of 12, find the percent increase.
34. The number of motels decreased from 55 to 14. Calculate the percent decrease in the number of motels.
35. The number of military personnel decreased from 10,750 to 8,900. Calculate the percent decrease in the number of military personnel.
36. The number of classrooms decreased from 152 to 132. Calculate the percent decrease in the number of classrooms.
37. While the school-aged population went from 6,750 to 9,023, the number of teachers went from 225 to 190. Calculate the percent increase in students and the percent decrease in teachers.

38. The tuition might double next semester. What percent increase would this be?

39. What percent increase will there be if the tuition triples?

Solutions

1. $\frac{14}{25}$ **2**. 0.059 **3**. 0.45% **4**. 80% **5**. 4.67

6. 389% **7**. 0.875 **8**. 50% **9**. 0.77% **10**. $\frac{1}{40}$

11. 50% **12**. 68% **13**. 51% **14**. 203%

15. 80% **16**. 250%

17. The Lexus increased more (by 38%) than the Geo (by 36%).

18. Let's make up our own numbers and see what happens. Suppose top sirloin was \$2.25/lb and then the price doubled to \$4.50/lb. Calculate the percent increase:

$$\text{percent increase} = \frac{\text{difference}}{\text{original}} = \frac{4.50-2.25}{2.25} = \frac{2.25}{2.25} = 1 = \mathbf{100\%}$$

Now make up your own numbers to see if the same result is obtained.

19. 200% **20**. 300% **21**. 400%

22. a. When something doubles, it increases by 100%. So if something more than doubles, it would increase by more than 100%.
b. Once it's decreased 100%, all of it is gone — it can't decrease any further.

23. 24% **24**. 55% **25**. 31% **26**. 97%

27. 22% **28**. 36% **29**. 69% **30**. 500%

31. 600% **32**. 800% **33**. 1,100% **34**. 75%

35. 17% **36**. 13%

37. Students increased by 34%. Teachers decreased by 16%.

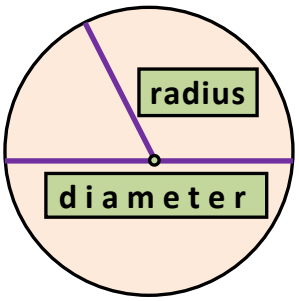
38. 100% **39**. 200%

"It is common sense to take a method and try it.
If it fails, admit it frankly and try another.
But above all, try something."

Franklin D. Roosevelt

CIRCLES – ARITHMETIC

❑ *RADIUS AND DIAMETER*



The distance from the center of a circle to any point on the circle is called its ***radius***. The distance from one point on a circle to another point on the circle, and passing through the center, is called its ***diameter***. It should be clear that the diameter is twice the radius:

$$\boldsymbol{d = 2r}$$

Equivalently, the radius is half the diameter $r = \frac{1}{2}d$, which can also be written as

$$\boldsymbol{r = \frac{d}{2}}$$

The terms *radius* and *diameter* can also refer to the line segments themselves, but their lengths are much more useful.

EXAMPLE 1:

A. **The radius of a circle is 89. Find the diameter.**

Since $d = 2r$, it follows that $d = 2(89) = \mathbf{178}$.

B. **The diameter of a circle is 13. Find the radius.**

The formula $r = \frac{d}{2}$ tells us that $r = \frac{13}{2} = \mathbf{6\frac{1}{2}}$, or **6.5**, if you prefer.

C. **The diameter of a circle is $\frac{3}{5}$. Find the radius.**

$$r = \frac{d}{2} = \frac{\frac{3}{5}}{2} = \frac{3}{5} \div 2 = \frac{3}{5} \div \frac{2}{1} = \frac{3}{5} \times \frac{1}{2} = \mathbf{\frac{3}{10}}$$

Homework

1. For each problem, find the diameter if the radius is given, and find the radius if the diameter is given:

   a. $r = 20$ b. $r = 7.6$ c. $r = \frac{1}{7}$ d. $r = 0.5$

   e. $d = 44$ f. $d = 0.25$ g. $d = 13$ h. $d = \frac{1}{5}$

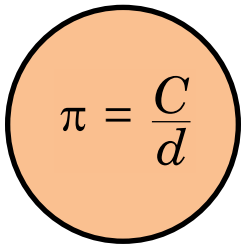
❑ *DEFINITION OF π*

A circle has a perimeter (the distance all the way around) just as squares and rectangles do, and we call this perimeter the ***circumference*** of the circle. The ***area*** of a circle is a measure of the region inside the circle. The formulas for circumference and area will be stated after a discussion of a very, very important number.

Choose any circle at all: tiny, mid-sized, or gigantic. When you divide the circumference by the diameter — no matter the circle — you always end up with the same number, a constant a little bigger than 3. This quotient (the circumference divided by the diameter), which is the same for all circles, is denoted by the Greek letter pi, "π", from the Greek word *perimetros* (though the Greeks themselves did not use the symbol π for this special number).

The modern symbol for pi
— π —
was first used in 1706 by William Jones.

The definition of π:



The circumference of a circle is a little over 3 times its diameter.

The decimal version of π has an infinite number of digits in it (and without a repeating pattern), and therefore any decimal number we write for π will only be an approximation. The calculator gives something like 3.141592654 for π. Two useful approximations of π are 3.14 and $\frac{22}{7}$. Since this class is getting you prepared for an algebra class, we'll stick (for now) with the exact value of π, which, of course, is simply written **π**.

Even though π has been calculated to about 202 trillion digits, according to NASA, 40 digits of π are all that is required to measure the distance around the universe to an accuracy equal to the diameter of a hydrogen atom. That's pretty darned accurate!

See Digits of Pi

❑ *Circumference and Area*

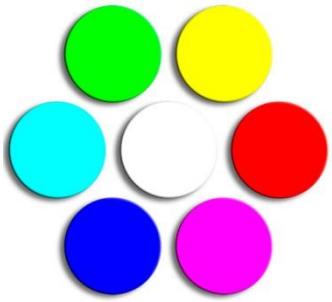
The circumference of a circle can be found by multiplying 2 times π times the radius:

$$C = 2\pi r$$

[See if you can derive this formula yourself, using the definition of π and the fact that $d = 2r$.]

As for area, the formula is difficult to derive, so we simply state it and use it:

$$A = \pi r^2$$



Note #1: Since the Order of Operations specifies that exponents have priority over multiplication, we note that the area formula tells us to first square the r, and then multiply by π. In other words, πr^2 means $\pi \times (r^2)$.

Note #2: Students sometimes mix up the circle formulas for circumference and area. Here's a way to remember which is which. The <u>area</u> of any geometric shape is always in <u>square</u> units, for example, square feet. And which formula, $2\pi r$ or πr^2, has the "square" in it? The πr^2 does, of course. So area is πr^2, and circumference is the one without the square in it, $2\pi r$.

Note #3: A previous teacher may have taught you that the circumference of a circle is given by the formula $C = \pi d$, instead of the formula $C = 2\pi r$ stated above. They are really the same formula, since $d = 2r$, so you can certainly use whichever one you like.

<u>EXAMPLE 2:</u> **The radius of a circle is 17.5. Find the circumference.**

<u>Solution:</u> Using the formula for the circumference of a circle, we proceed as follows:

$C = 2\pi r$ (the circumference formula)

$C = 2\pi(\mathbf{17.5})$ (plug in the given radius for r)

$C = 2(17.5)\pi$ (rearrange the factors)

$C = \boxed{35\pi}$ (multiply 2 by 17.5)

Note that this is the exact answer, because the symbol π is in the final answer. An approximation could easily be found by using 3.14 for π, and then multiplying 35 by 3.14 to get **109.9**.

EXAMPLE 3: **The radius of a circle is 17. Find the area.**

Solution: The relevant formula is $A = \pi r^2$. Notice that only the r is being squared in this formula, since the Order of Operations specifies that exponents have priority over multiplication. We therefore get

$A = \pi r^2$ (the area of a circle formula)

$A = \pi(\mathbf{17})^2$ (plug in the given radius for r)

$A = \pi(289)$ (square the 17, and we're basically done)

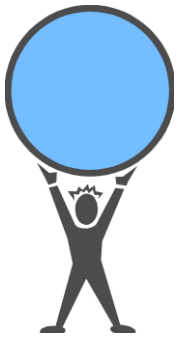
$A = \boxed{289\pi}$ (it looks prettier this way)

Homework

The Circle Formulas:

$$d = 2r \qquad r = \frac{d}{2} \qquad C = 2\pi r \qquad A = \pi r^2$$

2. Find the **circumference** of the circle with the given radius — leave your answers in exact form (that is, leave π in the answer):



a. $r = 8$ b. $r = 33.4$ c. $r = 0.07$ d. $r = 89$

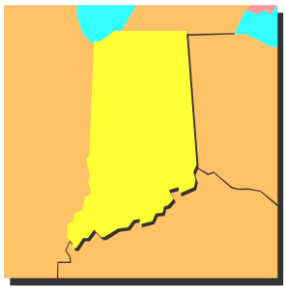
e. $r = 13$ f. $r = 0.5$ g. $r = 100$ h. $r = 77.5$

3. Find the **area** of the circle with the given radius — leave your answers in exact form (that is, leave π in the answer):

a. $r = 9$ b. $r = 100$ c. $r = 0.3$ d. $r = 3.5$

e. $r = 10$ f. $r = 2.5$ g. $r = 1$ h. $r = 0.08$

❑ THE DAY π WAS SAVED



In 1897, legislator Edwin J. Goodman of the Indiana legislature attempted to push through a law that would have indirectly declared the number 3.2 to be the exact value of π. Though Mr. Goodman's bill had unanimously passed the House of Representatives and had passed one reading in the Senate, a visitor, C.A. Waldo, a math professor from Purdue University, intervened and convinced the Senate not to pass the bill.

❑ A BIBLICAL VIEW OF π

From I Kings 7:23 comes the sentence:

> "*Then He made the molten sea, ten cubits from brim to brim, while a line of 30 cubits measured it around.*"

The word "sea" refers to a large container for holding water. "Brim to brim" refers to the diameter of 10 cubits, while the 30 cubits refers to the circumference. Thus, from this quote, we derive the value $\pi = \frac{C}{d} = \frac{30 \text{ cubits}}{10 \text{ cubits}} = \mathbf{3}$, which is quite a good estimate of π, given its great antiquity.

Review Problems

4. The radius of a circle is 2.3. Find the diameter.
5. The diameter of a circle is $\frac{4}{5}$. Find the radius.
6. The radius of a circle is 15. Find the area.
7. The radius of a circle is 12. Find the circumference.
8. Find the diameter of a circle if its radius is $\frac{3}{7}$.
9. Find the radius of a circle if its diameter is 0.28.
10. Find the circumference of a circle whose radius is 2.7.
11. Find the area of a circle whose radius is 8.8.
12. The radius of a circle is 0.8. Find the diameter.
13. The diameter of a circle is $\frac{3}{5}$. Find the radius.
14. The radius of a circle is 16. Find the area.
15. The radius of a circle is 25. Find the circumference.
16. Find the diameter of a circle if its radius is $\frac{3}{11}$.
17. Find the radius of a circle if its diameter is 0.86.
18. Find the circumference of a circle whose radius is 3.5.
19. Find the area of a circle whose radius is 2.3.
20. The radius of a circle is 0.002. Find the diameter.
21. The diameter of a circle is $\frac{9}{11}$. Find the radius.
22. The radius of a circle is 45. Find the area.
23. The radius of a circle is 126. Find the circumference.
24. Find the diameter of a circle if its radius is $\frac{4}{9}$.
25. Find the radius of a circle if its diameter is 0.27.
26. Find the circumference of a circle whose radius is 151.5.
27. Find the area of a circle whose radius is 1,000.

Solutions

1. a. $d = 40$ b. $d = 15.2$ c. $d = \frac{2}{7}$ d. $d = 1$
e. $r = 22$ f. $r = 0.125$ g. $r = 6\frac{1}{2}$ h. $r = \frac{1}{10}$

2. a. 16π b. 66.8π c. 0.14π d. 178π
e. 26π f. π g. 200π h. 155π

3. a. 81π b. $10{,}000\pi$ c. 0.09π d. 12.25π
e. 100π f. 6.25π g. π h. 0.0064π

4. 4.6 **5**. $\frac{2}{5}$ **6**. 225π **7**. 24π

8. $\frac{6}{7}$ **9**. 0.14 **10**. 5.4π **11**. 77.44π

12. 1.6 **13**. $\frac{3}{10}$ **14**. 256π **15**. 50π

16. $\frac{6}{11}$ **17**. 0.43 **18**. 7π **19**. 5.29π

20. 0.004 **21**. $\frac{9}{22}$ **22**. 2025π **23**. 252π

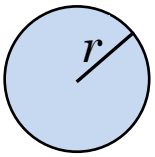
24. $\frac{8}{9}$ **25**. 0.135 **26**. 303π **27**. $1{,}000{,}000\pi$

“The greatest thing you’ll ever learn, is just to love, and be loved in return.”

eden ahbez

CIRCLES – ALGEBRA

❑ *REVIEW OF CIRCLE FORMULAS*



If r is the radius of a **circle**, then
its *diameter* is $\mathbf{2r}$,
its *circumference* is $\mathbf{2\pi r}$,
and its *area* is $\mathbf{\pi r^2}$.

Recall from the Order of Operations that in the formula πr^2, the best way to solve it is to square the r first, and then multiply by π.

You might see the area formula written as

$$A = \pi \times r \times r,$$

in which case you can multiply π by r, and then multiply that result by r.

π is defined as the ratio of the circumference of a circle to its diameter:

$$\pi = \frac{C}{d}$$

Homework

1. The radius of a circle is 10. Find the diameter, the circumference, and the area in exact form (that means leave π as π).
2. The radius of a circle is 1. Find the diameter, the circumference, and the area.
3. The radius of a circle is $\frac{1}{2}$. Find the diameter, the circumference, and the area.

❑ *DERIVING THE CIRCUMFERENCE FORMULA: $C = 2\pi r$*

The first problem we will solve using our algebra skills is to derive the formula $C = 2\pi r$ for the circumference of a circle whose radius is r.

We begin with the **definition** of π, namely that π is the ratio of the circumference of any circle to its diameter:

$$\pi = \frac{C}{d}$$

Multiplying each side of this equation by d gives:

$$\pi[d] = \frac{C}{\cancel{d}}[\cancel{d}]$$

Simplifying gives

$$\pi d = C$$

Now we turn the equation around and also change d into $2r$, since the diameter is twice the radius:

$$C = \pi(2r)$$

Applying the fact that multiplication is an associative operation, we can "shift" the parentheses to associate the π and the 2:

$$C = (\pi 2)r$$

Since multiplication is also a commutative operation, we can switch the π and the 2 to get:

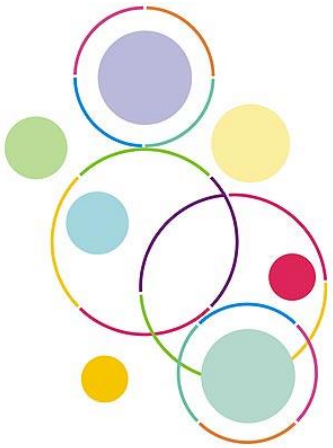
$$C = (2\pi)r$$

Again, since multiplication is an associative operation, the parentheses are redundant (unnecessary) and we can remove them. Our final formula:

$$\boxed{C = 2\pi r}$$

❑ *USING ALGEBRA TO SOLVE CIRCLE PROBLEMS*

EXAMPLE 1: **The diameter of a circle is 12. Find the area.**



Solution: The formula for the area of a circle, $A = \pi r^2$, requires that we have the radius, which was not explicitly given to us in the problem. But the diameter of the circle was given, so we can use this diameter to find the radius, and then use the radius to find the area:

given in the problem

$$r = \frac{d}{2} = \frac{12}{2} = \mathbf{6}$$

Now we know that $r = \mathbf{6}$, so the area is calculated as follows:

$$A = \pi r^2 = \pi \cdot \mathbf{6}^2 = \boxed{36\pi}$$

EXAMPLE 2: **Find the circumference of a circle whose diameter is 9.**

Solution: The process is similar to the previous problem.

$r = \frac{d}{2} = \frac{\mathbf{9}}{\mathbf{2}}$, and so

$$C = 2\pi r = 2\pi\left(\frac{9}{2}\right) = \cancel{2}\pi\left(\frac{9}{\cancel{2}}\right) = \boxed{9\pi}$$

EXAMPLE 3: **The circumference of a circle is 24π. Find the radius.**

Solution: The circumference of a circle is given by $C = 2\pi r$. Also, the circumference is given to be 24π. So if we set the

circumference formula ($2\pi r$) equal to the given circumference (24π), we should be able to solve for r:

$2\pi r = 24\pi$ ($C = 2\pi r$)

$\Rightarrow \quad \frac{2\pi r}{2\pi} = \frac{24\pi}{2\pi}$ (divide each side by 2π)

$\Rightarrow \quad \frac{\cancel{2\pi}\, r}{\cancel{2\pi}} = \frac{\overset{12}{\cancel{24}}\,\cancel{\pi}}{\underset{1}{\cancel{2}}\,\cancel{\pi}}$ (divide out common factors)

$\Rightarrow \quad \boxed{r = 12}$ (and we've isolated the r)

EXAMPLE 4: **Find the radius of a circle if it's known that its area is 121π.**

Solution: We set the area formula (πr^2) equal to the given area (121π), and solving for r will be as easy as π.

$$\pi r^2 = 121\pi \quad \Rightarrow \quad \frac{\cancel{\pi} r^2}{\cancel{\pi}} = \frac{121\cancel{\pi}}{\cancel{\pi}} \quad \Rightarrow \quad r^2 = 121$$

Using a calculator, or better yet your brain, we see that $\boxed{r = 11}$

Note: The equation $r^2 = 121$ actually has two solutions, 11 and –11. But we discard the negative solution since the radius of a circle must be a positive number.

Homework

4. Find the circumference and the area of the circle whose diameter is given:

a. $d = 10$ b. $d = 13$ c. $d = 2$ d. $d = 200$

5. Find the radius of the circle given its circumference or area:

a. $C = 30\pi$ b. $C = 19\pi$ c. $A = 100\pi$ d. $A = \pi$

EXAMPLE 5: **The circumference of a circle is 4.6π. Find the area.**

Solution: First ask yourself, What do I need to find the area? The radius, of course. But was the radius given to us? No, but the circumference was, so we can use the circumference to find the radius, and then use that radius to find the area.

Step 1: Use the circumference to find the radius

The circumference is given to be 4.6π.

$\Rightarrow \quad 2\pi r = 4.6\pi$ (circumference = $2\pi r$)

$\Rightarrow \quad \frac{2\pi r}{2\pi} = \frac{4.6\pi}{2\pi}$ (divide each side by 2π)

$\Rightarrow \quad r = 2.3$ (simplify each side)

Step 2: Use the radius to find the area

$A = \pi r^2 = \pi(2.3)^2 = \boxed{5.29\pi}$



EXAMPLE 6: **The area of a circle is 2.25π. Find the circumference.**

Solution: Reversing the logic in the previous problem, we use the given area to calculate the radius, and then use that radius to find the circumference:

$\pi r^2 = 2.25\pi \Rightarrow \frac{\pi r^2}{\pi} = \frac{2.25\pi}{\pi} \Rightarrow r^2 = 2.25 \Rightarrow r = \mathbf{1.5}$

It thus follows that $C = 2\pi r = 2\pi(1.5) = 2(1.5)\pi = \boxed{3\pi}$

Homework

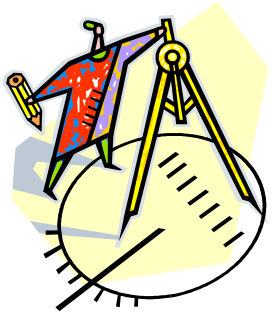
6. Find the area of the circle with the given circumference:

   a. $C = 10\pi$ b. $C = 2\pi$ c. $C = \pi$ d. $C = 5\pi$

7. Find the circumference of the circle with the given area:

   a. $A = 25\pi$ b. $A = \pi$ c. $A = 81\pi$ d. $A = 225\pi$

8. a. Find the circumference of a circle with diameter 2.
   b. Find the area of a circle with diameter 30.
   c. Find the radius of a circle with circumference 38π.
   d. Find the diameter of a circle with circumference 8π.
   e. Find the radius of a circle with area 576π.
   f. Find the diameter of a circle with area 400π.
   g. Find the area of a circle with circumference 2π.
   h. Find the circumference of a circle with area 16π.
   i. Find the circumference of a circle with diameter 28.
   j. Find the area of a circle with diameter 28.
   k. Find the radius of a circle with circumference 34π.
   l. Find the diameter of a circle with circumference 20π.
   m. Find the radius of a circle with area 529π.
   n. Find the diameter of a circle with area 81π.
   o. Find the area of a circle with circumference 18π.
   p. Find the circumference of a circle with area 169π.



Review Problems

9. The circumference of a circle is 20π. Find the area.

10. The area of a circle is 49π. Find the circumference.

11. True/False:

    a. If the diameter of a circle is 12, its area is 144π.

    b. If the diameter of a circle is 20, its circumference is 20π.

    c. If the circumference of a circle is 20π, its area is 100π.

    d. If the area of a circle is 49π, its diameter is 14.

    e. If the radius of a circle is 11, its area is 121.

    f. If the area of a circle is 36π, its circumference is 36π.

Solutions

1. $d = 20;\ C = 20\pi;\ A = 100\pi$

2. $d = 2;\ C = 2\pi;\ A = \pi$

3. $d = 1;\ C = \pi;\ A = \frac{1}{4}\pi$

4. a. $C = 10\pi;\ A = 25\pi$ b. $C = 13\pi;\ A = 42.25\pi$
c. $C = 2\pi;\ A = \pi$ d. $C = 200\pi;\ A = 10{,}000\pi$

5. a. $r = 15$ b. $r = 9.5$ c. $r = 10$ d. $r = 1$

6. a. $A = 25\pi$ b. $A = \pi$ c. $A = 0.25\pi$ d. $A = 6.25\pi$

7. a. $C = 10\pi$ b. $C = 2\pi$ c. $C = 18\pi$ d. $C = 30\pi$

8. a. $C = 2\pi$ b. $A = 225\pi$ c. $r = 19$ d. $d = 8$

e.	$r = 24$	f.	$d = 40$	g.	$A = \pi$	h.	$C = 8\pi$
i.	$C = 28\pi$	j.	$A = 196\pi$	k.	$r = 17$	l.	$d = 20$
m.	$r = 23$	n.	$d = 18$	o.	$A = 81\pi$	p.	$C = 26\pi$

9. 100π **10**. 14π

11. a. F b. T c. T d. T e. F f. F

❑ *To ∞ and Beyond*

A. Find a circle whose area is equal to its circumference (even though the units will be different).

B. Find the radius of a circle whose circumference is 20.

C. Find the radius of a circle whose area is 10.

D. By what factor must the radius of a circle be increased to increase the circumference by a factor of 4?

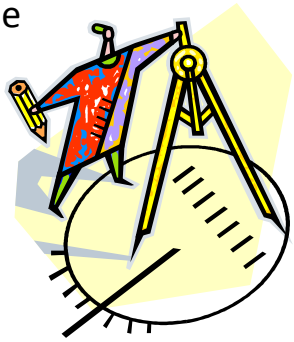
E. By what factor must the radius of a circle be increased to increase the area by a factor of 9?

"The wisest mind has something yet to learn."

George Santayana (1863 - 1952)

THE CIRCLE – CENTER *AT* THE ORIGIN

We're now ready for a new type of graph. In this chapter, we analyze "nature's perfect shape," the circle. Whereas the equation of a line has no squared variables, and a parabola has one squared variable, we will see that the equation of a circle has both variables squared.

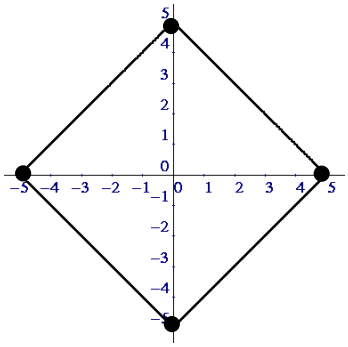


❑ *GRAPHING A CIRCLE*

EXAMPLE 1: **Graph:** $x^2 + y^2 = 25$

Solution: The graph is undoubtedly not a line, since the variables are squared. Let's plot points and see what we get.

Let's first check out the **intercepts**. If we set $x = 0$, the resulting equation is $y^2 = 25$, whose two solutions are $y = \pm 5$. Thus, there are two y-intercepts, **(0, 5)** and **(0, –5)**. You can set y to 0 and calculate the x-intercepts to be **(5, 0)** and **(–5, 0)**. We now have four points on our graph, but it's unclear how to connect them — maybe the graph looks like a diamond? We'll find some other points; for example, if we let $x = 3$, then



Are four points enough to make an accurate graph? Perhaps, perhaps not.

$$3^2 + y^2 = 25 \Rightarrow 9 + y^2 = 25$$

$$\Rightarrow y^2 = 16 \Rightarrow y = \pm 4$$

$$\Rightarrow \textbf{(3, 4)} \text{ and } \textbf{(3, –4)} \text{ are on the graph.}$$

If x is chosen to be -3, then

$$(-3)^2 + y^2 = 25 \Rightarrow 9 + y^2 = 25 \Rightarrow y^2 = 16 \Rightarrow y = \pm 4$$

$\Rightarrow$ **(–3, 4)** and **(–3, –4)** are also on the graph.

Now let $x = 4$:

$$4^2 + y^2 = 25 \Rightarrow 16 + y^2 = 25 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$$

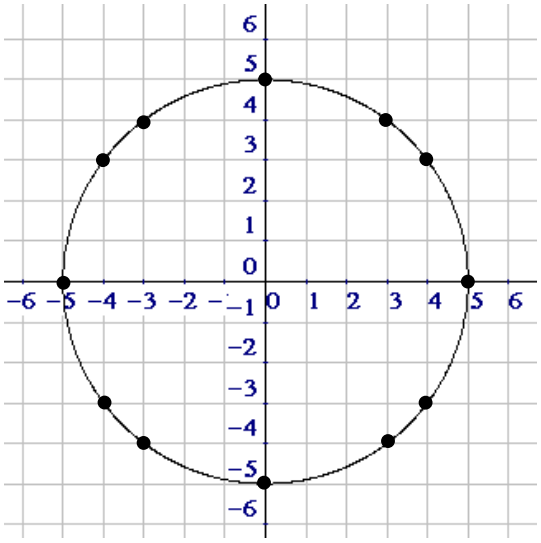
$\Rightarrow$ **(4, 3)** and **(4, –3)** are on the graph.

Our last choice for x will be -4:

$$(-4)^2 + y^2 = 25 \Rightarrow 16 + y^2 = 25 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$$

$\Rightarrow$ **(–4, 3)** and **(–4, –3)** are also on the graph.

This set of 12 points should be enough data to get a decent picture, which looks a lot like a circle:



Summary: The graph of

$$x^2 + y^2 = 25$$

is a circle with its center at the origin, (0, 0), and with a radius of 5.

Homework

1. For each circle, determine the four **intercepts**:

   a. $x^2 + y^2 = 1$ b. $x^2 + y^2 = 2$ c. $x^2 + y^2 = 4$

   d. $x^2 + y^2 = 7$ e. $x^2 + y^2 = 49$ f. $x^2 + y^2 = 60$

2. Given the circle and the *x*-value (i.e., an input), find the *y*-values (i.e., the outputs):

   a. $x^2 + y^2 = 100;\ x = 6$ b. $x^2 + y^2 = 100;\ x = -8$

   c. $x^2 + y^2 = 1;\ x = 1$ d. $x^2 + y^2 = 169;\ x = 5$

   e. $x^2 + y^2 = 169;\ x = -12$ f. $x^2 + y^2 = 169;\ x = -5$

   g. $x^2 + y^2 = 10;\ x = 2$ h. $x^2 + y^2 = 12;\ x = -2$

3. Consider again the circle $x^2 + y^2 = 25$ from Example 1. Its center is the origin and its radius is 5. Now look at the graph and notice that there's no point on the graph with an *x*-value of 8. Prove this fact algebraically (that is, use the equation).

EXAMPLE 2:

A. Consider the circle $x^2 + y^2 = 81$. Using Example 1 as a guide, we infer that its center is the **origin** and its radius is **9**.

B. Now look at the circle $x^2 + y^2 = 13$. The center is **(0, 0)**, and the radius is $\sqrt{\mathbf{13}}$.

C. If the center of a circle is the origin, and if its radius is 15, what is the equation of the circle? It's $\boldsymbol{x^2 + y^2 = 225}$.

D. What is the equation of the circle with center (0, 0) and radius $3\sqrt{7}$?

$x^2 + y^2 = \left(3\sqrt{7}\right)^2$, which is $\boldsymbol{x^2 + y^2 = 63}$.

Note: $\left(3\sqrt{7}\right)^2 = 3^2 \cdot \sqrt{7}^2 = 9 \cdot 7 = \mathbf{63}$

Homework

4. Find the center and radius of each circle:

a. $x^2 + y^2 = 25$ b. $x^2 + y^2 = 144$

c. $x^2 + y^2 = 1$ d. $x^2 + y^2 = 17$

e. $x^2 + y^2 = 27$ f. $x^2 + y^2 = 200$

g. $x^2 + y^2 = 0$ h. $x^2 + y^2 = -9$

i. $x + y = 10$ j. $x^2 + y = 49$

Note: Working with the radius can be confusing. If given the circle equation in standard form, the radius of the circle is found by taking the positive square root of the number to the right of the equal sign. On the other hand, if you know the radius, you square it when you put it into the formula.

5. Find the equation of the circle with center at the origin and the given radius:

a. $r = 10$ b. $r = 25$ c. $r = 1$

d. $r = 16$ e. $r = \sqrt{11}$ f. $r = \sqrt{18}$

g. $r = 4\sqrt{5}$ h. $r = 3\sqrt{7}$

6. The ***unit circle*** is the circle whose *center* is at the origin and whose *radius* is 1.

a. What are the coordinates of the center of the unit circle?
b. What is the radius of the unit circle?
c. What is the equation of the unit circle?
d. What is the area of the unit circle?
e. What is the circumference of the unit circle?

7. Consider the circle $x^2 + y^2 = 7$.

   a. What is the radius?
   b. What is the diameter?
   c. What is the circumference?
   d. What is the area?

8. Sketch a circle (whose center is not necessarily at the origin) that has exactly

   a. 4 intercepts
   b. 2 intercepts
   c. 3 intercepts
   d. 1 intercept
   e. 0 intercepts

9. Consider the equation $x^2 + y^2 = k$. Describe the graph of this equation for each situation:

   a. $k > 0$ b. $k = 0$ c. $k < 0$

Review Problems

10. Find the center and radius of the circle $x^2 + y^2 = 20$.

11. True/False:

   a. Every circle has at least one intercept.
   b. The radius of the circle $x^2 + y^2 = 1$ is 1.
   c. $x^2 + y^2 + 4 = 3$ is a circle.
   d. $x^2 + y^2 = 1$ is called the *unit circle*.
   e. The area of the circle $x^2 + y^2 = 25$ is 25π.

Solutions

1. a. $(\pm 1, 0)$ $(0, \pm 1)$ b. $(\pm\sqrt{2}, 0)$ $(0, \pm\sqrt{2})$

c. $(\pm 2, 0)$ $(0, \pm 2)$ d. $(\pm\sqrt{7}, 0)$ $(0, \pm\sqrt{7})$

e. $(\pm 7, 0)$ $(0, \pm 7)$ f. $(\pm 2\sqrt{15}, 0)$ $(0, \pm 2\sqrt{15})$

2. a. ± 8 b. ± 6 c. 0 d. ± 12

e. ± 5 f. ± 12 g. $\pm\sqrt{6}$ h. $\pm 2\sqrt{2}$

3. Hint: Let $x = 8$ in the circle equation and try to solve for y.

4. a. C(0, 0) $r = 5$ b. C(0, 0) $r = 12$ c. C(0, 0) $r = 1$

d. C(0, 0) $r = \sqrt{17}$ e. C(0, 0) $r = 3\sqrt{3}$ f. C(0, 0) $r = 10\sqrt{2}$

g. It's not a circle; the graph is just the origin.

h. It's not a circle; there are two reasons. First, the radius would be $\sqrt{-9}$, which is not a real number. Second, if two numbers are squared and then added together, there's no way that sum could be negative.

i. It's a line, not a circle.

j. It's not a circle — do you know what it is?

5. a. $x^2 + y^2 = 100$ b. $x^2 + y^2 = 625$ c. $x^2 + y^2 = 1$

d. $x^2 + y^2 = 256$ e. $x^2 + y^2 = 11$ f. $x^2 + y^2 = 18$

g. $x^2 + y^2 = 80$ h. $x^2 + y^2 = 63$

6. a. (0, 0) b. 1 c. $x^2 + y^2 = 1$

d. $A = \pi r^2 = \pi(1^2) = \pi(1) = \boldsymbol{\pi}$ e. $C = 2\pi r = 2\pi(1) = \boldsymbol{2\pi}$

7. a. $r = \sqrt{7}$ b. $d = 2\sqrt{7}$ c. $C = 2\sqrt{7}\,\pi$ d. $A = 7\pi$

8. You're on your own.

9. a. If $k > 0$, the graph is a circle with center at the origin and radius $\sqrt{k}$.

b. If $k = 0$, the graph is just the single point (0, 0); i.e., the origin.

c. If $k < 0$, the graph is empty (there's no graph at all).

10. C(0, 0); $r = 2\sqrt{5}$

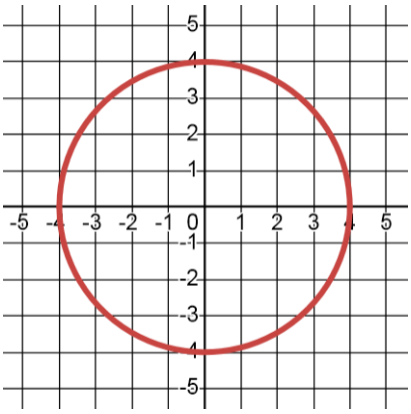
11. a. F b. T c. F d. T e. T

"Only those who dare
to fail greatly
can ever achieve greatly."

– Robert F. Kennedy



THE CIRCLE – CENTER *OFF* THE ORIGIN



You've graphed circles whose centers are at the origin in the Chapter *The Circle – Center at the Origin*. For example, the center of the circle $x^2 + y^2 = 16$ is (0, 0), and its radius is 4. Our goal in this chapter is to allow the center of a circle to be somewhere else in the plane.

What kind of equation would result in such a circle? The example in the next section,

$$(x - 2)^2 + (y + 1)^2 = 25$$

will provide us with one. But before we get there, we need to understand why this equation is likely a circle: If you square (expand) the binomials, note that you'll get an x^2 term, a y^2 term, some x's, some y's, and some numbers. The two squared terms lead us to believe that the graph of the equation is a circle, but that's only a theory.



❑ *USING THE ANNIHILATOR METHOD*

EXAMPLE 1: **Graph:** $(x-2)^2 + (y+1)^2 = 25$

Solution: First, it might be clear (due to the –2 and the 1) that the center of the circle is not the origin. Second, our best guess right now is that the radius is 5 (since 5 is the positive square root of 25). Let's check out these ideas.

It's hard to know what values of x we should choose to find our points to plot, but here's a neat trick to find four useful points.

We'll start with $x = 2$ (this rids us of the first term), and then solve for $\boldsymbol{y}$ in the circle equation $(x-2)^2 + (y+1)^2 = 25$:

$$\begin{aligned} x = \mathbf{2} \quad &\Rightarrow \quad (\mathbf{2}-2)^2 + (y+1)^2 = 25 \\ &\Rightarrow \quad 0^2 + (y+1)^2 = 25 \\ &\Rightarrow \quad (y+1)^2 = 25 \\ &\Rightarrow \quad y+1 = \pm 5 \qquad \text{[Don't forget: 25 has 2 square roots]} \\ &\Rightarrow \quad y = -1 \pm 5 \\ &\Rightarrow \quad y = 4 \text{ or } -6 \end{aligned}$$

Since letting $x = 2$ produced two y-values, we have the two points **(2, 4)** and **(2, –6)** on our circle.

Next, we'll let $y = -1$ (this annihilates the second term), and then solve for $\boldsymbol{x}$ in the circle equation $(x-2)^2 + (y+1)^2 = 25$:

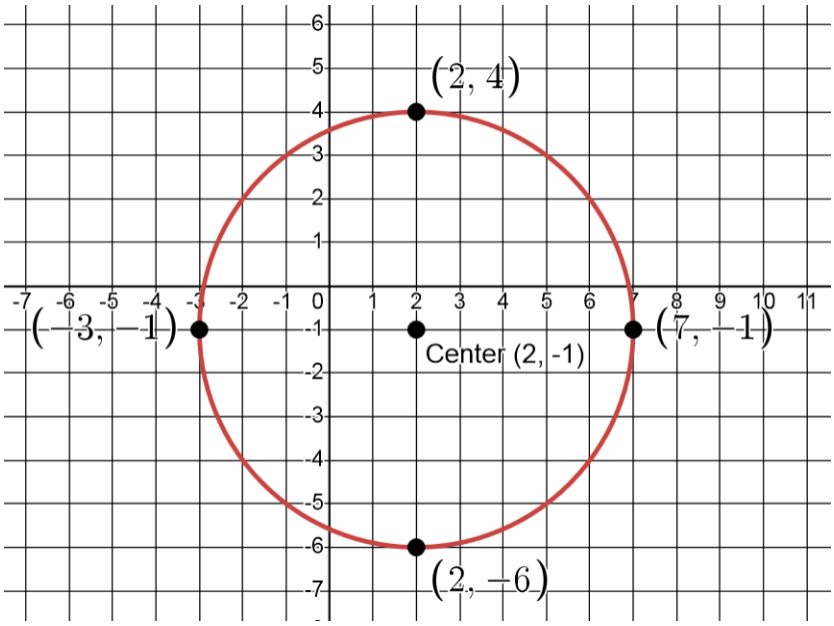
$$\begin{aligned} y = \mathbf{-1} \quad &\Rightarrow \quad (x-2)^2 + (\mathbf{-1}+1)^2 = 25 \\ &\Rightarrow \quad (x-2)^2 + 0^2 = 25 \\ &\Rightarrow \quad (x-2)^2 = 25 \\ &\Rightarrow \quad x-2 = \pm 5 \\ &\Rightarrow \quad x = 2 \pm 5 \\ &\Rightarrow \quad x = 7 \text{ or } -3 \end{aligned}$$

By letting $y = -1$, we determined that two more points on the circle are **(7, –1)** and **(–3, –1)**.

What have we done here? By cleverly choosing values of x and y (do you see how we chose them?), we have discovered these four points on our circle:

(2, 4) (2, –6) (7, –1) (–3, –1)

Let's plot these four points, connect them to make a circle, and then figure out the circle's center and radius.



Can you see that the center of the circle is the point (2, –1) and that the radius is 5? Let's summarize:

The circle $(x - 2)^2 + (y + 1)^2 = 25$ has its center at (2, –1) and has a radius of 5.

Homework

1. Find the **center** and **radius** of each circle by mimicking Example 1 (the *annihilator method*):

a. $x^2 + y^2 = 64$

b. $x^2 + y^2 = 24$

c. $x^2 + (y - 7)^2 = 4$

d. $(x - 3)^2 + y^2 = 5$

e. $(x + 1)^2 + (y + 8)^2 = 60$

f. $(x - 2)^2 + (y - 3)^2 = 99$

❑ A Quicker Approach

Look at the problem and the solution in Example 1. Do you see any connections there? The x-coordinate of the center (the 2) is the *opposite* of the number following the x in the circle equation. Also, the y-coordinate of the center (the –1) is the *opposite* of the number following the y in the circle equation. And last, the circle's radius, 5, is the positive square root of the 25 on the right side of the circle equation (just as we expected).

The following table shows the relationships between the equations of some circles and the circles' centers and radii. Study it carefully.

Equation of Circle	Center	Radius
$(x - 7)^2 + (y - 4)^2 = 121$	(7, 4)	11
$(x + 3)^2 + (y + 5)^2 = 49$	(–3, –5)	7
$(x - 1)^2 + (y + 12)^2 = 21$	(1, –12)	$\sqrt{21}$
$(x + 11)^2 + (y - 9)^2 = 93$	(–11, 9)	$\sqrt{93}$

Equation of Circle	Center	Radius
$x^2 + (y - 8)^2 = 1$	(0, 8)	1
$(x + 13)^2 + y^2 = 50$	(–13, 0)	$5\sqrt{2}$
$x^2 + y^2 = 108$	(0, 0)	$6\sqrt{3}$

Homework

2. Find the **center** and **radius** of each circle using the "shortcut" described in the chart above:

a. $(x + 5)^2 + (y - 3)^2 = 144$

b. $(x - 1)^2 + (y + 11)^2 = 48$

c. $(x + 7)^2 + (y + \pi)^2 = 50$

d. $(x - \sqrt{2})^2 + (y - 2.3)^2 = 1$

e. $(x + 3) + (y - 1) = 9$

f. $(x - 3)^2 + (y + 2) = 7$

3. Find the **equation** of the circle with the given *center* and *radius*:

a. C(0, 0) $r = 7$

b. C(0, 0) $r = \sqrt{10}$

c. C(3, 0) $r = 3$

d. C(0, 4) $r = 1$

e. C(0, –2) $r = \sqrt{3}$

f. C(–12, 0) $r = 12$

g. C(2, 7) $r = 10$

h. C(–1, –3) $r = 2\sqrt{3}$

i. C(3, –4) $r = 3\sqrt{5}$

j. C(–2, 9) $r = 5\sqrt{7}$

k. C(1, 2) $r = 0$

l. C(–3, 5) $r = -9$

❑ A New Twist for the Circle

Now for a tricky circle question:

Find the center and radius of the circle $x^2 + y^2 + 8x - 6y + 9 = 0$.

This circle is not in the standard form we've been using to extract the center and radius. So, how in the heck do we convert this circle equation containing no parentheses into the proper form with the two sets of parentheses? Any ideas before you read on?

EXAMPLE 2: **Graph the circle $x^2 + y^2 + 8x - 6y + 9 = 0$.**

Solution: Creating the graph will depend on finding the circle's center and radius. To calculate these, we need to convert the given equation of the circle into standard form.

Remember the "magic number" we used to solve quadratic equations by completing the square? We use the same trick here, except we will complete the square in both variables. Cool . . . we get to calculate two magic numbers!

Start with the given equation of the circle:

$$x^2 + y^2 + 8x - 6y + 9 = 0$$

Rearrange the terms, putting the x-terms next to each other and the y-terms next to each other:

$$x^2 + 8x + y^2 - 6y + 9 = 0$$

Take the constant 9 to the other side of the equation:

$$x^2 + 8x + y^2 - 6y = -9$$

Now calculate the two "magic numbers":

Half of 8 is 4, and $4^2 = \mathbf{16}$. This is the magic number for x.

Half of –6 is –3, and $(-3)^2 = \mathbf{9}$. This is the magic number for y.

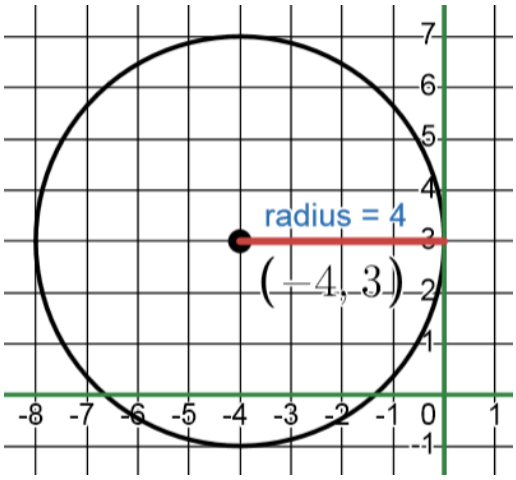
Now we add the magic numbers to both sides of the equation so that perfect square trinomials will be formed on the left side:

$$\underbrace{x^2 + 8x + \boxed{16}}_{\text{factorable}} + \underbrace{y^2 - 6y + \boxed{9}}_{\text{factorable}} = -9 + \boxed{16} + \boxed{9}$$

Now factor the first three terms, then factor the next three terms, and then do the arithmetic on the right side of the equation:

$$(x + 4)^2 + (y - 3)^2 = 16$$

We made it! Now that the circle equation is in standard form, we can read the center and radius directly from the equation. The center is (–4, 3) and the radius is 4 (the positive square root of 16). This info is just what we need to graph our circle:



Homework

4. Find the **center** and **radius** of each circle:

   a. $x^2 + y^2 + 8x - 10y - 40 = 0$

   b. $x^2 + y^2 - 16x + 8y + 31 = 0$

   c. $x^2 + y^2 + 12x + 14y + 49 = 0$

d. $x^2 + y^2 + 14x + 24 = 0$

e. $x^2 + y^2 - 6y - 216 = 0$

f. $x^2 + y^2 + 8x + 6y + 10 = 0$

g. $x^2 + y^2 - 2x - 4y - 3 = 0$

h. $x^2 + y^2 + 4x - 10y + 12 = 0$

i. $x^2 + y^2 - 20x + 6y + 101 = 0$

j. $x^2 + y^2 + 2x - 4y + 25 = 0$

Note: If you come across a circle equation like

$3x^2 + 3y^2 + 24x - 15y + 6 = 0$

just divide each side of the equation by 3 (meaning divide every term by 3), and then proceed as usual.

Review Problems

5. Describe the graph of each equation:

   a. $x^2 + y^2 = 1{,}000{,}000$ b. $x^2 + y^2 = 1$

   c. $x^2 + y^2 = 0$ d. $x^2 + y^2 = -9$

6. Consider the equation $(x - h)^2 + (y - k)^2 = C$. Describe the graph of this equation

   a. if $C > 0$ b. if $C = 0$ c. if $C < 0$

7. Find the center and radius of the circle $x^2 + y^2 = 20$.

8. Find the center and radius of the circle $x^2 + y^2 - 10x + 2y + 3 = 0$.

9. Matching:

   ____ $y = 3$ A. parabola

   ____ $2x^2 - 3y = 10$ B. circle

   ____ $x^2 + (y - 1)^2 = 2$ C. horizontal line

   ____ $x - y + 10 = 0$ D. non-horizontal line

10. True/False:

a. Every circle has at least one intercept.

b. The radius of the circle $x^2 + y^2 = 1$ is 1.

c. The graph of $y = x^2 - 9x + 17$ is a parabola.

d. $x^2 + y^2 + 4 = 3$ is a circle.

e. $x^2 + y^2 = 1$ is called the *unit circle*.

f. The center of the circle $(x - 2)^2 + (y - 5)^2 = 10$ is $(-2, -5)$.

g. The radius of the circle $x^2 + y^2 + 8x - 6y + 9 = 0$ is 4.

h. The graph of $10x^2 + 10y^2 = 37$ is a circle.

i. The graph of $10x^2 - 10y^2 = 37$ is a circle.

j. The graph of $10x^2 + 9y^2 = 37$ is a circle.

k. The area of the circle $x^2 + y^2 = 25$ is 25π.

Solutions

1. a. C(0, 0) $r = 8$ b. C(0, 0) $r = 2\sqrt{6}$

c. C(0, 7) $r = 2$ d. C(3, 0) $r = \sqrt{5}$

e. C(–1, –8) $r = 2\sqrt{15}$ f. C(2, 3) $r = 3\sqrt{11}$

2. a. C(–5, 3) $r = 12$ b. C(1, –11) $r = 4\sqrt{3}$

c. C(–7, $-\pi$) $r = 5\sqrt{2}$ d. C($\sqrt{2}$, 2.3) $r = 1$

e. Trick: It's a line f. Trick: It's a parabola

3. a. $x^2 + y^2 = 49$ b. $x^2 + y^2 = 10$

c. $(x - 3)^2 + y^2 = 9$ d. $x^2 + (y - 4)^2 = 1$

e. $x^2 + (y + 2)^2 = 3$ f. $(x + 12)^2 + y^2 = 144$

g. $(x - 2)^2 + (y - 7)^2 = 100$ h. $(x + 1)^2 + (y + 3)^2 = 12$

i. $(x-3)^2 + (y+4)^2 = 45$ j. $(x+2)^2 + (y-9)^2 = 175$

k. Not a circle; just the point (1, 2)

l. Not a circle; in fact, no graph at all

4. a. C(–4, 5) $r = 9$ b. C(8, –4) $r = 7$

c. C(–6, –7) $r = 6$ d. C(–7, 0) $r = 5$

e. C(0, 3) $r = 15$ f. C(–4, –3) $r = \sqrt{15}$

g. C(1, 2) $r = 2\sqrt{2}$ h. C(–2, 5) $r = \sqrt{17}$

i. C(10, –3) $r = 2\sqrt{2}$ j. Not a circle

5. a. A circle with center at the origin and a radius of 1,000.

b. A circle with center at the origin and a radius of 1.

c. The point (0, 0), and that's it.

d. Since the sum of two squares is <u>never</u> negative, there is NO graph.

6. a. If $C > 0$, the graph is a circle with center at the origin and radius $\sqrt{C}$.

b. If $C = 0$, the graph is just the single point (h, k); i.e., the "center" of a phantom circle.

c. If $C < 0$, the graph is empty (there's no graph at all).

7. C(0, 0); $r = 2\sqrt{5}$ **8.** C(5, –1); $r = \sqrt{23}$

9. C, A, B, D

10. a. F b. T c. T d. F e. T f. F

g. T h. T i. F j. F k. T

"The educated differ from the uneducated as much as the living from the dead."

Αριστοτλε (Aristotle)

ADVANCED CIRCLE PROBLEMS

Homework

1. Find the equation of the circle with center at the origin and the given **diameter**:

   a. $d = 10$ b. $d = 6\sqrt{5}$ c. $d = 3\sqrt{2}$
   d. $d = 7$ e. $d = 5\sqrt{3}$ f. $d = 0$

2. The ***unit circle*** is the circle whose center is at the origin and whose radius is 1.

   a. Find the equation of the unit circle.
   b. What is the area of the unit circle? [$A = \pi r^2$]
   c. What is the circumference of the unit circle? [$C = 2\pi r$]

3. a. Find two horizontal tangent lines to the circle $x^2 + y^2 = 16$.
   b. Find two vertical tangent lines to the circle $x^2 + y^2 = 49$.

4. Find the **midpoint** of the segment connecting the given pair of points:

   a. (2, 7) and (6, 13) b. (–8, 5) and (0, 11)
   c. (0, 0) and (–22, 9) d. (3, –4) and (–8, 9)

EXAMPLE 1: **Find the equation of the circle whose center is (9, –12) and whose diameter is $8\sqrt{5}$.**

Solution: We're given the diameter instead of the radius, but there's no need to worry. Since the radius is always half the diameter, we can calculate the radius:

$$r = \frac{d}{2} = \frac{8\sqrt{5}}{2} = \frac{\cancel{2}\cdot 4\sqrt{5}}{\cancel{2}} = 4\sqrt{5}$$

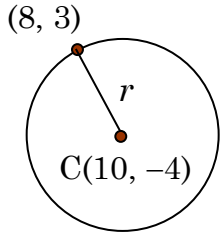
When we know the radius, recall that it has to be squared before we can put it into the formula: $\left(4\sqrt{5}\right)^2 = 4^2\cdot\sqrt{5}^{\,2} = 16\cdot 5 = 80$.

Using the center given in the problem, the equation of the circle becomes

$$\boxed{(x-9)^2+(y+12)^2 = 80}$$

EXAMPLE 2: **Find the equation of the circle given that its center is at the point (10, –4) and that it passes through the point (8, 3).**

Solution: We're given the center, so that's no problem. However, we don't have the radius — we don't even have the diameter. Can we calculate the radius from the picture at the right?



We know that the point (8, 3) is on the circle, and remember that the radius of a circle is the distance from the center of the circle to any point on the circle. So, we can figure out the radius merely by calculating the distance from the center (10, –4) to the point (8, 3). The Distance Formula rescues us again:

$$r = \sqrt{(\Delta x)^2+(\Delta y)^2}$$

$$\Rightarrow\quad r = \sqrt{(10-8)^2+(-4-3)^2}$$

$$\Rightarrow\quad r = \sqrt{2^2+(-7)^2} = \sqrt{4+49} = \sqrt{53}$$

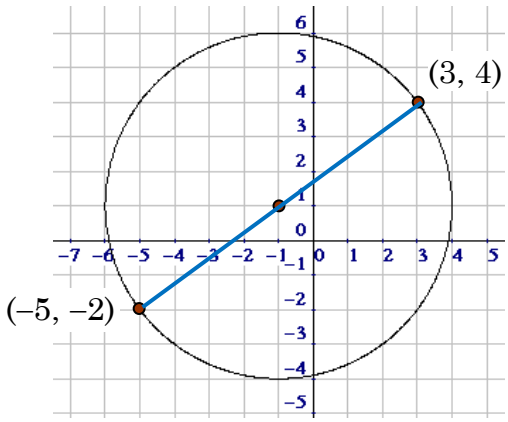
It's time to recap what we have so far: First, the center (10, –4) was given to us; second, we have just calculated the radius to be $\sqrt{53}$. Placing these values into the standard form of a circle gives

$(x-10)^2+(y+4)^2 = \left(\sqrt{53}\right)^2$; the final answer is

$$\boxed{(x-10)^2+(y+4)^2 = 53}$$

EXAMPLE 3: **Find the equation of the circle given that the endpoints of a diameter are the points (–5, –2) and (3, 4).**

Solution: As you know by now, the definition of a circle requires that we know the center and the radius. This problem gives us neither of these. Let's draw a picture and see what we can see.



Look at the diameter with endpoints (–5, –2) and (3, 4). Is it clear that the center must be halfway between the endpoints? What have you learned that will give us the point that is midway between two other points? The Midpoint Formula is just perfect. So the center must be

$$\left(\frac{-5+3}{2}, \frac{-2+4}{2}\right) = \left(\frac{-2}{2}, \frac{2}{2}\right) = (-1, 1)$$

Now look back at the picture again. As in the previous example, the radius of the circle is the distance between the center (–1, 1) and either one of the diameter endpoints. Let's use the endpoint (3, 4). Then,

$$r = \sqrt{[3-(-1)]^2+(4-1)^2} = \sqrt{(3+1)^2+(4-1)^2}$$
$$= \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5$$

Finally, with a center of (–1, 1) and a radius of 5, the equation of the circle must be

$$(x+1)^2+(y-1)^2 = 25$$

Homework

5. Find the equation of the circle with the given center and diameter:

a. C(2, 3) $d = 12$ b. C(–4, 1) $d = 3$

c. C(–1, –5) $d = 2\sqrt{7}$ d. C(7, –3) $d = 5\sqrt{2}$

6. In each problem below, the center C of a circle is given, and a point P on the circle is given. Find the equation of the circle.

a. C(2, 5) P(3, 7) b. C(–1, 4) P(–2, 5)

c. C(0, –3) P(1, –1) d. C(7, 0) P(7, –5)

e. C(–3, 6) P(–2, –3) f. C(7, –2) P(–2, –2)

7. Each pair of points below represents the endpoints of a diameter of a given circle. Find the equation of the circle.

a. (2, 5) (4, –1) b. (–3, –2) (–1, 8)

c. (5, 0) (7, –4) d. (0, –3) (0, –7)

e. (2, –1) (6, –3) f. (–3, 8) (–1, –2)

❑ *SEMICIRCLES*

EXAMPLE 4: **Find the equation of the top-half of the circle $x^2 + y^2 = 10$.**

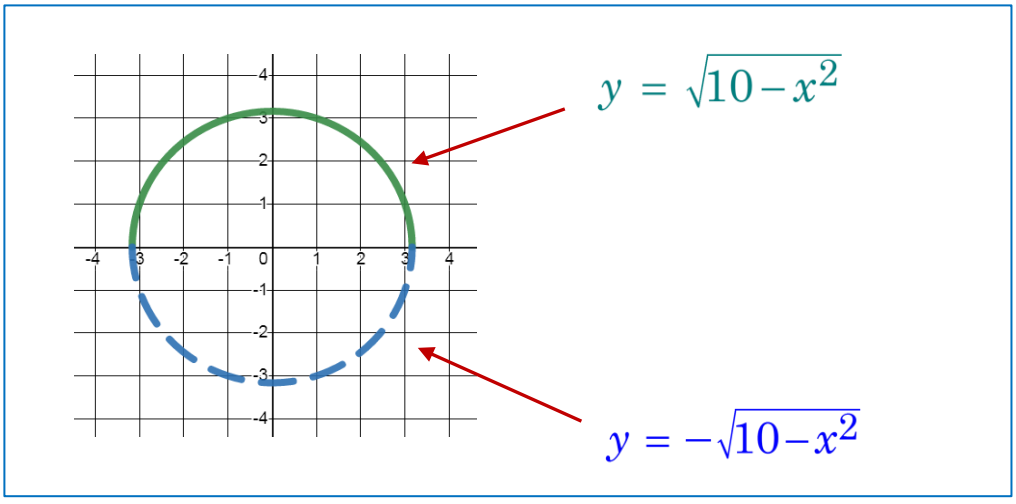
Solution: First, we notice that the given circle has its center at the origin and has a radius of $\sqrt{10}$. If you picture the graph of this circle, you will see that the top half of the circle can be described as that part of the circle which lies in Quadrants I and II (touching the x-axis). Equivalently, this is where the y-coordinate of any point on the circle is never negative, which can be written $y \geq 0$.

Second, we solve the circle equation for y:

$$\begin{aligned} & x^2 + y^2 = 10 \\ \Rightarrow \quad & y^2 = 10 - x^2 \\ \Rightarrow \quad & y = \pm\sqrt{10 - x^2} \end{aligned}$$

Since we require that the y-value never be negative, we will take a "subset" to ensure that $y \geq 0$. To do this, simply take the non-negative square root — that is, remove the ± sign.

$$\boxed{y = \sqrt{10 - x^2}}$$



Homework

8. a. Graph $y = \sqrt{25 - x^2}$ b. Graph $y = -\sqrt{25 - x^2}$

   c. T/F: The graphs are disjoint (have no points in common).

9. Describe the graph of $y = \sqrt{20 - x^2}$.

10. Describe the graph of $y = -\sqrt{84 - x^2}$.

11. Find the equation of the bottom half of the circle $x^2 + y^2 = 100$.

12. Find the equation of the top half of the circle $x^2 + y^2 = 30$.

Review Problems

13. Find the equation of the circle given that the endpoints of a diameter are (2, –8) and (8, 0).

14. Find two horizontal tangent lines to the circle in the preceding problem.

15. Find the equation of the top half of the circle with center at the origin and radius 5.

16. Find the equation of the bottom half of the circle with center at the origin and radius 12.

17. Find the center and radius of the circle $x^2 + y^2 - 10x + 2y + 3 = 0$.

18. Find the equation of the circle whose center is (–1, –3) and whose diameter is $12\sqrt{11}$.

19. Assume that the center of a circle is (9, –3), and also assume that the point (7, –8) is on the circle. Find the equation of the circle.

20. In terms of exponents and coefficients, explain why each of the following is <u>not</u> a circle.

    a. $2x + 3y = 1$ b. $y = x^2$ c. $y = x^3$

    d. $x^2 + 3y^2 = 4$ e. $x^2 - y^2 = 9$ f. $y = \sqrt{x}$

21. True/False:

    a. Every circle has at least one intercept.
    b. The radius of the circle $x^2 + y^2 = 1$ is 1.
    c. A semicircle can be a function.
    d. A semicircle must be a function.
    e. The derivation of the circle formula is based upon the Distance Formula.
    f. $x^2 + y^2 + 4 = 0$ is a circle.
    g. $x^2 + y^2 = 1$ is called the unit circle.
    h. The center of the circle $(x - 2)^2 + (y - 5)^2 = 10$ is (–2, –5).
    i. The radius of the circle $x^2 + y^2 + 8x - 6y + 9 = 0$ is 4.
    j. The radius of a circle whose diameter is $25\sqrt{2}$ is $5\sqrt{2}$.
    k. The graph of $y = \sqrt{19 - x^2}$ is the top half of a circle.
    l. The radius of the above circle is 19.
    m. The graph of $10x^2 + 10y^2 = 37$ is a circle.
    n. The graph of $10x^2 - 10y^2 = 37$ is a circle.
    o. The graph of $10x^2 + 9y^2 = 37$ is a circle.

p. The area of the circle $x^2 + y^2 = 25$ is 25π.

Solutions

1. a. $x^2 + y^2 = 25$ b. $x^2 + y^2 = 45$ c. $x^2 + y^2 = \frac{9}{2}$

d. $x^2 + y^2 = \frac{49}{4}$ e. $x^2 + y^2 = \frac{75}{4}$ f. Not a circle

2. a. $x^2 + y^2 = 1$ b. $A = \pi r^2 = \pi(1^2) = \pi(1) = \pi$

c. $C = 2\pi r = 2\pi(1) = 2\pi$

3. a. $y = 4$ and $y = -4$ b. $x = 7$ and $x = -7$

4. a. (4, 10) b. (–4, 8) c. (–11, 9/2) d. (–5/2, 5/2)

5. a. $(x - 2)^2 + (y - 3)^2 = 36$ b. $(x + 4)^2 + (y - 1)^2 = \frac{9}{4}$

c. $(x + 1)^2 + (y + 5)^2 = 7$ d. $(x - 7)^2 + (y + 3)^2 = \frac{25}{2}$

6. a. $(x - 2)^2 + (y - 5)^2 = 5$ b. $(x + 1)^2 + (y - 4)^2 = 2$

c. $x^2 + (y + 3)^2 = 5$ d. $(x - 7)^2 + y^2 = 25$

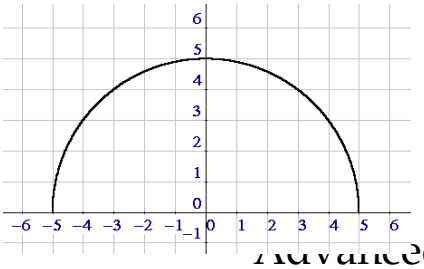
e. $(x + 3)^2 + (y - 6)^2 = 82$ f. $(x - 7)^2 + (y + 2)^2 = 81$

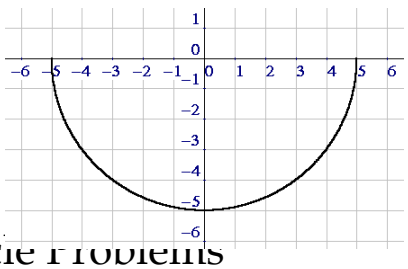
7. a. $(x - 3)^2 + (y - 2)^2 = 10$ b. $(x + 2)^2 + (y - 3)^2 = 26$

c. $(x - 6)^2 + (y + 2)^2 = 5$ d. $x^2 + (y + 5)^2 = 4$

e. $(x - 4)^2 + (y + 2)^2 = 5$ f. $(x + 2)^2 + (y - 3)^2 = 26$

8. a. b.





c. False; they intersect at the points (5, 0) and (–5, 0).

9. The top-half of a circle with center at the origin and radius $2\sqrt{5}$.

10. The bottom-half of a circle with center at the origin and radius $2\sqrt{21}$.

11. $y = -\sqrt{100 - x^2}$

12. $y = \sqrt{30 - x^2}$

13. $(x - 5)^2 + (y + 4)^2 = 25$

14. $y = 1$ and $y = -9$

15. $y = \sqrt{25 - x^2}$

16. $y = -\sqrt{144 - x^2}$

17. C(5, –1); $r = \sqrt{23}$

18. $(x + 1)^2 + (y + 3)^2 = 396$

19. $(x - 9)^2 + (y + 3)^2 = 29$

20.

a. $2x + 3y = 1$: At the very least, a circle has both variables squared, which the given equation certainly does not have. In fact, the given equation is a line.

b. $y = x^2$: Again, at the very least both variables need to be squared in order to be a circle. Indeed, the equation is a parabola.

c. $y = x^3$: This is a cubic function, whose graph was discussed in Chapter 20.

d. $x^2 + 3y^2 = 4$: This equation does have both variables squared, but the coefficients of the squared terms (the 1 and the 3) are <u>not</u> the same, so there's no way we could put the equation into the standard form of the circle: $(x - h)^2 + (y - k)^2 = r^2$. The next chapter will show you what the given equation actually looks like.

e. $x^2 - y^2 = 9$: This equation strongly resembles the equation of a circle, but the minus sign between the squared terms kills the deal. Remember that the equation of a circle ultimately comes from the Distance Formula, which has a plus sign between the squared parts. You might see this equation later in your studies.

f. $y = \sqrt{x}$: Certainly, the exponents aren't right for this to be a circle, since the y has an exponent of 1 and the x has an exponent of 1/2. In fact, the graph of this function is the top half of the parabola opening to the right.

21. a. F b. T c. T d. F e. T f. F g. T h. F
i. T j. F k. T l. F m. T n. F o. F p. T

"Education is the vaccine for violence."

—Edward James Olmos

CONIC SECTIONS – THE CIRCLE

Why this chapter? Because in previous chapters on the circle, we used patterns to determine the equation of a circle. This chapter will prove beyond a doubt that our circle equation is always correct.

We'll need the **Distance Formula** in the plane for this chapter:

> If (x_1, y_1) and (x_2, y_2) are points in the plane, then the distance, ***d***, between them is given by the formula:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$\left(\text{which equals } \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right)$$



Patterns don't always pan out. Consider:

3, 5, 7, . . .

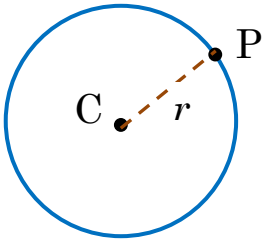
Is the next number **9**, because 3, 5, and 7 are consecutive *odd* numbers?

Or is the next number **11**, because they're consecutive *prime* numbers?

Who's to tell?

❑ *DEFINITION OF THE CIRCLE*

Now for the official definition of the circle. Let C be any point in the plane, and let r be any positive real number. Then a ***circle*** is the set of points in the plane satisfying the following condition: *A point is on the circle if its distance to the point* C *is* r. The point C is called the **center** of the circle and the number r is called its **radius**.



If P is a point on the circle, then its distance to the center C must be r.

Equivalently:

A **circle** is the set of all points in the plane which are *equidistant* from its center.

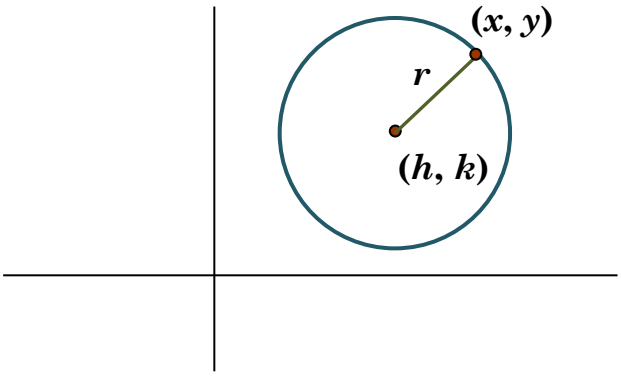
Homework

1. Why do we assume that r is a positive real number?

2. Let P be a point on a circle with center C, and assume that the distance between P and C is 18. Now let Q be another point on the circle. Find the distance between Q and C.

❑ *The Equation of the Circle*

We label the center of the circle (h, k). And we use the generic point (x, y) to represent <u>all</u> the points on the circle.



r is the distance between the points (x, y) and (h, k).

We note that the **distance** between the point (x, y) on the circle and the center (h, k) must be $\boldsymbol{r}$, the radius of the circle. We can thus write, using the Distance Formula:

$$\sqrt{(x-h)^2+(y-k)^2} = r$$

Squaring both sides of this equation produces our circle equation:

The equation of the circle with center $\boldsymbol{(h, k)}$ and radius $\boldsymbol{r}$ is

$$(x-h)^2+(y-k)^2 = r^2$$

The Standard Form of a Circle

EXAMPLE 1: **Use the standard form of a circle to find the center and radius of the circle**

$$\boldsymbol{(x-2)^2+(y+7)^2 = 10}$$

Solution: Transform the given equation into the following form, designed to match the standard form:

$$(x-\boxed{2})^2+\left(y-(\boxed{-7})\right)^2 = \left(\boxed{\sqrt{10}}\right)^2$$

Standard Form: $(x-\boxed{h})^2+(y-\boxed{k})^2 = \boxed{r}^2$

We can deduce: $h = 2, \quad k = -7, \quad r = \sqrt{10}$

And we have our center and radius:

$$\text{C}(2, -7) \quad r = \sqrt{10}$$

EXAMPLE 2: **Find the equation of the circle in standard form if the center of the circle is (9, –4) and the radius is 5.**

Solution: From the given center and radius, we get the values

$$h = -9, \quad k = 4, \quad r = 5$$

So, if we take the standard form of a circle:

$$(x - \boxed{h})^2 + (y - \boxed{k})^2 = \boxed{r}^2$$

and then place the three values into their appropriate positions, we get

$$\left(x - (\boxed{-9})\right)^2 + (y - \boxed{4})^2 = \boxed{5}^2$$

Simplify, and we're done:

$$\boxed{(x+9)^2 + (y-4)^2 = 25}$$

Homework

3. Use the standard form of a circle to find the center and radius of each circle:

a. $x^2 + y^2 = 64$

b. $x^2 + y^2 = 24$

c. $x^2 + (y - 7)^2 = 4$

d. $(x - 3)^2 + y^2 = 5$

e. $(x + 1)^2 + (y + 8)^2 = 60$

f. $(x - 2)^2 + (y - 3)^2 = 99$

g. $(x + 5)^2 + (y - 3)^2 = 144$

h. $(x - 1)^2 + (y + 11)^2 = 48$

i. $(x + 3) + (y - 1) = 9$

j. $(x - 3)^2 + (y + 2) = 7$

4. Find the equation of the circle with the given center and radius:

a.	C(0, 0)	$r = 7$	b.	C(0, 0)	$r = \sqrt{10}$
c.	C(3, 0)	$r = 3$	d.	C(0, 4)	$r = 1$
e.	C(0, –2)	$r = \sqrt{3}$	f.	C(–12, 0)	$r = 12$
g.	C(2, 7)	$r = 10$	h.	C(–1, –3)	$r = 2\sqrt{3}$
i.	C(3, –4)	$r = 3\sqrt{5}$	j.	C(–2, 9)	$r = 5\sqrt{7}$
k.	C(1, 2)	$r = 0$	l.	C(–3, 5)	$r = -9$

Review Problems

5. Give the geometric definition of the circle.

6. Consider the equation $x^2 + y^2 = k$. Describe the graph of this equation if $k > 0$, if $k = 0$, and if $k < 0$.

7. Find the center and radius of the circle $x^2 + y^2 = 20$.

8. Find the center and radius of the circle $x^2 + y^2 - 10x + 2y + 3 = 0$.

9. In terms of exponents and coefficients, explain why each of the following is <u>not</u> a circle:

a. $2x + 3y = 1$	b. $y = x^2$	c. $y = x^3$
d. $x^2 + 3y^2 = 4$	e. $x^2 - y^2 = 9$	f. $y = \sqrt{x}$

10. True/False:

a. Every circle has at least one intercept.

b. The radius of the circle $x^2 + y^2 = 1$ is 1.

c. The derivation of the circle formula is based upon the Distance Formula.

d. $x^2 + y^2 + 4 = 0$ is a circle.

e. $x^2 + y^2 = 1$ is called the unit circle.

f. The center of the circle $(x - 2)^2 + (y - 5)^2 = 10$ is $(-2, -5)$.

g. The radius of the circle $x^2 + y^2 + 8x - 6y + 9 = 0$ is 4.

h. The radius of a circle whose diameter is $25\sqrt{2}$ is $5\sqrt{2}$.

i. The graph of $10x^2 + 10y^2 = 37$ is a circle.

j. The graph of $10x^2 - 10y^2 = 37$ is a circle.

k. The graph of $10x^2 + 9y^2 = 37$ is a circle.

l. The area of the circle $x^2 + y^2 = 25$ is 25π.

Solutions

1. r represents the radius (which is a distance), and thus makes sense only if $r > 0$ [a 'circle' with zero or negative radius wouldn't be a circle].

2. It must be 18, since the distance from any point on the circle to the center must be the radius, which is 18.

3.

a. C(0, 0) $r = 8$

b. C(0, 0) $r = 2\sqrt{6}$

c. C(0, 7) $r = 2$

d. C(3, 0) $r = \sqrt{5}$

e. C(–1, –8) $r = 2\sqrt{15}$

f. C(2, 3) $r = 3\sqrt{11}$

g. C(–5, 3) $r = 12$

h. C(1, –11) $r = 4\sqrt{3}$

i. It's a line

j. It's a parabola

4.

a. $x^2 + y^2 = 49$

b. $x^2 + y^2 = 10$

c. $(x - 3)^2 + y^2 = 9$

d. $x^2 + (y - 4)^2 = 1$

e. $x^2 + (y + 2)^2 = 3$

f. $(x + 12)^2 + y^2 = 144$

g. $(x-2)^2 + (y-7)^2 = 100$ h. $(x+1)^2 + (y+3)^2 = 12$

i. $(x-3)^2 + (y+4)^2 = 45$ j. $(x+2)^2 + (y-9)^2 = 175$

k. Just the point (1, 2) l. Not a circle; no graph at all

5. A circle is the set of all points in the plane that are equidistant from a fixed point in the plane.

6. If $k > 0$, the graph is a circle with center at the origin and radius $\sqrt{k}$.
If $k = 0$, the graph is just the single point (0, 0); i.e., the origin.
If $k < 0$, the graph is empty (there's no graph at all).

7. C(0, 0); $r = 2\sqrt{5}$ **8**. $(x-5)^2 + (y+4)^2 = 25$

9.

a. $2x + 3y = 1$: At the very least, a circle has both variables squared, which the given equation certainly does not have. In fact, the given equation is a line.

b. $y = x^2$: Again, at the very least, both variables need to be squared to be a circle. Indeed, the equation is a parabola.

c. $y = x^3$: This is a cubic function.

d. $x^2 + 3y^2 = 4$: This equation does have both variables squared, but the coefficients of the squared terms (the 1 and the 3) are <u>not</u> the same, so there's no way we could put the equation into the standard form of the circle: $(x-h)^2 + (y-k)^2 = r^2$. The chapter *Ellipses* will show you what the given equation actually looks like.

e. $x^2 - y^2 = 9$: This equation resembles a circle's equation, but the minus sign between the squared terms kills the deal. Remember that the equation of a circle ultimately comes from the Distance Formula, which has a <u>plus</u> sign between the squared parts. We'll work with formulas like this in the chapter *Hyperbolas*.

f. $y = \sqrt{x}$: Certainly the exponents aren't appropriate for a circle, since the y has an exponent of 1 and the x has an exponent of 1/2.

In fact, the graph of this function is the top half of a parabola opening to the right.

11. a. F b. T c. T d. F e. T f. F
g. T h. F i. T j. F k. F l. T

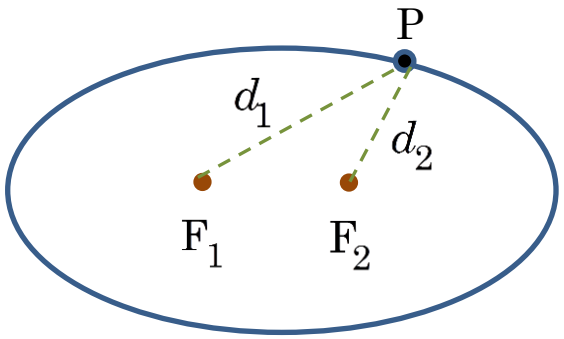
Only those who dare to fail greatly can ever achieve greatly.

Robert F. Kennedy

CONIC SECTIONS – THE ELLIPSE

❑ *DEFINITION OF THE ELLIPSE*

Let F_1 and F_2 be two distinct points in the plane and let C be a positive constant. Then an ***ellipse*** is the set of points in the plane satisfying the following condition: A point is on the ellipse if the sum of its distances to F_1 and F_2 is C, the given constant.



For any point P on the ellipse:

$$d_1 + d_2 = C$$

Each of the points F_1 and F_2 is called a **focus** (plural: **foci**, pronounced fō-sī).

In the elliptical orbit of the Earth about the Sun, the Sun is one of the two foci of the ellipse.

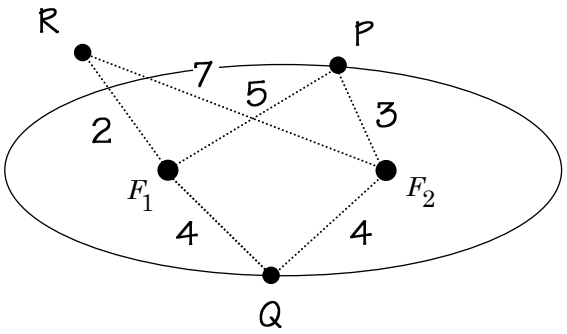
EXAMPLE 1: **In the ellipse graphed below, assume that the constant mentioned in the definition of the ellipse is 8. Explain why points P and Q are on the ellipse and why point R is not on the ellipse.**

Solution: Let's use the notation $d(A, B)$ to denote the distance between the points A and B. By definition, a point is on the ellipse if the sum of its distances to the two foci is constant — in this example, 8.

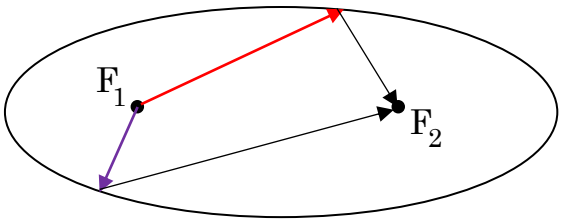
Point P: $d(P, F_1) = 5$ and $d(P, F_2) = 3$, and so the sum of the distances is 8. Therefore, P is on the ellipse.

Point Q: $d(Q, F_1) = 4$ and $d(Q, F_2) = 4$, and so the sum of the distances is 8. Thus, Q is on the ellipse.

Point R: $d(R, F_1) = 2$ and $d(R, F_2) = 7$, which means that the sum of the distances is 9. We conclude that R is not on the ellipse.



The ellipse has a very interesting reflecting property: Any form of energy (for example, sound or light) that emanates from one focus of the ellipse will reflect off the inside of the ellipse right to the other focus. See the two "rays" starting at F_1? Each one leaves the focus, reflects off the inside of the ellipse, and then bounces to F_2, the other focus.





This is the property which makes lithotripsy work: A device which produces sound waves is positioned at one focus, while the patient is positioned so that his kidney stone is positioned at the other. The sound waves that are produced at the first focus reflect off the inside of the machine and are all "focused" at the kidney stone, providing enough sonic energy to break the stone into pieces small enough to pass naturally through the body.

Homework

1. The closer the foci, the more the ellipse looks like a ____.

   [What happens when the foci are the same point? It becomes a circle, with the foci becoming the center of the circle. This brings up the question: Is a circle just a special kind of ellipse? Some books say yes; the foci coincide to become the center. Some say no, a circle is not an ellipse because their definition requires that the foci must be distinct.]

2. Sketch an ellipse where the foci are on the same vertical line.

3. Is it possible for an ellipse to be a function?

4. Let F_1 and F_2 represent the foci of an ellipse. Also suppose that P is a point on the ellipse. Assume further that the distance between P and F_1 is 20 and the distance between P and F_2 is 12. Now let Q be a point on the ellipse such that the distance from Q to F_1 is 30. What is the distance between Q and F_2?

❑ *DERIVING THE FORMULA FOR THE ELLIPSE*

LINK TO A WEBSITE

ELLIPSES WITH CENTER AT THE ORIGIN

EXAMPLE 2: **Graph the ellipse and state its domain and range:**

$$\frac{x^2}{4}+\frac{y^2}{25} = 1$$

Solution: Let's begin by finding all the intercepts of the ellipse. These points and a couple of others will convince us that the graph is indeed elliptical.

***x*-intercepts:** Set $y = 0$ to get

$$\frac{x^2}{4} = 1 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm\sqrt{4} = \pm 2$$

and so the x-intercepts are (2, 0) and (–2, 0).

***y*-intercepts:** Set $x = 0$ to get

$$\frac{y^2}{25} = 1 \quad \Rightarrow \quad y^2 = 25 \quad \Rightarrow \quad y = \pm\sqrt{25} = \pm 5$$

and thus the y-intercepts are (0, 5) and (0, –5).

additional points: Let $x = 1$. Then

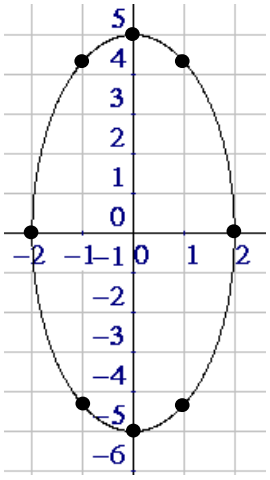
$$\frac{1^2}{4}+\frac{y^2}{25} = 1 \quad \Rightarrow \quad \frac{y^2}{25} = \frac{3}{4} \quad \Rightarrow \quad y^2 = \frac{75}{4}$$

$$\Rightarrow \quad y = \pm\sqrt{\frac{75}{4}} \approx \pm 4.33$$

This yields the points (1, 4.33) and (1, –4.33).

Note that if we let $x = -1$, we obtain the same y-values. Thus, two additional points are (–1, 4.33) and (–1, –4.33).

We now have a total of eight points we can plot. When we do, we get the graph of the ellipse:



Domain = [–2, 2]

Range = [–5, 5]

Note: It's not a function.

Note: The **center** of the ellipse is the origin.

A final note: Since the domain of the ellipse is [–2, 2], we know that an x-value of 3, for example, would not be allowed in the formula; indeed, there's no point on the ellipse whose x-coordinate is 3. But how can we verify this fact if we don't already know the domain? Let's let $x = 3$ and see what happens:

$$\frac{x^2}{4}+\frac{y^2}{25} = 1 \quad \Rightarrow \quad \frac{3^2}{4}+\frac{y^2}{25} = 1 \quad \Rightarrow \quad \frac{9}{4}+\frac{y^2}{25} = 1$$

$$\Rightarrow \quad \frac{y^2}{25} = 1-\frac{9}{4} \quad \Rightarrow \quad \frac{y^2}{25} = -\frac{5}{4} \quad \Rightarrow \quad y^2 = -\frac{125}{4}$$

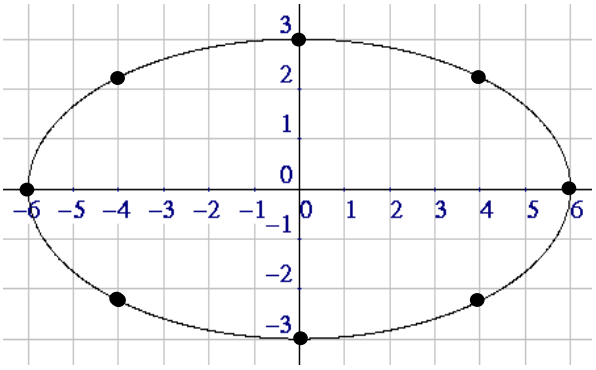
$\Rightarrow y = \pm\sqrt{-\frac{125}{4}}$, which are not real numbers. Since letting $x = 3$ resulted in no solution for y, we conclude that 3 is definitely not in the domain of the ellipse.

EXAMPLE 3: **Graph the ellipse and state its domain and range:**

$$\frac{x^2}{36}+\frac{y^2}{9} = 1$$

Solution: Since this equation is very similar to the previous one, it's likely that its graph is an ellipse with center at the origin. You can calculate the intercepts to be (6, 0), (–6, 0), (0, 3), and (0, –3).

For extra resolution, we can find four more points by letting $x = \pm 4$, and then calculating y to be $\pm\sqrt{5}$. We thus get (4, 2.24), (4, –2.24), (–4, 2.24), and (–4, –2.24). Here's the graph:



Center = (0, 0)

Domain = [–6, 6]

Range = [–3, 3]

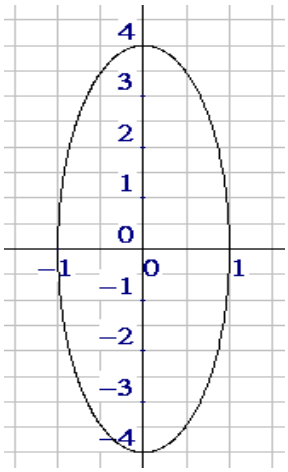
EXAMPLE 4: **Graph the ellipse and state its domain and range:**

$$\mathbf{16}x^2 + y^2 = \mathbf{16}$$

Solution: Is this an ellipse? It doesn't have the same form as the equations in Examples 2 and 3. But we can turn it into that form by dividing each side by 16 (the 16 on the right side of the equation).

$$\frac{16x^2 + y^2}{16} = \frac{16}{16} \Rightarrow x^2 + \frac{y^2}{16} = 1$$

The intercepts are (1, 0), (–1, 0), (0, 4), and (0, –4). That's enough points — we know what the graph looks like:



Center = (0, 0)

Domain = [−1, 1]

Range = [−4, 4]

Homework

5. Prove that $x = 12$ is <u>not</u> in the domain of the ellipse $\frac{x^2}{100} + \frac{y^2}{49} = 1$.

6. Consider the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

   a. Find all four intercepts.
   b. Find two more points on the ellipse by letting $x = 2$.
   c. Find two more points on the ellipse by letting $x = -3$.
   d. Sketch the ellipse.
   e. State the domain and range.
   f. Explain why it's not a function.

Graph each ellipse and state its domain and range:

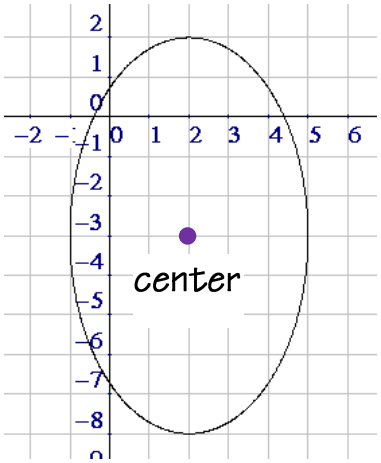
7. $\frac{x^2}{25} + \frac{y^2}{9} = 1$ 8. $x^2 + \frac{y^2}{49} = 1$ 9. $x^2 + 9y^2 = 9$

❑ *ELLIPSES WITH CENTER OFF THE ORIGIN*

EXAMPLE 5: **Graph the ellipse and state its center, domain and range:**

$$\frac{(x-2)^2}{9} + \frac{(y+3)^2}{25} = 1$$

Solution: If the equation were $\frac{x^2}{9} + \frac{y^2}{25} = 1$, we know that it would be an ellipse with center at the origin. The ellipse would extend 3 units right and left (because of the 9), and 5 units up and down (because of the 25), from its center. It's exactly the same idea for the given ellipse, except that the center has been moved 2 units to the right (because of the –2) and 3 units down (because of the +3). [The origin center has been *horizontally translated* to the right and *vertically translated* down.]



Center = (2, –3)

Domain = [–1, 5]

Range = [–8, 2]

EXAMPLE 6: **Graph the ellipse $x^2 + 9y^2 + 4x - 18y + 4 = 0$ and state its center, domain and range.**

Solution: It's not so obvious that the graph is an ellipse. Recall that in the formula for a circle, both variables are squared, and each squared variable has the same coefficient. What makes a formula an ellipse is that both variables are squared, but the coefficients are different. In addition, the two coefficients must have the same sign (this won't make sense until you study hyperbolas).

To find the center of this ellipse, we wish it had been written in the form of Example 5 — but it wasn't, so we'll have to deal with it. Our goal, then, is to take the equation $x^2 + 9y^2 + 4x - 18y + 4 = 0$ and change it into a form where we can recognize its center. As we did with circles, we'll complete the square in both variables; but being that it's an ellipse, we'll make sure that we have a "**1**" on the right side of the equation when we're done.

We take a deep breath and start with the original equation:

$$x^2 + 9y^2 + 4x - 18y + 4 = 0$$

Rearrange the terms and pull the constant to the right:

$$x^2 + 4x + 9y^2 - 18y = -4$$

Factor out the GCF:

$$x^2 + 4x + 9(y^2 - 2y) = -4$$

Complete the squares:

$$x^2 + 4x + \boxed{4} + 9(y^2 - 2y + \boxed{1}) = -4 + \boxed{4} + \boxed{9}$$

Note: We put the magic number of 1 inside the parentheses; therefore, we really added $\mathbf{9 \cdot 1}$ to the left side of the equation. This maneuver, of course, is balanced by adding **9** to the right side of the equation.

Factor the perfect square trinomials and simplify the right side:

$$(x + 2)^2 + 9(y - 1)^2 = 9$$

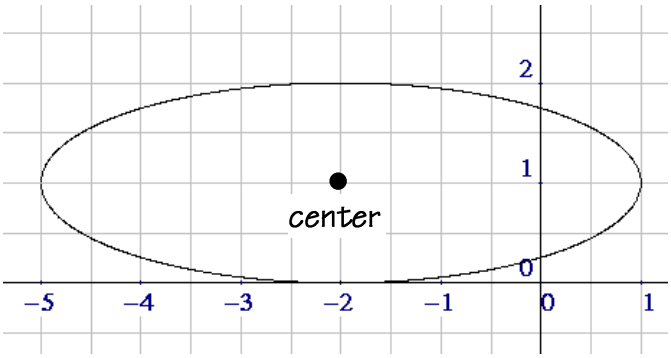
Divide through by 9:

$$\frac{(x+2)^2}{9} + \frac{9(y-1)^2}{9} = \frac{9}{9}$$

which simplifies to

$$\frac{(x+2)^2}{9} + \frac{(y-1)^2}{1} = 1$$

It's now in the proper ellipse form — we see that the center is the point (–2, 1), that it extends 3 units left and right from the center, and 1 unit up and down from the center.



Center = (–2, 1)

Domain = [–5, 1]

Range = [0, 2]

Homework

10. Find the center, the domain, and the range of each ellipse:

a. $\frac{(x-2)^2}{100} + \frac{(y+7)^2}{49} = 1$

b. $\frac{(x+3)^2}{4} + \frac{(y-9)^2}{64} = 1$

c. $(x+3)^2 + \frac{(y-3)^2}{25} = 1$

d. $\frac{(x-5)^2}{81} + y^2 = 1$

Graph each ellipse and state its domain and range:

11. $9x^2 + 4y^2 - 54x + 40y + 37 = 0$

12. $49x^2 + 16y^2 - 294x + 32y - 327 = 0$

13. $4x^2 + y^2 + 24x + 8y + 16 = 0$

14. $x^2 + 16y^2 - 10x + 64y + 73 = 0$

15. $25x^2 + 9y^2 - 50x - 108y + 124 = 0$

16. $x^2 + 4y^2 + 14x - 32y + 97 = 0$

17.** Find the domain of $\frac{5}{9}x^2 + \frac{11}{2}y^2 + \frac{5}{9}x - \frac{22}{3}y + \frac{19}{12} = 0$

Review Problems

18. State the geometric definition of the ellipse.

19. a. Does an ellipse have one or two foci?
    b. Describe where the center of an ellipse is relative to the foci.
    c. The foci of an ellipse are at (2, 10) and (20, 10). What is the center of the ellipse?

20. Let F_1 and F_2 represent the foci of an ellipse. Also suppose that P is a point on the ellipse. Assume further that the distance between P and F_1 is 88 and the distance between P and F_2 is 122. Now let Q be a point on the ellipse such that the distance from Q to F_1 is 115. What is the distance between Q and F_2?

21. Consider the ellipse $\frac{x^2}{9} + \frac{y^2}{100} = 1$.

    a. Prove that $x = 0$ is in the domain.

b. Prove that $x = 4$ is <u>not</u> in the domain.

22. Find all the intercepts, the domain, and the range of $25x^2 + 4y^2 = 100$.

23. Find the center, the domain, and the range of $25x^2 + 4y^2 - 50x + 56y + 121 = 0$.

24. Do you think that the graph of $\frac{x^2}{123} + \frac{y^2}{123} = 1$ is an ellipse?

25. An elliptical pool table has its only pocket at one of the foci. Where should you place the cue ball so that <u>no matter</u> what direction you hit it, you'll sink it?

26. True/False:

a. An ellipse is the set of points in the plane such that the sum of the distances to two fixed points is a constant.

b. The foci of an ellipse always lie on the same horizontal line.

c. The further apart the foci, the more oblong the ellipse.

d. There's an ellipse that's a function.

e. The ellipse $25x^2 + 4y^2 = 100$ has four intercepts.

f. The range of the ellipse above is $[-5, 5]$.

g. $x = 12$ is in the domain of the ellipse $\frac{x^2}{100} + \frac{y^2}{49} = 1$.

h. The center of the ellipse $x^2 + 9y^2 + 4x - 18y + 4 = 0$ is $(-2, 1)$.

i. The graph of $9x^2 - 16y^2 = 144$ is an ellipse.

j. The graph of $10x^2 = y^2 + 100$ is an ellipse.

Solutions

1. Circle **2**. ⬭ **3**. No **4**. 2

5. Let $x = 12$ in the ellipse equation and you should end up with y^2 equaling a negative number, resulting in no real solutions.

6. a. (4, 0) (–4, 0) (0, 2) (0, –2)

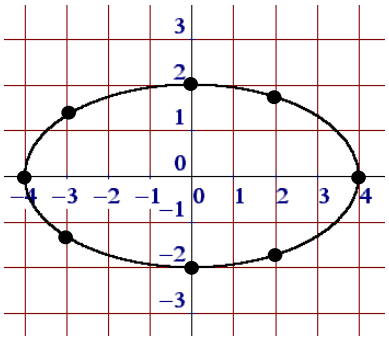
b. Letting $x = 2$ gives

$$\frac{2^2}{16}+\frac{y^2}{4} = 1 \quad \Rightarrow \quad \frac{1}{4}+\frac{y^2}{4} = 1 \quad \Rightarrow \quad \frac{y^2}{4} = \frac{3}{4}$$

$\Rightarrow \quad y^2 = 3 \quad \Rightarrow \quad y = \pm\sqrt{3}$, which give us the two approximate points (2, 1.732) and (2, –1.732).

c. Letting $x = -3$ gives the points (–3, 1.333) and (–3, –1.333).

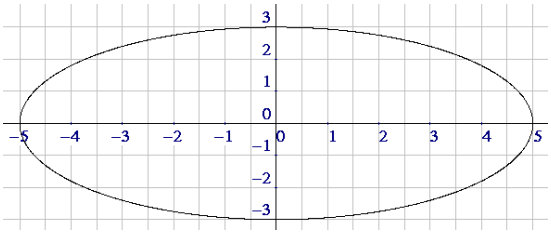
d.



e. Domain = [–4, 4] Range = [–2, 2]

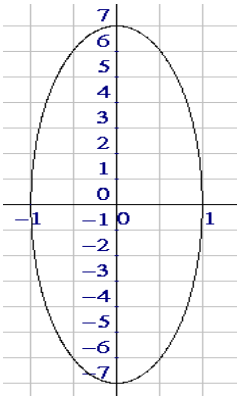
f. Letting the input (x-value) be 0, for example, we get the <u>two</u> outputs (y-values) of 2 and –2. Thus, the ellipse is not a function.

7.



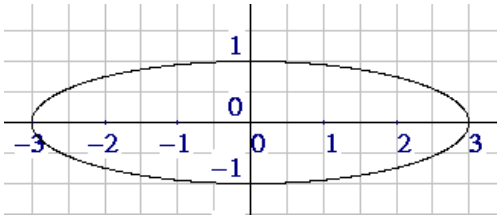
Domain = [–5, 5]
Range = [–3, 3]

8.



Domain = [–1, 1]
Range = [–7, 7]

9.



Domain = [–3, 3]
Range = [–1, 1]

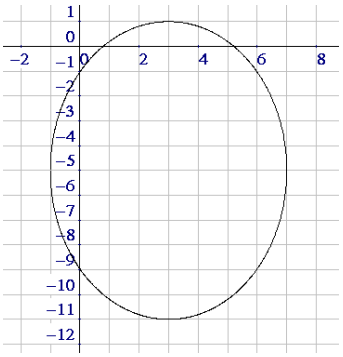
10. a. Center = (2, –7) Domain = [–8, 12] Range = [–14, 0]

b. Center = (–3, 9) Domain = [–5, –1] Range = [1, 17]

c. Center = (–3, 3) Domain = [–4, –2] Range = [–2, 8]

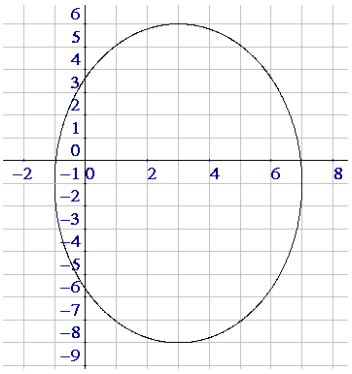
d. Center = (5, 0) Domain = [–4, 14] Range = [–1, 1]

11.



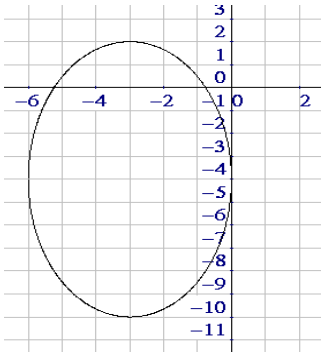
Domain = [–1, 7]
Range = [–11, 1]

12.



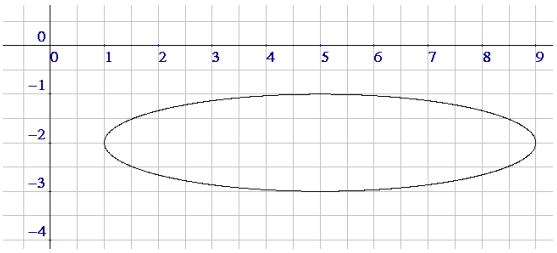
Domain = [–1, 7]
Range = [–8, 6]

13.



Domain = [–6, 0]
Range = [–10, 2]

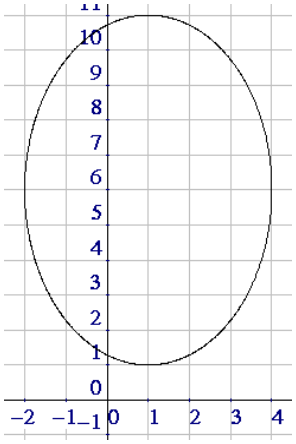
14.



Domain = [1, 9]

Range = [–3, –1]

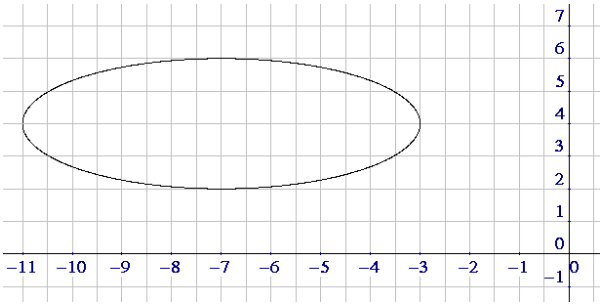
15.



Domain = [–2, 4]

Range = [1, 11]

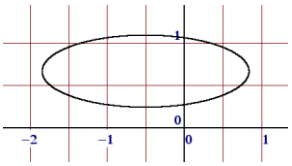
16.



Domain = [–11, –3]

Range = [2, 6]

17. $\left[\dfrac{-5-6\sqrt{5}}{10}, \dfrac{-5+6\sqrt{5}}{10}\right]$



18. An ellipse is the set of points in the plane such that the <u>sum</u> of the distances to two fixed points (the foci) is a constant.

19. a. Most sources say two foci.
b. The center is halfway between the foci, or, the center is the midpoint of the line segment connecting the foci.
c. (11, 10)

20. 95

21. a. Letting $x = 0$ in the ellipse equation results in $y = \pm 10$, which of course are real numbers. Therefore, 0 is in the domain of the ellipse.

b. Letting $x = 4$ in the equation results in $y^2 = -\frac{700}{9}$, which has no solution. Thus, 4 is not in the domain.

22. $(\pm 2, 0)$ (0 ± 5); $[-2, 2]$; $[-5, 5]$

23. $\frac{(x-1)^2}{4} + \frac{(y+7)^2}{25} = 1$ Center: $(1, -7)$

Domain: $[-1, 3]$ Range = $[-12, -2]$

24. Maybe it is, and maybe it isn't.

25. At the other focus.

26. a. T b. F c. T d. F e. T f. T
g. F h. T i. F j. F

Courage and perseverance have a magical talisman, before which difficulties disappear and obstacles vanish into air.

John Quincy Adams

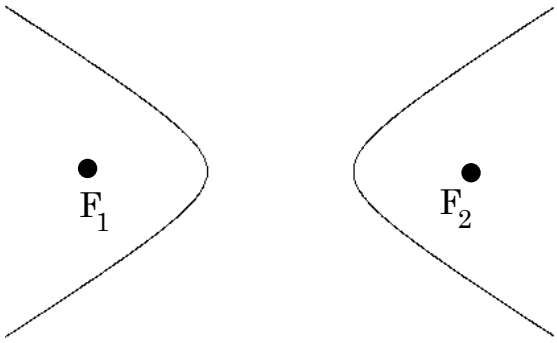
CONIC SECTIONS – THE HYPERBOLA

MOST of this is in Hyperbolas.docx.

❑ Introduction

❑ Definition of a Hyperbola

Let F_1 and F_2 be two distinct points in the plane and let c be a positive constant. Then a ***hyperbola*** (hy-PER-bo-la) is the set of points in the plane satisfying the following condition: A point is on the hyperbola if the <u>difference</u> of its distances to F_1 and F_2 is c, the given constant. Each of the points F_1 and F_2 is called a **focus**.



Apollonius (Greek: 262 BC – 190 BC) gave the hyperbola its present name. See how the graph of the hyperbola seems to "fly away" in all directions? Look up the word "hyperbol<u>e</u>" in a dictionary and see if its

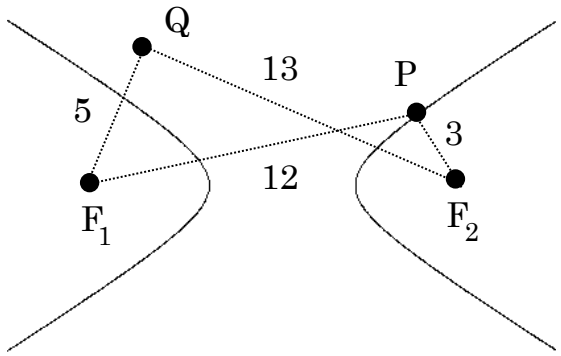
meaning has any connection to the graph of a hyperbola.

EXAMPLE 1: **In the hyperbola below assume that the constant mentioned in the definition of the hyperbola is 9. Explain why point P is on the hyperbola and why point Q is not on the hyperbola.**

Solution: Again we use the notation d(A, B) to denote the distance between the points A and B. By definition, a point is on the hyperbola if the difference of its distances to the two foci is constant — in this example, **9**.

Point P: $d(P, F_1) = 12$ and $d(P, F_2) = 3$, and so the difference of the distances is 9. Therefore, P is on the hyperbola.

Point Q: $d(Q, F_1) = 5$ and $d(Q, F_2) = 13$, which means that the difference of the distances is 8, not 9. We conclude that Q is not on the hyperbola.



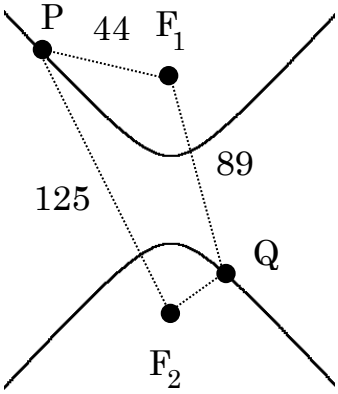
Homework

1. Sketch a hyperbola with the foci on the same vertical line.

2. Let F_1 and F_2 represent the foci of a hyperbola. Also suppose that P is a point on the hyperbola. Assume further that the distance between P and F_1 is 20 and the distance between P and F_2 is 12. Now let Q be a point on the hyperbola such that the distance from Q to F_1 is 30. What is the distance between Q and F_2?

   a. 1 b. 2 c. 22 d. 15 e. 14

3. In the hyperbola below, find the distance between Q and F_2.



4. Is it possible for a hyperbola to be a function? (We assume just the two kinds of hyperbolas discussed in this chapter — there are others.)

❑ The Main Example

Graph the hyperbola $x^2 - y^2 = 9$.

First, let's find the **domain**. We solve for *y*, and then analyze all the legal values of *x*:

$x^2 - y^2 = 9$	(the original equation)
$\Rightarrow\ -y^2 = -x^2 + 9$	(subtract *x* from each side)
$\Rightarrow\ y^2 = x^2 - 9$	(multiply each side by –1)
$\Rightarrow\ y = \pm\sqrt{x^2 - 9}$	(take both square roots)

Because of the square root, y is a real number only when the radicand, $x^2 - 9$, is greater than or equal to zero: $x^2 - 9 \geq 0$. You can solve this inequality using the Boundary Point Method. When you do, you will conclude that the domain of the graph is $\mathbf{(-\infty, -3]} \cup \mathbf{[3, \infty)}$. By the way, can you explain why the hyperbola is <u>not</u> a function?

Second, we'll check out **intercepts**. Since 0 is not in the domain, there cannot possibly be any y-intercepts. Indeed, letting $x = 0$ gives $0^2 - y^2 = 9 \Rightarrow y^2 = -9 \Rightarrow y = \pm\sqrt{-9}$, not members of $\mathbb{R}$. Thus, there are no y-intercepts. As for x-intercepts, let $y = 0$; this gives $x^2 - 0^2 = 9 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$. Therefore, there are two x-intercepts: **(3, 0)** and **(–3, 0)**.

Third, let's tackle **symmetry**. Replace x with $-x$: $(-x)^2 - y^2 = 9$, which is equivalent to $x^2 - y^2 = 9$, the original equation. Thus, the graph has y-axis symmetry. Similarly, you can show that the graph has x-axis symmetry. In fact, the graph of the hyperbola also has origin symmetry.

Fourth, we analyze **asymptotes**. Since there are no fractions in the hyperbola formula, the graph appears to have no vertical asymptotes. To find other possible asymptotes, we'll let $x \to \infty$ and see if y approaches anything interesting.

Let's construct a little x-y chart, and then choose really big values for x. Use your calculator to confirm these results.

x	y
5	± 4
10	± 9.5394
100	± 99.5499
1,000	± 999.9955
10,000	$\pm 9{,}999.9996$

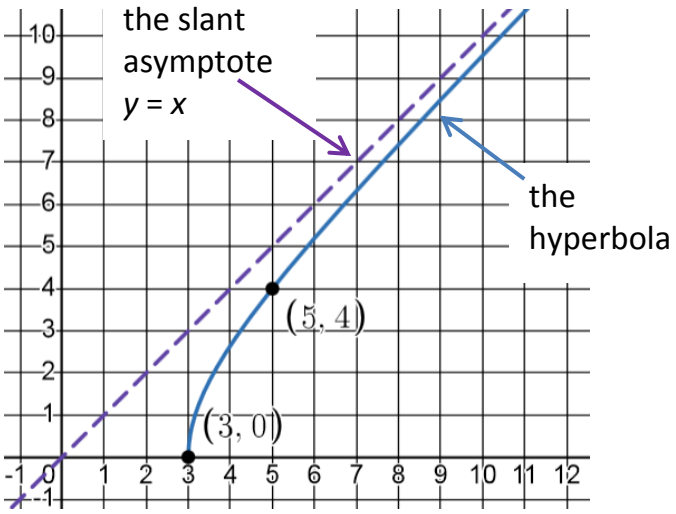
What's going on here? Let's look just at the positive y-values. On the surface, it appears that as x grows larger and larger, the y-values grow

larger and larger. But look more carefully; the y-value is actually approaching the x-value! That is, as $\boldsymbol{x \to \infty, y \to x}$. We can write this limit as

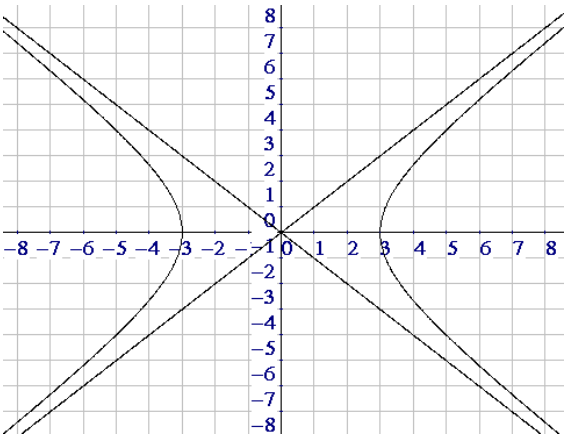
$$\lim_{x \to \infty} \sqrt{x^2 - 9} = x$$

This means that as x grows larger and larger, the curve is approaching the line $y = x$ (a 45° line passing through the origin). Since this line is neither vertical nor horizontal, we need a new adjective. We call this kind of asymptote a ***slant***, or *oblique*, asymptote.

The completed analysis gives a picture of the hyperbola in the first quadrant. It has an x-intercept at (3, 0), and a slant asymptote of $y = x$. Also, from the chart above, the hyperbola passes through the point (5, 4).



Since the hyperbola has y-axis symmetry, another piece of the hyperbola lies in the second quadrant. But the pieces of the hyperbola in Quadrants I and II have symmetric pieces in Quadrants III and IV, due to the x-axis symmetry of the hyperbola. We're now ready to graph the



entire hyperbola. Note that the lines $y = x$ and $y = -x$ are the slant asymptotes of the hyperbola, and are graphed just for reference — they are <u>not</u> part of the hyperbola. Finally, the graph should convince you of the fact deduced earlier that the domain is $(-\infty, -3] \cup [3, \infty)$. In addition, the graph shows us that the **range** of the hyperbola is $\mathbb{R}$.

Homework

5. Regarding the main example above, $x^2 - y^2 = 9$, explain, using the formula, why it isn't a line, a parabola, a circle, or an ellipse.

6. Find the slant asymptotes for each hyperbola:

   a. $x^2 - y^2 = 25$ b. $x^2 - 9y^2 = 9$

   c. $36x^2 - y^2 = 36$ d. $9x^2 - 4y^2 = 36$

7. Find the domain and range of each hyperbola in the previous problem.

8. Perform a complete analysis (including a graph) on the hyperbola $x^2 - y^2 = 49$.

❑ Another Example

Graph the hyperbola $y^2 - 4x^2 = 25$.

First we solve for y — this will allow us to find the domain and the slant asymptotes.

$$y^2 - 4x^2 = 25 \Rightarrow y^2 = 25 + 4x^2 \Rightarrow y = \pm\sqrt{25 + 4x^2}$$

Look at the radicand. No matter what kind of number x is, its square x^2 is non-negative, which implies that $4x^2$ is non-negative, which

implies that $25 + 4x^2$ is <u>at least</u> 25. So surely the radicand can <u>never</u> be negative. Thus, the **domain** of the hyperbola is $\mathbb{R}$.

Now it's time for the **intercepts**. Let $x = 0$; then $y^2 = 25$, or $y = \pm 5$. So we have a couple of y-intercepts: (0, 5) and (0, –5). Now let $y = 0$; then $-4x^2 = 25 \Rightarrow x^2 = -\frac{25}{4} \Rightarrow$ no real solution for x. Therefore, there are no x-intercepts.

I'll leave it to you to verify that the hyperbola possesses all three types of **symmetry**.

To find the slant **asymptotes**, we refer to the form of the equation calculated above,

$$y = \pm\sqrt{25 + 4x^2}$$

Use your calculator to confirm the following ordered pairs (using just the positive square root.):

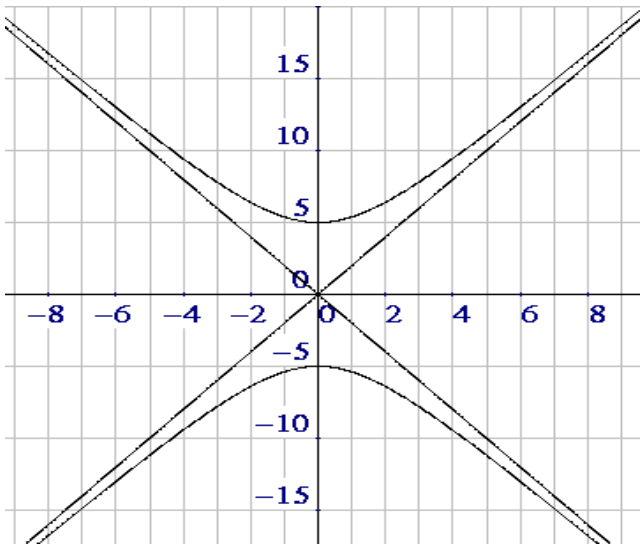
(10, 20.616) (100, 200.062) (1000, 2000.006)

As x grows larger and larger, the y-values appear to be growing closer and closer to whatever <u>twice x</u> is. That is, as $x \to \infty$, $y \to 2x$. We write

$$\lim_{x \to \infty} \sqrt{25 + 4x^2} = 2x.$$

Thus, one of the slant asymptotes is the line $y = 2x$.

The graph goes through the intercept (0, 5) and then approaches the slant asymptote $y = 2x$. The x-axis and y-axis symmetries then give us the rest of the hyperbola in the other three quadrants. As before, the asymptotes are graphed for reference.



Last, we discuss the **range** of the hyperbola. The y-values appear to be numbers that are either less than or equal to –5, or greater than or equal to 5. That is, the range of the hyperbola is $\{y \in \mathbb{R} \mid y \le -5 \text{ OR } y \ge 5\}$, which is **$(-\infty, -5] \cup [5, \infty)$**.

Homework

9. Find the slant asymptotes for each hyperbola:

   a. $y^2 - x^2 = 144$ b. $y^2 - 16x^2 = 16$

   c. $9y^2 - x^2 = 9$ d. $25y^2 - 9x^2 = 225$

10. Find the domain and range of each hyperbola in the previous problem.

Perform a complete analysis (including a graph) on each hyperbola:

11. $y^2 - x^2 = 25$

12. $y^2 - 9x^2 = 16$

13. $16y^2 - 9x^2 = 144$

14. $\frac{x^2}{16} - \frac{y^2}{4} = 1$

15. Classify each of the following as a line, parabola, circle, ellipse, or hyperbola:

   a. $7x^2 - \pi x + 2y + 9 = 0$ b. $83x + 99y = 34$

   c. $7x^2 + 2y^2 - x = 10$ d. $y^2 + 7y - x^2 - 2x - 100 = 0$

   e. $7x + 99 = 400$ f. $32 - 9y + \pi = 0$

   g. $\sqrt{2}\,x^2 + \sqrt{2}\,y^2 - \sqrt{e}\,x + \sqrt[3]{\pi}\,y - 10^6 = 0$

16.** Find the slant asympotes for $25x^2 - 9y^2 + 50x + 90y - 425 = 0$.

Review Problems

17. State the geometric definition of the hyperbola.

18. Classify as parabola, circle, ellipse, hyperbola, or none of them:

a. $x^2 + y^2 = -1$ b. $y^2 - x^2 = 5$ c. $2x^2 + 4y^2 = 7$

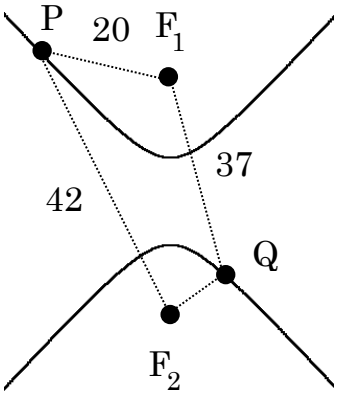
d. $x^2 - y = 3$ e. $2x^2 - \pi y^2 = \sqrt{2}$ f. $3y^2 = x^2 + 5$

19. How many foci does a hyperbola have?

20. Let F_1 and F_2 represent the foci of a hyperbola. Also suppose that P is a point on the hyperbola. Assume further that the distance between P and F_1 is 300 and the distance between P and F_2 is 160. Now let Q be a point on the hyperbola such that the distance from Q to F_1 is 110. What is the distance between Q and F_2?

a. 160 b. 180 c. 540 d. 250 e. 360

21. In the hyperbola below, find the distance between Q and F_2.



22. a. Find the intercepts of $3x^2 - 2y^2 = 2$.

b. Find the intercepts of $10y^2 - 30x^2 = 3$.

c. Find the domain of $x^2 - 14y^2 = 144$.

d. Find the range of $9x^2 - 36y^2 = 41$.

e. Find the asymptotes of $49x^2 - y^2 = 10$.

f. Find the asymptotes of $36y^2 - 4x^2 = 1$.

g. Find all symmetries of $4x^2 - 9y^2 = 36$.

23. Find the intercepts, domain, range, asymptotes, and symmetry of the hyperbola $x^2 - y^2 = 4$. Graph the hyperbola.

24. Find the asymptotes and the range of $y^2 - 9x^2 = 16$. Sketch the graph.

25. True/False:

a. A hyperbola is the set of points in the plane such that the difference of the distances to two fixed points is a constant.

b. Some of the hyperbolas in this chapter are functions.

c. The domain of the hyperbola is $x^2 - y^2 = 9$ is $[-3, 3]$.

d. Hyperbolas in this chapter always possess all three symmetries.

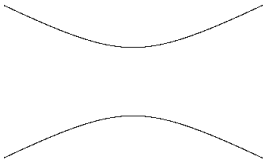
e. The slant asymptotes for $16x^2 - 9y^2 = 1$ are $y = \pm\frac{4}{3}x$.

f. There are no slant asymptotes for $25x^2 + 49y^2 = 1$.

g. The range of a hyperbola may consist of the union of two intervals.

h. Some of the hyperbolas in this chapter pass through the origin.

Solutions

1. 

2. c.

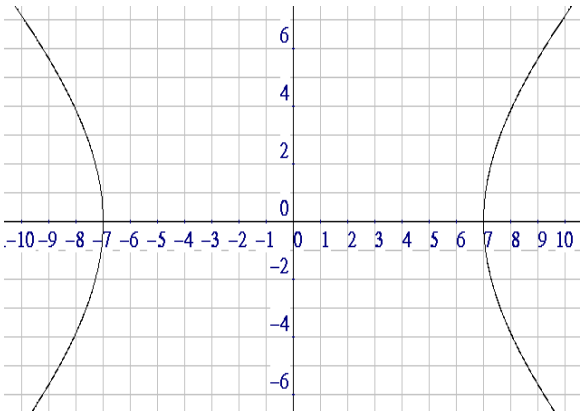
3. 8

4. No

5. Any letter squared $\Rightarrow$ it's not a line
Both letters squared $\Rightarrow$ not a parabola
Different coefficients on the squares $\Rightarrow$ not a circle
Opposite signs on the coefficients on the squares $\Rightarrow$ not an ellipse

6. a. $y = \pm x$ b. $y = \pm \frac{1}{3}x$ c. $y = \pm 6x$ d. $y = \pm \frac{3}{2}x$

7. a. $(-\infty, -5] \cup [5, \infty)$; $\mathbb{R}$ b. $(-\infty, -3] \cup [3, \infty)$; $\mathbb{R}$
c. $(-\infty, -1] \cup [1, \infty)$; $\mathbb{R}$ d. $(-\infty, -2] \cup [2, \infty)$; $\mathbb{R}$

8. 

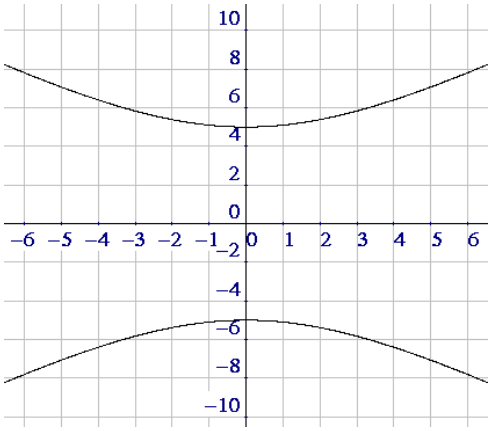
Domain = $(-\infty, -7] \cup [7, \infty)$
Range = $\mathbb{R}$ Not a function
No y-intercepts
x-intercepts: $(-7, 0)$ and $(7, 0)$
Symmetry: all three
$\lim_{x \to \infty} \sqrt{x^2 - 49} = x$
Slant asymptotes: $y = x$ and $y = -x$

9. a. $y = \pm x$ b. $y = \pm 4x$ c. $y = \pm \frac{1}{3}x$ d. $y = \pm \frac{3}{5}x$

10. a. $\mathbb{R}$; $(-\infty, -12] \cup [12, \infty)$ b. $\mathbb{R}$; $(-\infty, -4] \cup [4, \infty)$

c. $\mathbb{R}$; $(-\infty, -1] \cup [1, \infty)$ d. $\mathbb{R}$; $(-\infty, -3] \cup [3, \infty)$

11.

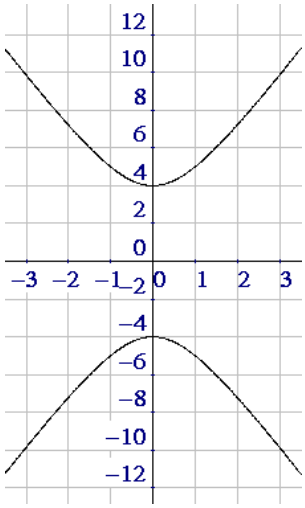


Domain = $\mathbb{R}$ Range = $(-\infty -5] \cup [5, \infty)$ Not a function

No x-intercept; y-intercepts: (0, 5) and (0, –5)

Symmetry: all three Asymptotes: $y = x$ and $y = -x$

12.



Domain = $\mathbb{R}$

Range = $(-\infty, -4] \cup [4, \infty)$

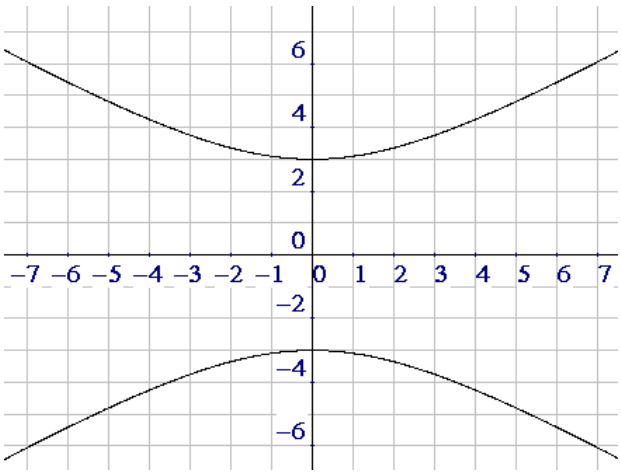
Not a function

No x-intercepts; y-intercepts: (0, 4) and (0, –4)

Symmetry: all three

Asymptotes: $y = 3x$ and $y = -3x$

13.



Domain = $\mathbb{R}$

Range = $(-\infty\ -3] \cup [3, \infty)$

Not a function

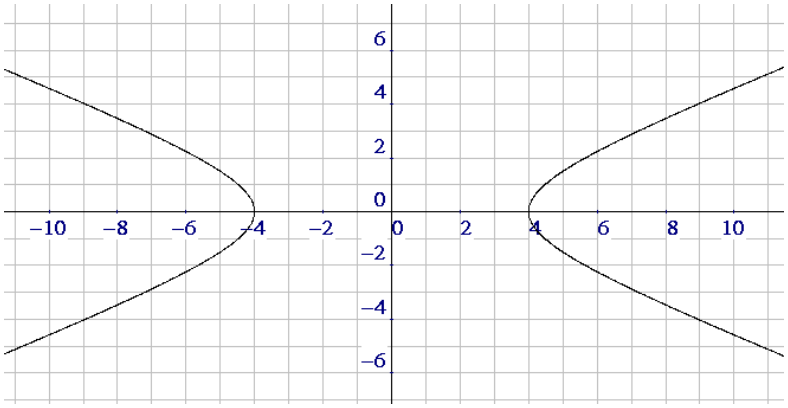
No x-intercepts

y-intercepts: (0, 3) and (0, −3)

Symmetry: all three

Asymptotes: $y = \pm\frac{3}{4}x$

14.



Domain = $(-\infty, -4] \cup [4, \infty)$ Range = $\mathbb{R}$ Not a function

Intercepts: (−4, 0) and (4, 0) Symmetry: all three

Slant asymptotes: $y = \pm\frac{1}{2}x$

15. a. parabola b. line c. ellipse d. hyperbola
e. (vertical) line f. (horizontal) line g. circle

16. $y = \frac{5}{3}x + \frac{20}{3}$ $y = -\frac{5}{3}x + \frac{10}{3}$

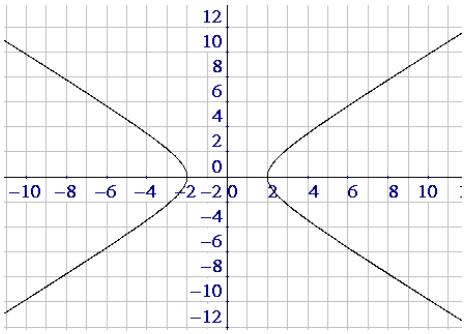
17. A hyperbola is the set of points in the plane such that the <u>difference</u> of the distances to two fixed points is a constant.

18. a. none b. hyperbola c. ellipse
d. parabola e. hyperbola f. hyperbola

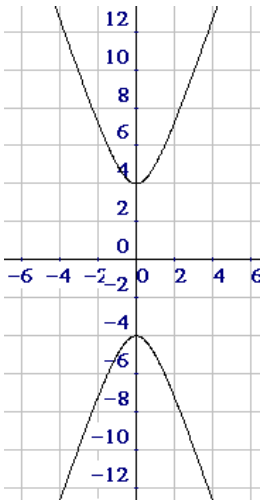
19. 2 **20**. d. **21**. 15

22. a. $\left(\pm\sqrt{\frac{2}{3}},\ 0\right)$ b. $\left(0,\ \pm\sqrt{\frac{3}{10}}\right)$ c. $(-\infty, -12] \cup [12, \infty)$ d. $\mathbb{R}$
e. $y = \pm 7x$ f. $y = \pm\frac{1}{3}x$ g. x-axis, y-axis, and origin

23. $(\pm 2, 0)$ Dom = $(-\infty, -2] \cup [2, \infty)$ Rg = $\mathbb{R}$ asym: $y = \pm x$
all 3 symmetries



24. $y = \pm 3x$; $(-\infty, -4] \cup [4, \infty)$



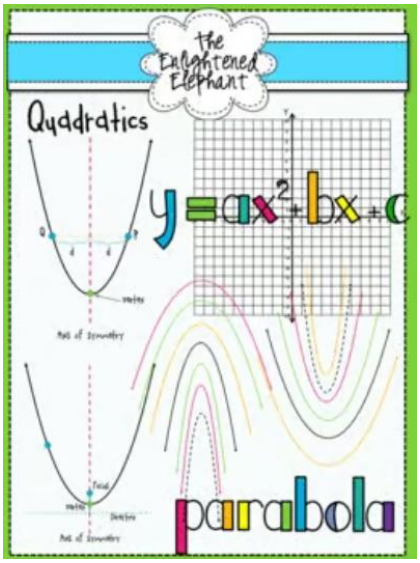
"Effort only fully releases its reward after a person refuses to quit."

Napoleon Hill

25. a. T b. F c. F d. T e. T f. T g. T h. F

CONIC SECTIONS – THE PARABOLA

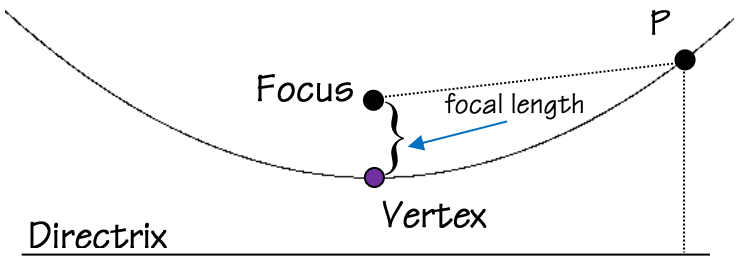
If you've studied the chapter on Parabolas, you will remember that we graphed them by plotting lots of points, and then connected those points to make a smooth curve. That curve then allowed us to estimate the vertex (extreme point) and the intercepts of the graph. In this chapter we get down and dirty and learn where the parabola comes from.



❑ *DEFINITION OF THE PARABOLA*

Let F be a point in the plane and let L be a line in the plane not containing the point F. Then a ***parabola*** is the set of points satisfying the following condition: *A point is on the parabola if the distance to the point* F *equals the distance to the line* L.

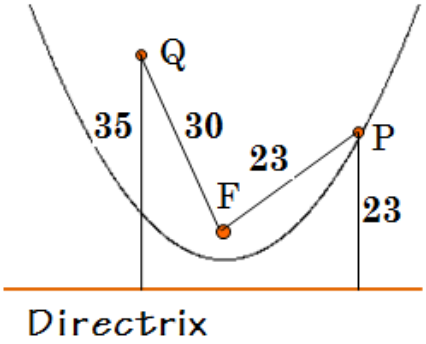
The point F is called the **focus** and the line L is called the **directrix**. The point that is halfway between the focus and the directrix is called the **vertex**. The distance between the vertex and focus (or between the vertex and the directrix) is called the **focal length**.



If point P is on the parabola, then the distance from P to the focus must equal the distance from P to the directrix.

If the focus is above the directrix, we will agree that the focal length is positive, but if the focus is below the directrix, we'll regard the focal length as negative.

For example, suppose that the point P is on the parabola to the right, and assume that the distance between P and the focus, F, of the parabola is 23. Then what is the distance between P and the directrix? It must be **23**, because if P is on the parabola, the distance between P and the focus must equal the distance between P and the directrix.



On the other hand, point Q is not on the parabola because it is 30 units from Q to the focus, whereas it is 35 units from Q to the directrix. Since the distances don't match, Q cannot be on the parabola.

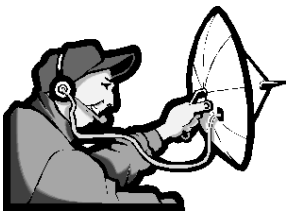
Homework

1. Here's a simpler way to write the definition of the parabola:
   A parabola is the set of points in the plane which are equidistant from a given point and a given line.
   a. What is the given point called?
   b. What is the given line called?

2. a. Sketch a parabola whose focus is below the directrix.

   b. Sketch a parabola whose focus is to the right of the directrix.

   c. Sketch a parabola whose focus is to the left of the directrix.



   d. What might be the reason that in this chapter we won't work with parabolas like those in parts b. and c. of this question?

3. a. If the focal length is positive, the parabola opens ____.

   b. If the focal length is negative, the parabola opens ____.

4. Suppose Q is a point on a parabola, and assume that the distance between Q and the directrix is 197. What is the distance between Q and the focus?

5. Let F be the focus of a parabola, and let P be a point in the plane such that the distance between P and F is 30 and the distance between P and the directrix is 25. Explain why point P is <u>not</u> on the parabola.

6. In each problem, the vertex and focal length of a parabola are given. Find the focus and the directrix.

   a. V(0, 0) $p = 3$ b. V(0, 0) $p = -7$

   c. V(2, 3) $p = 5$ d. V(2, 3) $p = -5$

   e. V(–5, 1) $p = 2$ f. V(–3, –7) $p = -1$

7. In each problem, the focus and the directrix are given. Find the vertex.

   a. F(2, 3) directrix: $y = 1$ b. F(3, 7) directrix: $y = 9$

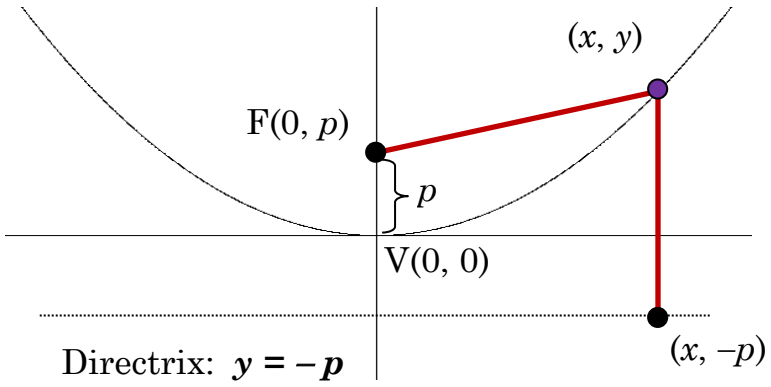
   c. F(–8, –4) directrix $y = 2$ d. F(0, –10) directrix: $y = -30$

❑ THE FUNDAMENTAL PARABOLA FORMULA

THEOREM: Consider the parabola whose vertex is at the origin (0, 0) and which opens up. Let p represent the focal length (note that, in this case, $p > 0$). Then the formula for the parabola is given by

$$y = \frac{1}{4p}x^2$$

PROOF: Consider the following diagram of a parabola with vertex at the origin, opening up, with focal length p. Since p is the distance between the vertex and the focus, the coordinates of the focus are $(0, p)$. Since the directrix is p units down from the origin, its equation is the horizontal line $y = -p$. A generic point (x, y) has been labeled on the parabola — this point, in a sense, represents all the points on the parabola. Also note the point $(x, -p)$ on the directrix. Its first coordinate must be the first coordinate of the point directly above it, so it's x. The second coordinate must be $-p$, since it's on the line $y = -p$.



In order that the point (x, y) be on the parabola, it must satisfy the definition of the parabola: The distance from (x, y) to $(0, p)$

must equal the distance from (x, y) to $(x, -p)$. Recalling the Distance Formula, we can write

$$\sqrt{(x-0)^2+(y-p)^2} = \sqrt{(x-x)^2+[y-(-p)]^2}$$

$$\Rightarrow \quad \sqrt{x^2+y^2-2py+p^2} = \sqrt{0^2+(y+p)^2}$$

$$\Rightarrow \quad \sqrt{x^2+y^2-2py+p^2} = \sqrt{y^2+2py+p^2}$$

$$\Rightarrow \quad x^2+y^2-2py+p^2 = y^2+2py+p^2$$

$$\Rightarrow \quad x^2+\cancel{y^2}-2py+\cancel{p^2} = \cancel{y^2}+2py+\cancel{p^2}$$

$$\Rightarrow \quad x^2-4py = 0$$

$$\Rightarrow \quad -4py = -x^2$$

$$\Rightarrow \quad y = \frac{-x^2}{-4p}$$

$$\Rightarrow \quad \boxed{y = \frac{1}{4p}x^2}$$

Q.E.D.

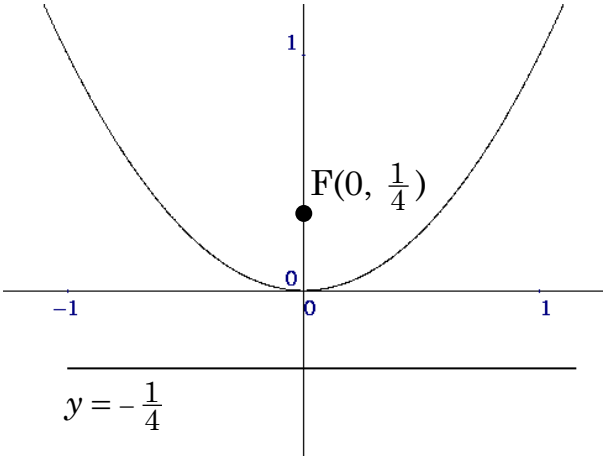
EXAMPLE 1: **Find the focal length of the classic parabola $y = x^2$. Now state the vertex, the focus, and directrix of the parabola.**

Solution: In the formula $y = \frac{1}{4p}x^2$, the coefficient of the x^2 term is $\frac{1}{4p}$. For the classic parabola in this example, the coefficient of the x^2 term is 1. Hence, setting the coefficients equal to each other will yield us the value of p, the focal length.

$$\frac{1}{4p} = 1 \quad \Rightarrow \quad 1 = 4p \quad \Rightarrow \quad \boldsymbol{p = \frac{1}{4}}$$

Since $p > 0$, the parabola opens up. Because $p = \frac{1}{4}$, and the vertex is at the origin, the focus is $\frac{1}{4}$ unit up from (0, 0), which is **(0, $\frac{1}{4}$)**.

The directrix is therefore $\frac{1}{4}$ unit down from (0, 0), and so the equation of the directrix is the horizontal line $\boldsymbol{y = -\frac{1}{4}}$.



EXAMPLE 2: **Find the vertex, the focus, and the directrix of the parabola $y = \frac{1}{8}x^2$.**

Solution: Comparing $y = \frac{1}{8}x^2$ with $y = \frac{1}{4p}x^2$, we can write

$$\frac{1}{4p} = \frac{1}{8} \Rightarrow 8 = 4p \Rightarrow p = 2$$

So the focus is 2 units above the vertex (0, 0): **F(0, 2)**.

The directrix is 2 units below the vertex (0, 0): $\boldsymbol{y = -2}$.

EXAMPLE 3: **Find the vertex, the focus, and the directrix of the parabola $y = -2x^2$.**

Solution: Now that we're getting the hang of it, we see that

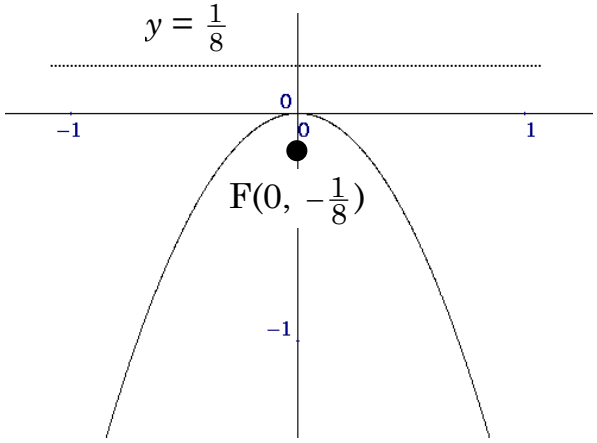
1) The vertex is **(0, 0)**.

2) $\frac{1}{4p} = -2 \;\Rightarrow\; 1 = -8p \;\Rightarrow\; p = -\frac{1}{8}$

Note that $p < 0$; thus the parabola opens down.

3) The focus is $\frac{1}{8}$ unit below (0, 0); therefore, **F(0, $-\frac{1}{8}$).**

4) The directrix is $\frac{1}{8}$ unit above (0, 0); thus, $\boldsymbol{y = \frac{1}{8}}$.



Homework

8. Find the vertex, the focus, and the directrix of each parabola:

a. $y = -x^2$ b. $y = \frac{1}{12}x^2$ c. $y = -\frac{1}{16}x^2$

d. $y = 3x^2$ e. $y = -4x^2$ f. $y = 2x + 3$

❑ *SHIFTING THE PARABOLA*

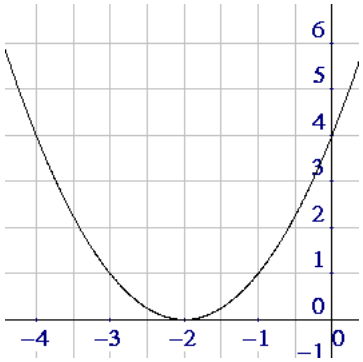
As we did with square root graphs, absolute value graphs, and cube root graphs, we now analyze what happens when the parabola formula we've been using has extra numbers hanging around.

EXAMPLE 4: **Graph the two parabolas $y = (x + 2)^2$ and $y = x^2 - 3$.**

Solution: We'll graph these two functions by plotting some points, kind of pretending that we don't already know that they're parabolas.

$$y = (x + 2)^2$$

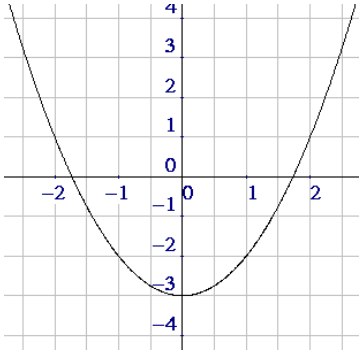
x	y
−4	4
−3	1
−2	0
−1	1
0	4



We notice that this parabola is the parabola $y = x^2$ shifted 2 units to the left. This is the same concept we've seen with other functions. Now for the other parabola:

$$y = x^2 - 3$$

x	y
−2	1
−1	−2
0	−3
1	−2
2	1

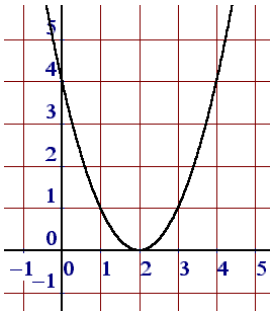


Again we see a familiar shift, in this case 3 units down. We reach the same conclusion we have in the past. When there's a number

next to the x (being added or subtracted), there's a horizontal shift in the direction opposite of the sign of the number. When there's a number hanging off the end of the function, we get a vertical shift in the same direction as the sign of the number.

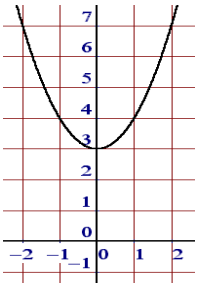
EXAMPLE 5: **Find the vertex of each parabola:**

A. $y = (x - 2)^2$



The vertex of the standard parabola $y = x^2$ is (0, 0). Since there's a –2 next to the x, the vertex is shifted 2 units to the right. Therefore, the vertex of $y = (x - 2)^2$ is **(2, 0)**.

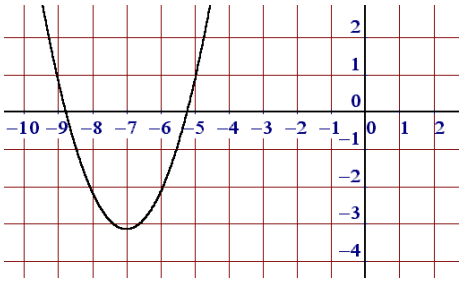
B. $y = x^2 + 3$



The vertex of the standard parabola $y = x^2$ is (0, 0). Because there's a 3 hanging off the end of the formula, the vertex is shifted 3 units up. Thus, the vertex of $y = x^2 + 3$ is **(0, 3).**

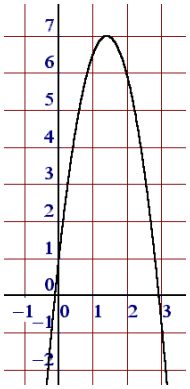
C. $y = (x + 7)^2 - \pi$

The 7 yields a shift to the left of 7 units. The $-\pi$ produces a shift down of π units. And so the graph has its vertex at the point **$(-7, -\pi)$**.



D. $y = -3(x - \sqrt{2})^2 + 7$

The -3 simply makes the parabola open down and appear skinnier than $y = x^2$, but it does not affect the vertex. The usual vertex of (0, 0) is shifted $\sqrt{2}$ units to the right and 7 units up. Therefore, the new vertex is at **$(\sqrt{2}, 7)$.**



EXAMPLE 6: **Analyze the parabola $y = \frac{1}{32}(x-3)^2 + 2$.**

Include the domain, the *y*-intercept, the *x*-intercept(s), the vertex, the focus, the directrix, and the graph.

Solution: This is one tall order, but if we proceed one step at a time, and remember everything we've learned so far, we can solve this problem.

Domain: We've seen before that the domain of this function is all real numbers; after all, there's no division to worry about, and no square roots to concern us. Thus, x can be any number at all, and so **the domain is $\mathbb{R}$**.

y-intercept: As always, y-intercepts are found by setting $x = 0$ and solving for y. We get

$$y = \frac{1}{32}(\mathbf{0}-3)^2+2 = \frac{1}{32}(9)+2 = \frac{9}{32}+\frac{64}{32} = \frac{73}{32}$$

and so **the *y*-intercept is $\mathbf{(0, \frac{73}{32})}$**.

***x*-intercept(s):** Setting $y = 0$ gives the equation

$$0 = \frac{1}{32}(x-3)^2+2 \;\Rightarrow\; -2 = \frac{1}{32}(x-3)^2 \;\Rightarrow\; -64 = (x-3)^2.$$

But this last equation has no solution, since the square of a quantity can never be negative. Since we were seeking *x*-intercepts, and we ran into an unsolvable equation, we conclude that our parabola has **no *x*-intercepts**.

Vertex: By looking at the equation, we see a horizontal shift of 3 units to the right and a vertical shift of 2 units up. Therefore, **the vertex is (3, 2)**.

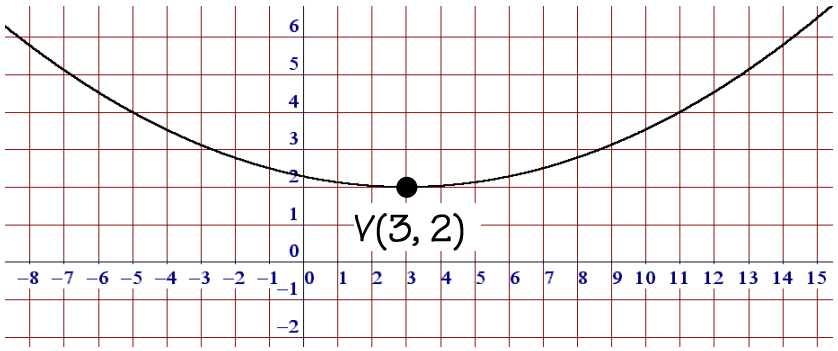
Focus: To find the focus we need the focal length. We know that the coefficient $\frac{1}{32}$ must equal $\frac{1}{4p}$. Thus,

$$\frac{1}{4p} = \frac{1}{32} \;\Rightarrow\; 4p = 32 \;\Rightarrow\; p = 8$$

Since $p > 0$, we know that the parabola opens up. And the focus must therefore be 8 units above the vertex. Since the vertex is at (3, 2), **the focus is at (3, 10)**.

Directrix: In a parabola which opens up, the directrix is the horizontal line p units below the vertex. So, the line which is 8 units below the vertex (3, 2) is ***y* = –6**.

Graph: Plotting the vertex, then the *y*-intercept, approximately (0, 2.3), and a few other points yields the following graph:



EXAMPLE 7: **Determine the vertex, the focus and the directrix of the parabola $y = -4(x+1)^2 - 3$.**

Solution: Since there's a constant of +1 to the right of the x and a –3 hanging off the end of the function, the vertex is **(–1, –3)**.

To locate the focus, we need the focal length:

$$\frac{1}{4p} = -4 \;\Rightarrow\; 1 = -16p \;\Rightarrow\; p = -\frac{1}{16}$$

Since $p < 0$, we know that the parabola opens down — and the focus must lie $\frac{1}{16}$ of a unit below the vertex. Since the vertex is (–1, –3), the focus is $\mathbf{\left(-1, -3\frac{1}{16}\right)}$.

The directrix is $\frac{1}{16}$ of a unit above the vertex. The directrix is therefore the horizontal line $\mathbf{y = -2\frac{15}{16}}$.

Homework

9. Find the vertex, the focus, and the directrix of each parabola:

a. $y = 4x^2 + 5$

b. $y = -\frac{1}{4}(x+2)^2$

c. $y = -2(x-1)^2 - 3$

d. $y = \frac{1}{2}(x-5)^2 + 8$

e. $y = -\frac{1}{12}(x+3)^2 - 10$

f. $y = \frac{1}{20}(x-3)^2 + 8$

❑ *NEW FORM OF THE PARABOLA*

Consider the parabola $y = 2x^2 + 12x + 13$. We know it's a parabola because one variable is squared and the other isn't. But it doesn't have any parentheses in it — that is, it doesn't look anything like the

parabolas in the previous section. So how can we find the vertex, focus, and directrix of the parabola in this form?

EXAMPLE 8: **Determine the vertex, the focus and the directrix of the parabola $y = 2x^2 + 12x + 13$.**

Solution: To make the given parabola look like the ones in the previous section, we proceed as follows.

Start with the given parabola:

$$y = 2x^2 + 12x + 13$$

Factor 2 out of the $2x^2$ and the $12x$. Don't worry about the 13 yet:

$$y = 2(x^2 + 6x) + 13$$

Complete the Square in the inner parentheses:

1) take half of 6 to get 3
2) square the 3 to get **9**

Add the "magic number" 9 to the inside of the parentheses. This will produce $x^2 + 6x + 9$, a perfect square trinomial which factors into $(x + 3)^2$. However, how do we justify this adding of 9 to the quantity inside the parentheses?

The first issue to consider is that we did not really add 9 to the right side of the equation. Noting that the 9 we added is actually being multiplied by the 2 in front of the parentheses, we actually added **18** to the right side of the equation.

The second issue is exactly how to balance the adding of 18 to the right side of the equation. The quickest way is to also subtract 18 from the right side of the equation (since adding and subtracting 18 is really doing nothing — it's just adding 0 to the right side of the equation). Let's do it:

$$y = 2(x^2 + 6x \boxed{+9}) + 13 \boxed{-18}$$

We actually added 18

This balances the adding of 18

Now factor and then do the arithmetic:

$$y = 2(x+3)^2 - 5$$

It finally looks the way it ought to. We easily see that **the vertex is (–3, –5)**, and the focal length can be calculated as

$$\frac{1}{4p} = 2 \;\Rightarrow\; 1 = 8p \;\Rightarrow\; \boldsymbol{p = \frac{1}{8}}$$

Since $p > 0$, the parabola opens up, and therefore the focus is $\frac{1}{8}$ of a unit above the vertex, making **the focus (–3, $-\frac{39}{8}$)**. The directrix is $\frac{1}{8}$ of a unit below the vertex, and so **the directrix is** $\boldsymbol{y = -\frac{41}{8}}$.

EXAMPLE 9: **Determine the vertex, the focus and the directrix of the parabola** $\boldsymbol{y = -\frac{1}{4}x^2 + \frac{5}{2}x - \frac{1}{4}}$.

Solution: Completing the square will lead the way to a solution.

$$y = -\frac{1}{4}x^2 + \frac{5}{2}x - \frac{1}{4} \qquad \text{(the given parabola)}$$

$$\Rightarrow \quad y = -\frac{1}{4}(x^2 - 10x) - \frac{1}{4} \qquad \text{(factor)}$$

[One way to find the –10 is to divide $\frac{5}{2}$ by $-\frac{1}{4}$]

$$\Rightarrow \quad y = -\frac{1}{4}(x^2 - 10x \boxed{+25}) - \frac{1}{4}\boxed{+\frac{25}{4}} \qquad \text{(complete the square)}$$

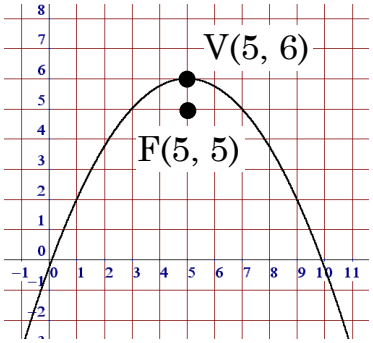
[Note: $-\frac{1}{4}\times 25 = -\frac{25}{4}$; this is balanced by adding $\frac{25}{4}$.]

$\Rightarrow \quad y = -\frac{1}{4}(x-5)^2 + 6$ (factor & arithmetic)

We now have the parabola in the form we like, from which we deduce that **the vertex is (5, 6)**. The focal length is calculated as:

$$\frac{1}{4p} = -\frac{1}{4} \Rightarrow -4p = 4 \Rightarrow \boldsymbol{p = -1},$$

and so the parabola opens down. **The focus is (5, 5)**, 1 unit below the vertex, and **the directrix is $y = 7$**, 1 unit above the vertex.



Homework

10. For each parabola, determine whether the parabola opens up or down, and then compute its focal length.

a. $y = \frac{1}{8}x^2 + \frac{3}{4}x - \frac{31}{8}$ b. $6y = x^2 - 8x + 22$

c. $x^2 + 4x + 4y + 16 = 0$ d. $y = -x^2 + 10x - 21$

11. Find the vertex, the focus, and the directrix of each parabola:

a. $5y - 54 = x^2 + 14x$ b. $x^2 + 2x + 3y + 16 = 0$

c. $y - x^2 + 8x - 9 = 0$ d. $y = -\frac{1}{9}x^2 + \frac{20}{9}x - \frac{73}{9}$

12. Find the vertex, the focus, and the directrix of each parabola:

a. $(x-3)^2 = 20(y+2)$ b. $(x+5)^2 = -8(y-1)$

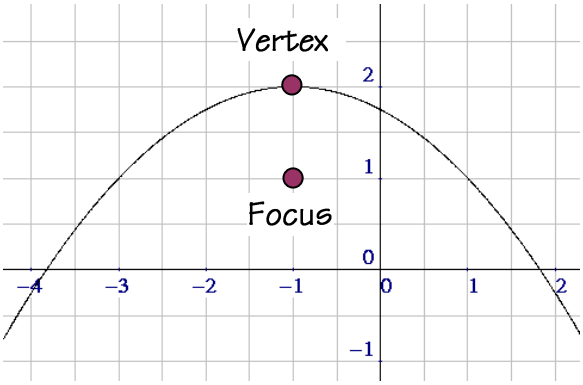
c. $x^2 + 2x + 4y - 15 = 0$ d. $y = 2x^2 - 8x + 7$

13. Graph the parabola $y = \dfrac{x^2 - 8x - 8}{12}$. Label the vertex and the focus. Also give the y-intercept and use the Quadratic Formula to estimate the x-intercepts.

14. Find the equation of the parabola given that

a. its vertex is $(2, -7)$ and $p = 5$.
b. its focus is $(-3, 4)$ and $p = -3$.
c. its vertex is $(-3, 1)$ and its focus is $(-3, 7)$.
d. its vertex is $(4, -1)$ and its focus is $(4, -2)$.

15. Find the equation of the following parabola, and state the equation of the directrix:



Review Problems

16. Give the geometric definition of the parabola.

17. Suppose R is a point on a parabola, and assume that the distance between R and the focus is 250. What is the distance between R and the directrix?

18. The vertex of a parabola is V(3, –10) and the focal length is $p = -4$. Calculate the focus and the directrix of the parabola.

19. Find the focal length of $y = -\frac{7}{8}x^2$.

20. Consider the parabola $y = -9(x+7)^2 - 20$.
    a. Does it open up or down?
    b. Is it skinnier or wider than the $y = x^2$ parabola?
    c. Find the vertex.

21. Give a full analysis of the parabola $y = \frac{1}{24}(x-3)^2 - 5$.

22. Find the vertex and the focal length of the parabola $y = -3x^2 - 24x + 8$.

23. a. Let $y = -\frac{1}{12}x^2 + \frac{1}{3}x - \frac{13}{3}$. Find the domain and all the intercepts.
    b. Find the vertex, the focus, and the directrix.

24. Find the equation of the parabola with vertex (–2, 1) and focus (–2, 7).

25. Derive the parabola formula $y = \frac{1}{4p}x^2$.

26. True/False:

a. A parabola is the set of all points in the plane such that the sum of the distances to a point and a line is a constant.

b. A parabola is the set of all points in the plane that are equidistant from a point and a line.

c. All parabolas are functions.

d. If the vertex of a parabola is V(2, 3) and the focal length is $p = 5$, then the focus is F(2, 8).

e. The focus is always midway between the vertex and the directrix.

f. In the standard parabola formula $y = \frac{1}{4p}x^2$, the p represents the focal length.

g. The derivation of the formula above is based upon the Distance Formula.

h. The focal length of the parabola $y = x^2$ is 1/2.

i. The vertex of the parabola $y = x^2$ is the origin.

j. The vertex of the parabola $y = 2(x - 3)^2 + 4$ is V(–3, 4).

k. The parabola above has no x-intercepts.

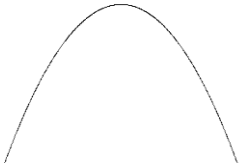
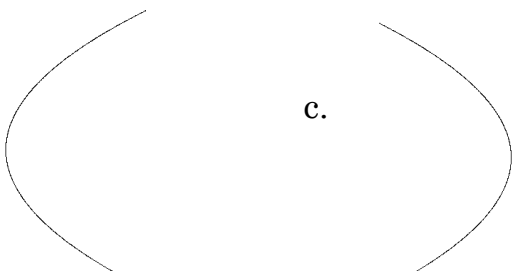
l. The vertex of the parabola $y = 2x^2 + 12x + 13$ is V(–3, –5).

m. The parabola $x = -y^2$ opens down.

n. The graph of $y^2 = 2(x - 3)^2 + 7$ is a parabola.

Solutions

1. a. Focus b. Directrix

2. a.  b.  c.

d. Hint: Look at the chapter title

3. a. Up b. Down

4. 197

5. In order for a point to be on the parabola, the distance between that point and the focus must equal the distance from that point to the directrix. Since the given distances don't match, the point cannot possibly be on the parabola.

6. a. F(0, 3); $y = -3$ b. F(0, –7); $y = 7$ c. F(2, 8); $y = -2$
d. F(2, –2); $y = 8$ e. F(–5, 3); $y = -1$ f. F(–3, –8); $y = -6$

7. a. V(2, 2) b. V(3, 8) c. V(–8, –1) d. V(0, –20)

8.

a.	V(0, 0)	F(0, $-\frac{1}{4}$)	$y = \frac{1}{4}$
b.	V(0, 0)	F(0, 3)	$y = -3$
c.	V(0, 0)	F(0, –4)	$y = 4$
d.	V(0, 0)	F(0, $\frac{1}{12}$)	$y = -\frac{1}{12}$
e.	V(0, 0)	F(0, $-\frac{1}{16}$)	$y = \frac{1}{16}$

f. A line has no vertex, focus, or directrix

9.

a.	V(0, 5)	F(0, $5\frac{1}{16}$)	$y = 4\frac{15}{16}$
b.	V(–2, 0)	F(–2, –1)	$y = 1$
c.	V(1, –3)	F(1, $-3\frac{1}{8}$)	$y = -2\frac{7}{8}$
d.	V(5, 8)	F(5, $8\frac{1}{2}$)	$y = 7\frac{1}{2}$
e.	V(–3, –10)	F(–3, –13)	$y = -7$
f.	V(3, 8)	F(3, 13)	$y = 3$

10. a. Up; 2 b. Up; $\frac{3}{2}$ c. Down; -1 d. Down; $-\frac{1}{4}$

11. a. V$(-7, 1)$ F$(-7, \frac{9}{4})$ $y = -\frac{1}{4}$

b. V$(-1, -5)$ F$(-1, -\frac{23}{4})$ $y = -\frac{17}{4}$

c. V$(4, -7)$ F$(4, -\frac{27}{4})$ $y = -\frac{29}{4}$

d. V$(10, 3)$ F$(10, \frac{3}{4})$ $y = \frac{21}{4}$

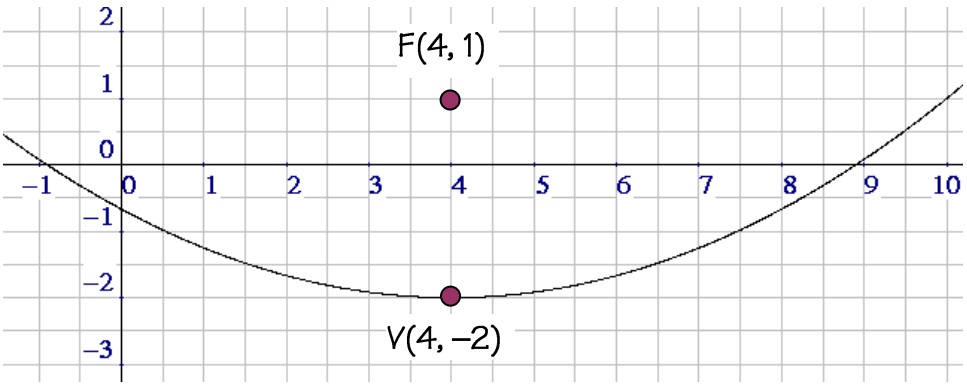
12. a. V$(3, -2)$ F$(3, 3)$ $y = -7$

b. V$(-5, 1)$ F$(-5, -1)$ $y = 3$

c. V$(-1, 4)$ F$(-1, 3)$ $y = 5$

d. V$(2, -1)$ F$(2, -\frac{7}{8})$ $y = -\frac{9}{8}$

13.



The y-intercept is $(0, -\frac{2}{3})$.

The x-intercepts are approximately at $(8.9, 0)$ and $(-0.9, 0)$.

14. a. $y = \frac{1}{20}(x-2)^2 - 7$ b. $y = -\frac{1}{12}(x+3)^2 + 7$

c. $y = \frac{1}{24}(x+3)^2 + 1$ d. $y = -\frac{1}{4}(x-4)^2 - 1$

15. $y = -\frac{1}{4}(x+1)^2 + 2$; The directrix is $y = 3$.

16. The set of all points in the plane which are equidistant from a given point and a given line.

17. 250 **18**. F(3, –14) directrix: $y = -6$ **19**. $p = -\frac{2}{7}$

20. a. Down b. Skinnier c. V(–7, –20)

21. y-intercept: setting $x = 0$ gives

$$y = \frac{1}{24}(-3)^2 - 5 = \frac{9}{24} - 5 = \frac{3}{8} - 5 = -\frac{37}{8}.$$

The y-intercept is $\left(0, -\frac{37}{8}\right)$.

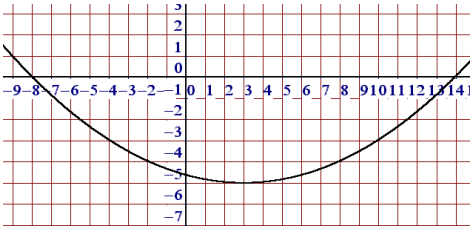
x-intercepts: setting $y = 0$ gives

$$\frac{1}{24}(x-3)^2 - 5 = 0 \Rightarrow \frac{1}{24}(x-3)^2 = 5 \Rightarrow (x-3)^2 = 120$$

$$\Rightarrow x - 3 = \pm\sqrt{120} \Rightarrow x = 3 \pm 2\sqrt{30}.$$

The x-intercepts are $(3 \pm 2\sqrt{30},\ 0)$.

The parabola opens up. The vertex is V(3, –5).
The focal length is $p = 6$. The focus is F(3, 1).
The directrix is $y = -11$.



22. $y = -3x^2 - 24x + 8 = -3(x^2 + 8x) + 8 = -3(x^2 + 8x + 16) + 8 + 48$
$= -3(x + 4)^2 + 56$

The vertex is (–4, 56) and the focal length is $p = -\frac{1}{12}$.

23. a. Domain = $\mathbb{R}$; no x-intercepts; y-int: $(0, -\frac{13}{3})$

b. V(2, –4); F(2, –7); directrix: $y = -1$

24. $y = \frac{1}{24}(x+2)^2 + 1$

25. It's in the book.

26. a. F b. T c. F d. T e. F f. T g. T
h. F i. T j. F k. T l. T m. F n. F

Consider the postage stamp.
It secures success through its ability
to stick to one thing till it gets there.

Josh Billings

CONSECUTIVE INTEGERS – LINEAR

❑ *INTRODUCTION*

Consecutive Integers

An ***integer*** is either a positive whole number, zero, or a negative whole number; in other words, it's the collection of numbers:

. . . –4, –3, –2, –1, 0, 1, 2, 3, 4 . . . The set of ***integers***

An example of four **consecutive integers** might be **11, 12, 13, 14**. Notice that consecutive integers differ by 1 — each integer is 1 more than the previous integer (assuming they're listed in ascending order). Better yet, we can think of the 12, the 13, and the 14 in the following way:



Start with **11** (the smallest of the 4 numbers)
$11 + \underline{1} = \mathbf{12}$
$11 + \underline{2} = \mathbf{13}$
$11 + \underline{3} = \mathbf{14}$

This means that if we have four unknown consecutive integers, and if we call the smallest one x, then the next one would be written $x + 1$, and the one after that, $x + 2$, and the largest one, $x + 3$.

x
$x + 1$
$x + 2$
$x + 3$

Four **consecutive** integers

Consecutive *Even* Integers

An example of three **consecutive <u>even</u> integers** might be **8, 10, 12**. Notice that consecutive even integers differ by 2 and that we usually write them in ascending order (smallest to largest).

Thus, in a sequence of consecutive even integers, each number after the first number is 2 larger than the preceding number. So, assuming the smallest of them is called x, they could be written

x
$x + 2$ Three **consecutive even** integers
$x + 4$

Consecutive *Odd* Integers

An example of five **consecutive <u>odd</u> integers** might be **–11, –9, –7, –5, –3**. Notice that consecutive odd integers differ by 2. Now comes a curious little fact: If n is an odd integer, then the odd integer that comes right after n must be $n + 2$, since they must differ by 2. (For example, –11 is odd, and the next odd integer is –11 **+ 2** = –9. ✓) Therefore, if x represents the smallest of five consecutive odd integers, for instance, they could be denoted like this:

x
$x + 2$
$x + 4$ Five **consecutive odd** integers
$x + 6$
$x + 8$

❑ *EXAMPLES*

<u>EXAMPLE 1:</u> **Four consecutive odd integers have a sum of 32. Find the four numbers.**

<u>Solution:</u> If we let n represent the smallest of the four consecutive odd integers, then the four integers can be represented like this: n, $n + 2$, $n + 4$, and $n + 6$. Since the given condition is that their sum must be 32, the following equation seems appropriate:

$$n + (n + 2) + (n + 4) + (n + 6) = 32$$

Dropping the parentheses (which weren't really necessary):

$$4n + 12 = 32 \Rightarrow 4n = 20 \Rightarrow n = 5$$

Since n, the smallest of the four numbers, is 5, our final answer is

5, 7, 9, 11

EXAMPLE 2: **Find three consecutive integers such that the largest is 18 less than 3 times the smallest.**

Solution: We can represent the three consecutive integers with a, $a + 1$, and $a + 2$. Is it clear that a is the smallest and $a + 2$ is the largest? The relationship between the largest and the smallest, as stated in the problem, can be transformed into the following equation:

$$a + 2 = 3a - 18$$

$$\Rightarrow \quad -2a = -20$$

$$\Rightarrow \quad a = 10$$

Since the smallest of the three consecutive integers is 10, the three numbers must be

10, 11, 12

EXAMPLE 3: **If the mean (average) of five consecutive even integers is 2, find the largest of the integers.**

Solution: We can represent the five consecutive even integers with x, $x + 2$, $x + 4$, $x + 6$, and $x + 8$. Recall that the mean of five numbers is found by dividing the sum of the numbers by 5:

$$\frac{x+(x+2)+(x+4)+(x+6)+(x+8)}{5} = 2$$

$\Rightarrow \quad \frac{5x+20}{5} = 2$ (combine like terms)

$\Rightarrow \quad 5x + 20 = 10$ (multiply each side by 5)

$\Rightarrow \quad 5x = -10$ (subtract 20 from each side)

$\Rightarrow \quad x = -2$ (divide each side by 5)

Since $x = -2$, and the largest of the five integers is $x + 8$, it follows that the largest integer is $-2 + 8 =$ $\boxed{-6}$

Homework

1. Find three consecutive integers whose sum is 57.
2. Find four consecutive <u>even</u> integers whose sum is 52.
3. Find three consecutive <u>odd</u> integers whose sum is 39.
4. Find three consecutive integers such that the sum of the smallest and largest is 100.
5. Find five consecutive even integers such that the largest is 3 times the smallest.
6. Find four consecutive odd integers such that the largest is 3 less than twice the smallest.
7. Find three consecutive integers such that the largest number is twice the middle number.
8. Find four consecutive even integers such that twice the smallest plus 4 times the largest equals 0.

9. Find three consecutive even integers such that the sum of the smallest and the largest is 20.

10. Find five consecutive odd integers such that the largest is 4 less than 5 times the smallest.

11. The mean (average) of four consecutive integers is 8.5. Find the 4 numbers. Be sure you write and solve an equation.

12. Find three consecutive odd integers whose mean is 21.

13. Find five consecutive even integers whose mean is 0.

Solutions

1. 18, 19, 20

2. 10, 12, 14, 16

3. 11, 13, 15

4. 49, 50, 51

5. 4, 6, 8, 10, 12

6. 9, 11, 13, 15

7. 0, 1, 2

8. –4, –2, 0, 2

9. 8, 10, 12

10. 3, 5, 7, 9, 11

11. 7, 8, 9, 10

12. 19, 21, 23

13. –4, –2, 0, 2, 4

"Education is the passport to the future, for tomorrow belongs to those who prepare for it today."



Malcolm X (1925–1965)

CONSECUTIVE INTEGERS – QUADRATIC

❑ *INTRODUCTION*

Recall: Consecutive Integers: $n,\ n+1,\ n+2,\ n+3,\ \ldots$

Consecutive Even Integers: $n,\ n+2,\ n+4,\ n+6,\ \ldots$

Consecutive Odd Integers: $n,\ n+2,\ n+4,\ n+6,\ \ldots$

If this layout above is unclear, check out Consecutive Integers — Linear.

Note: This chapter's problems will result in quadratic equations, all of which can be solved by the factoring method. But if you've learned the Quadratic Formula, you might use that method to give yourself extra practice.

❑ *EXAMPLES*

EXAMPLE 1: **Find four consecutive <u>negative</u> even integers such that the <u>product</u> of the smallest and the largest is 40.**

<u>Solution:</u> If we let n represent the smallest of the four even integers, then the four integers are n, $n + 2$, $n + 4$, and $n + 6$. The problem then stipulates that

$$n(n+6) = 40$$

$$\Rightarrow \quad n^2 + 6n = 40$$

$\Rightarrow \quad n^2 + 6n - 40 = 0$

$\Rightarrow \quad (n + 10)(n - 4) = 0$

$\Rightarrow \quad n = -10 \text{ OR } n = 4$

Since the problem is asking for negative integers, we'll use the solution $n = -10$, thus giving us our four integers:

$$\boxed{-10, -8, -6, -4}$$

EXAMPLE 2: **Find three consecutive positive odd integers such that the product of the two largest integers is 13 more than 10 times the smallest.**

Solution: Letting n, $n + 2$, and $n + 4$ represent the integers, we get

$(n + 2)(n + 4) = 10n + 13$

$\Rightarrow \quad n^2 + 6n + 8 = 10n + 13$

$\Rightarrow \quad n^2 - 4n - 5 = 0$

$\Rightarrow \quad (n - 5)(n + 1) = 0$

$\Rightarrow \quad n = 5 \text{ OR } n = -1$

Taking the positive solution, $n = 5$, we get our integers:

$$\boxed{5, 7, 9}$$

Homework

1. Find five consecutive positive integers such that the product of the two largest integers is 54 more than the sum of the first three integers.

Solutions

1. 5, 6, 7, 8, 9

“Change is the result of all true learning.”

Leo Buscaglia

CONSTANTS – FINDING THE NUMBERS

❑ *FINDING A PAIR OF NUMBERS*

To succeed in the homework in this section, you need only the following four math terms:

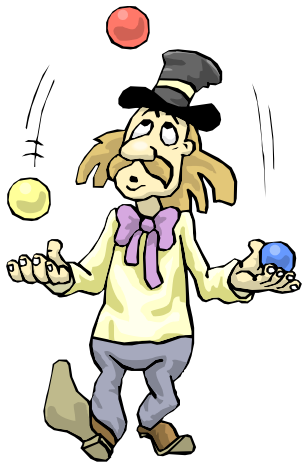
The ***sum*** of two numbers is the result of adding them.

The ***difference*** of two numbers is the result of subtracting them.

The ***product*** of two numbers is the result of multiplying them.

The ***quotient*** of two numbers is the result of dividing them.

EXAMPLE 1: **Find two numbers with a sum of 11 and a difference of 5.**



Solution: What? How can we have an algebra problem with unknown values on the first page of an algebra book? Here's how: We don't care about algebra yet. Let's use our knowledge of the words *sum* and *difference* and a little trial and error (guessing!).

Well, 10 and 1 have a sum of 11, so that's a possibility:

$10 + 1 = 11$ ✓

$10 - 1 = 9$, not 5, like the problem requires.

We didn't get the right two numbers, so let's try 7 and 4:

$7 + 4 = 11$ ✓

$7 - 4 = 3$, but the difference is supposed to be 5.

Consider 8 and 3:

$8 + 3 = 11$ ✓

$8 - 3 = 5$, as required by the problem!

In short, the numbers 8 and 3 have a sum of 11 and a difference of 5. Both conditions of the problem have been met, and so our final answer is the pair of numbers



8 & 3

"Only education is capable of saving our societies from possible collapse, whether violent, or gradual."

– Jean Piaget

Homework

1. Find two numbers with a product of 20 and a sum of 12.
2. Find two numbers with a sum of 2 and a difference of 0.
3. Find two numbers with a product of 12 and a difference of 4.
4. Find two numbers with a product of 7 and a sum of 8.
5. Find two numbers with a difference of 9 and a sum of 11.
6. Find two numbers with a product of 42 and a sum of 13.

7. Find two numbers with a difference of 7 and a product of 18.
8. Find two numbers with a product of 16 and a difference of 6.
9. Find two numbers with a difference of 2 and a product of 48.
10. Find two numbers with a sum of 8 and a difference of 0.
11. Find two numbers with a difference of 14 and a quotient of 8.
12. Find two numbers with a product of 16 and a sum of 8.
13. Find two numbers with a sum of 19 and a product of 90.
14. Find two numbers with a difference of 4 and a sum of 10.
15. Find two numbers with a quotient of 2 and a product of 50.
16. Find two numbers with a sum of 11 and a difference of 7.
17. Find two numbers with a difference of 9 and a quotient of 2.
18. Find two numbers with a sum of 3 and a difference of 1.
19. Find two numbers with a product of 12 and a sum of 7.
20. Find two numbers with a quotient of 4 and a difference of 27.

❑ *I'M THINKING OF A NUMBER*

I'm thinking of a number. If I add 10 to the number, the result is 45. What number was I thinking of ?

Our goal in this section is to solve problems like the one above using just our common sense. We want to "reason" it out using just a bit of logic. Can you find the number? The number I was thinking of was **35**. Next, we need some more terminology.

To ***double*** a number means to multiply it by **2**.

To ***triple*** a number means to multiply it by **3**.

To ***quadruple*** a number means to multiply it by **4**.

To ***quintuple*** a number means to multiply it by **5**.

There's no process that I intend to teach you at this point in the course, so on to the homework. One hint: If you see a phrase like "*subtract 7 from 20*," it means 20 – 7 = 13. [Not the other way around; that would result in –13, which is *20 subtracted from 7*.]

Homework

21. Solve each of the following "*I'm thinking of a number*" problems using whatever reasoning you like:



a. If I multiply the number by 2, the result is 24.
b. If I subtract 7 from the number, the result is 20.
c. If I divide the number by 5, the result is 10.
d. If I add 19 to the number, the result is 100.
e. If I multiply the number by 3, the result is 24.
f. If I subtract 17 from the number, the result is 20.
g. If I divide the number by 6, the result is 12.
h. If I add 29 to the number, the result is 50.
i. If I multiply the number by 20, the result is 300.
j. If I subtract 27 from the number, the result is 40.
k. If I divide the number by 9, the result is 8.
l. If I add 199 to the number, the result is 200.
m. If I double the number, the result is 34.
n. If I triple the number, the result is 72.
o. If I quadruple the number, the result is 300.
p. If I quintuple the number, the result is 95.



22. I'm thinking of a number . . .

    a. If I triple the number and then add 4, the result is 34.

    b. If I multiply the number by 7, and then subtract 8, the result is 20.

    c. If I add 3 to the number, and then double that result, the final result is 16.

    d. If I subtract 2 from the number, and then triple that result, the final result is 99.

    e. If I quintuple the number and then add 7, the result is 57.

    f. If I quadruple the number, and then subtract 8, the result is 40.

    g. If I add 13 to the number, and then double that result, the final result is 100.

    h. If I subtract 20 from the number, and then triple that result, the final result is 51.

    i. If I add 11 to the number, and then quadruple that result, the final result is 200.

    j. If I subtract 22 from the number, and then quintuple that result, the final result is 75.

Review Problems

23. Find two numbers with a difference of 0 and a product of 9.

24. Find two numbers with a product of 16 and a quotient of 4.

25. Find two numbers with a sum of 7 and a product of 10.



26. Find two numbers with a quotient of 3 and a difference of 14.
27. Find two numbers with a quotient of 4 and a product of 4.
28. Find two numbers with a quotient of 5 and a sum of 36.
29. Find two numbers with a quotient of 4 and a sum of 10.
30. I'm thinking of a number. If 3 is added to the number, and then that result is doubled, the final answer is 14. What is the number?
31. I'm thinking of a number. If 4 is subtracted from the number, and then that result is multiplied by 6, the final answer is 18. What is the number?

> ***"The only real mistake is the one from which we learn nothing."***
>
> *– John Powell*

32. I'm thinking of a number. If 2 is added to the number, and then that result is divided by 4, the final answer is 3. What is the number?
33. I'm thinking of a number. If I double the number and then add 8, the final result will be 10. What is the number?
34. I'm thinking of a number. If I divide the number by 3 and then add 5, the final result will be 10. What is the number?
35. I'm thinking of a number. If 3 is subtracted from the number, and then that result is divided by 7, the final answer is 2. What is the number?
36. I'm thinking of a number. If I divide the number by 7 and then subtract 5, the final result will be 3. What is the number?
37. I'm thinking of a number. If the number is tripled, and then 9 is subtracted, the final result will be 15. What is the number?

Solutions

1. 2 & 10
2. 1 & 1
3. 2 & 6
4. 7 & 1
5. 1 & 10
6. 6 & 7
7. 9 & 2
8. 8 & 2
9. 8 & 6
10. 4 & 4
11. 16 & 2
12. 4 & 4
13. 10 & 9
14. 3 & 7
15. 10 & 5
16. 2 & 9
17. 18 & 9
18. 1 & 2
19. 4 & 3
20. 36 & 9
21. a. 12 b. 27 c. 50 d. 81 e. 8
    f. 37 g. 72 h. 21 i. 15 j. 67
    k. 72 l. 1 m. 17 n. 24 o. 75
    p. 19
22. a. 10 b. 4 c. 5 d. 35 e. 10
    f. 12 g. 37 h. 37 i. 39 j. 37
23. 3 & 3
24. 8 & 2
25. 2 & 5
26. 21 & 7
27. 4 & 1
28. 30 & 6
29. 8 & 2
30. 4
31. 7
32. 10
33. 1
34. 15
35. 17
36. 56
37. 8

❑ *To ∞ and Beyond*

A. Find two numbers whose sum is 45 and whose product is 500.

B. Find two numbers whose difference is $\frac{1}{4}$ and whose product is $\frac{1}{8}$.

C. Find two numbers whose sum is 16 and whose difference is $1\frac{1}{3}$.

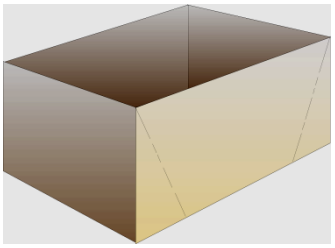
"Live as if you were to die tomorrow. Learn as if you were to live forever."

Mahatma Gandhi



CUBIC FUNCTIONS

You're given a square piece of cardboard, 8 cm on a side, and told to cut squares of equal size out of each corner and fold up the resulting flaps, all to create an open box (no top) with the *largest* possible volume. What size corners should you cut from the cardboard? Solving this problem involves a cubic function.

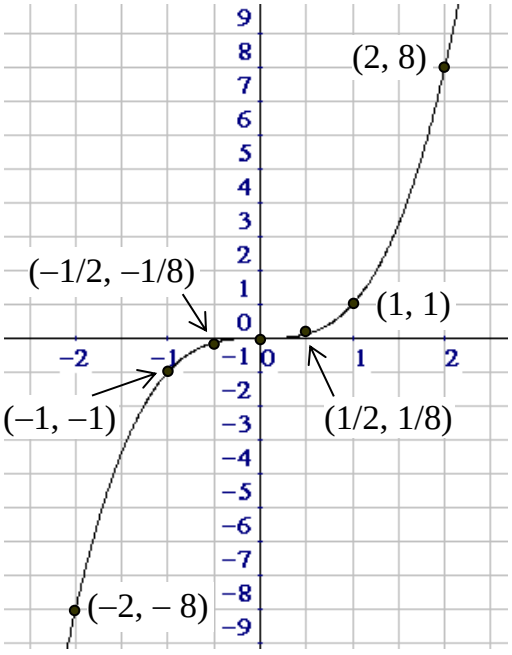


❑ *INTRODUCTORY EXAMPLE*

EXAMPLE 1: **Graph:** $y = x^3$

Solution: This is the classic cubic function — it's as simple as a cubic can be. The **domain** is $\mathbb{R}$ (since any number can be cubed without a problem). Confirm the values in the following x-y table.

x	y
–3	–27
–2	–8
–1	–1
$-\frac{1}{2}$	$-\frac{1}{8}$
0	0
$\frac{1}{2}$	$\frac{1}{8}$
1	1
2	8
3	27



Note that when $x = 0$, $y = 0$; and when $y = 0$, $x = 0$. Therefore, the only **intercept** is the origin. Also notice that the **range** of the function is $\mathbb{R}$.

Also, the graph of the function appears to have **origin symmetry**. To prove this, replace x with $-x$ and y with $-y$, giving

$$-y = (-x)^3 \quad \Rightarrow \quad -y = -x^3 \quad \Rightarrow \quad y = x^3, \text{ the original equation.}$$

In addition, **these limits** should be clear:

As $x \to \infty$, $y \to \infty$ As $x \to -\infty$, $y \to -\infty$

Homework

1. a. In Example 1, we claimed that the domain is $\mathbb{R}$, so we should be able to cube any real number. Cube each of these numbers: 10, π, $\sqrt[3]{7}$, $-\sqrt[6]{11}$.

   b. It's also the case that y can be any real number. Find an x-value that will yield the given y-value: 64, $\frac{1}{27}$, -125, π, $-e$. [e is a number we'll learn about later in the course.]

2. Graph $y = -x^3$. Remember, $-x^3$ means cube x first, and then attach the minus sign.

3. Graph $f(x) = x^3 + 2$ and $g(x) = x^3 - 3$, and then explain what effect the 2 and the -3 have on the graph of $y = x^3$.

4. Graph $f(x) = (x-1)^3$ and $g(x) = (x+4)^3$, and then explain what effect the -1 and the 4 have on the graph of $y = x^3$.

5. For the function $y = x^3$, verify the limit "As $x \to \infty$, $y \to \infty$" by finding the x-value needed to guarantee that y can be made as large as one billion.

6. What is the slope of the curve $y = x^3$ at the origin?

7. This problem will prepare us for an issue in the next section. Let's solve the quadratic equation $x^2 + 7x - 18 = 0$ by factoring.

$$\begin{aligned} & x^2 + 7x - 18 = 0 \\ \Rightarrow \quad & (x + 9)(x - 2) = 0 \\ \Rightarrow \quad & x + 9 = 0 \text{ or } x - 2 = 0 \\ \Rightarrow \quad & x = -9 \text{ or } x = 2 \end{aligned}$$

Notice that the solution $x = -9$ came from the factor $x + 9$, while the solution $x = 2$ came from the factor $x - 2$. So here's the question: What quadratic equation has the solutions $x = 7$ and $x = -5$? If 7 and -5 are solutions, then $x - 7$ and $x + 5$ must be factors: $(x - 7)(x + 5) = 0$, or $x^2 - 2x - 35 = 0$.

Find a quadratic equation whose solutions are given:

a. $x = 5, 10$ b. $x = -4, 9$ c. $x = -2, -11$

d. $x = 12$ e. $x = \pm 7$ f. $x = 0, 5$

❑ *INTERCEPTS AND GRAPHING*

EXAMPLE 2: **Graph: $y = x^3 - 9x$**

Solution: This is another cubic function whose **domain** is $\mathbb{R}$.

Let's analyze **intercepts**. If we let $x = 0$, then $y = 0$, and so the graph has a y-intercept at the origin. Setting y to 0 gives

$$0 = x^3 - 9x = x(x^2 - 9) = x(x + 3)(x - 3),$$

whose solutions are $x = 0$, $x = -3$, and $x = 3$. Thus the x-intercepts are the points (0, 0), (–3, 0), and (3, 0).

Let's check for **symmetry** — we'll try y-axis symmetry first. Replace x with $-x$ and we get $y = (-x)^3 - 9(-x) = -x^3 + 9x$. This is not the same formula as the original, so we do not have y-axis symmetry. But you can check that if we replace x with $-x$ and y

with $-y$, then we do get the same equation, and we can conclude that the graph has origin symmetry.

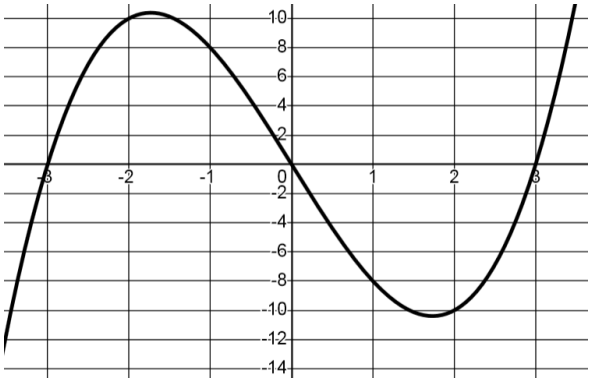
A few more calculations will give us the points

$$(-4, -28) \quad (-2, 10) \quad (-1, 8) \quad (1, -8) \quad (2, -10) \quad (4, 28).$$

Now let x be really big and really small, and you should be convinced of **these limits**:

$$\text{As } x \to \infty,\ y \to \infty \qquad \text{As } x \to -\infty,\ y \to -\infty$$

These points, along with the intercepts, the origin symmetry, and the limits, lead us to the following graph:



EXAMPLE 3: **Find all the x-intercepts of**

$$y = x^3 - x^2 - 14x + 24.$$

Hint: One of them is (2, 0).

Solution: Why the hint? To find the x-intercepts of any graph, we set $y = 0$. This yields the equation

$$x^3 - x^2 - 14x + 24 = 0$$

We can't use the Quadratic Formula since our equation is cubic, not quadratic. That leaves factoring as the only viable technique, but factoring a cubic is difficult.

It's time to take advantage of the hint: (2, 0) is one of the x-intercepts. This means that the point (2, 0) is on the graph of the cubic function, which implies that the coordinates $x = 2$ and $y = 0$ must satisfy the equation of the cubic function. Let's verify this:

$$\mathbf{0} = \mathbf{2}^3 - \mathbf{2}^2 - 14(\mathbf{2}) + 24$$
$$\Rightarrow \quad 0 = 8 - 4 - 28 + 24$$
$$\Rightarrow \quad 0 = 0 \quad \checkmark$$

Now back to the cubic equation above:

$$x^3 - x^2 - 14x + 24 = 0$$

Here's the clue we need to solve this equation: Since $x = 2$ is one of the solutions of this equation (verified above), then $x - 2$ must be one factor of the left side of the equation. [See the previous homework problem.]

To find the other factors, we'll divide $x^3 - x^2 - 14x + 24$ by $x - 2$, the details of which are left to you.

$$x - 2 \overline{)\, x^3 - x^2 - 14x + 24} \qquad x^2 + x - 12$$

Since the quotient is $x^2 + x - 12$ and the divisor is $x - 2$, we can write

$$x^3 - x^2 - 14x + 24 = (x^2 + x - 12)(x - 2)$$

By factoring the quadratic, $x^2 + x - 12$, we get the complete factorization. Here's the process from the beginning:

	$x^3 - x^2 - 14x + 24 = 0$	(the equation we're trying to solve)
$\Rightarrow$	$(x^2 + x - 12)(x - 2) = 0$	(from the long division)
$\Rightarrow$	$(x + 4)(x - 3)(x - 2) = 0$	(factor the quadratic)
$\Rightarrow$	$x = -4$ or $x = 3$ or $x = 2$	(set each factor to 0)

And we thus have all three x-intercepts:

(–4, 0)	(3, 0)	(2, 0)

Can you see that the y-intercept is (0, 24)?

EXAMPLE 4: **Consider the cubic function $y = x^3 + 2x^2 - x - 2$. Use the fact that one of the x-intercepts is (–1, 0) to find all the intercepts, and then graph the function.**

Solution: First, the **domain** of the function is $\mathbb{R}$. Second, as in the previous example, we realize that an x-intercept of (–1, 0) means that $x + 1$ is one factor of the cubic. Dividing $x + 1$ into $x^3 + 2x^2 - x - 2$ gives a quotient of $x^2 + x - 2$, which itself factors into $(x + 2)(x - 1)$. The complete factorization of the cubic function $y = x^3 + 2x^2 - x - 2$ is

$$y = (x + 1)(x + 2)(x - 1).$$

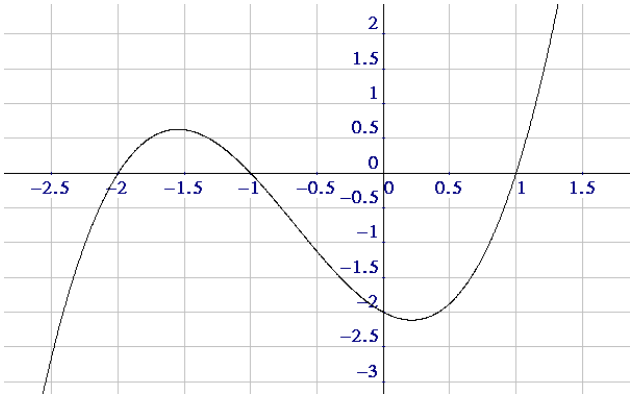
Setting y to 0 gives the following three ***x*-intercepts**:

(–1, 0) (–2, 0) (1, 0)

Setting x to 0 gives a ***y*-intercept** of (0, –2). You can verify (via calculator) that these points are on the graph, too:

(–2.25, –1.015625) (–1.5, 0.625) (0.25, –2.109375)

If you plot the three x-intercepts, the y-intercept, and the three points above, you'll get a graph like the following:



The graph does not seem to have any **symmetries**. Also, we note **these two limits**: As $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to -\infty$.

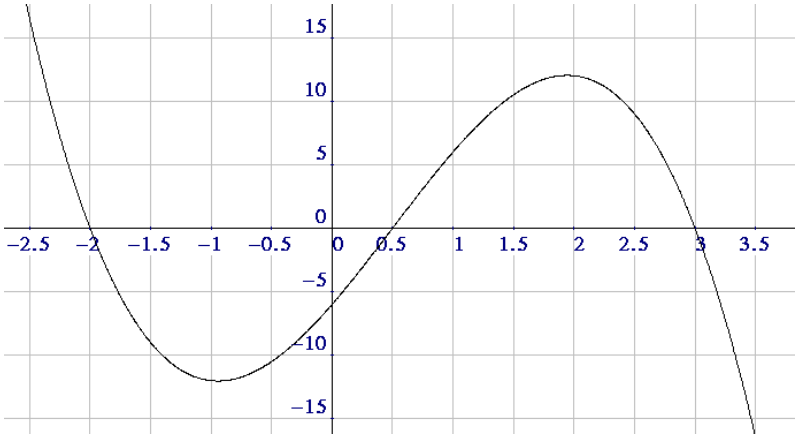
Postscript: It's impossible to accurately determine the *maximum point* (the top of the hill) in the second quadrant, or the *minimum point* (the bottom of the valley) in the fourth quadrant. Calculus is the subject where we find the tools needed to determine these two *extreme points* precisely.

EXAMPLE 5: **Find the intercepts and then sketch the graph of $y = -2x^3 + 3x^2 + 11x - 6$.**
Hint: One of the x-intercepts is (3, 0).

Solution: An x-intercept of (3, 0) implies that $x - 3$ is one factor of the cubic. Dividing the cubic by $x - 3$ yields a quotient of $-2x^2 - 3x + 2$, which factors into $-(2x - 1)(x + 2)$. When this product is set to 0, we find that two additional x-intercepts are $(\frac{1}{2}, 0)$ and $(-2, 0)$. When we throw in the given x-intercept, and then calculate the y-intercept, we get four intercepts:

$(-2, 0)$ $(\frac{1}{2}, 0)$ $(3, 0)$ $(0, -6)$

With these four intercepts and a few other points which I'll leave for you to plot, we get the following graph:



The limits for this graph are different from the previous examples: As $x \to \infty$, $y \to -\infty$, and as $x \to -\infty$, $y \to \infty$. Do you know what it is about the cubic equation which produces these limits?

Homework

8. Graph $y = x^3 - 4x$.

9. Find all the x-intercepts of $y = x^3 - 4x^2 - 7x + 10$. Hint: One of them is (1, 0).

10. In Example 4, perform the long division to verify the factorization of the cubic. Now verify the calculations of the three additional points on the graph. Finally, without referring to the graph, prove that the graph has no symmetries.

11. Graph $y = x^3 - 3x^2 + 2x$. Label all the intercepts clearly. Estimate the maximum and minimum points on the graph.

12. Graph $y = -x^3 + 3x - 2$. Hint: One of the x-intercepts is (–2, 0). Estimate the maximum and minimum points on the graph. As $x \to \infty$, $y \to$ _____. As $x \to -\infty$, $y \to$ _____.

13. Graph $y = x^3 - 2x^2 - 5x + 6$. Hint: (–2, 0) is an x-intercept.

14. Graph $y = -x^3 - 4x^2 - 4x$.
As $x \to \infty$, $y \to$ _____. As $x \to -\infty$, $y \to$ _____.

❑ *An Application of Cubic Functions*

Remember the box question described in this chapter's Introduction? Now we're ready to answer that query. You may recall from Gertrude's pigpen problem from a while back that a given amount of fence could

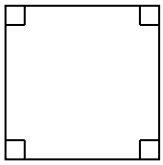
produce different areas, depending on what dimensions we chose for the rectangle. The same concept applies to the box problem. We'll begin with two specific scenarios which should convince you that we could get different volumes with the same square piece of cardboard.

EXAMPLE 6: **Start with a cardboard square 8 cm by 8 cm. First, cut out 1-cm squares from the corners and fold up the flaps to create an open box, and compute its volume. Second, use the original 8-cm square, but this time cut out 2-cm squares from the corners and compute the volume of the resulting box. Prove that the two boxes have <u>different</u> volumes.**

Solution: The volume of a box with dimensions l, w, and h is given by

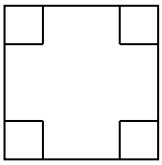
$$V = lwh$$

1-cm corners removed leaves a box 6 cm on each side and 1 cm high



$V = 6 \text{ cm} \times 6 \text{ cm} \times 1 \text{ cm}$
$= \mathbf{36 \text{ cm}^3}$

2-cm corners removed leaves a box 4 cm on each side and 2 cm high

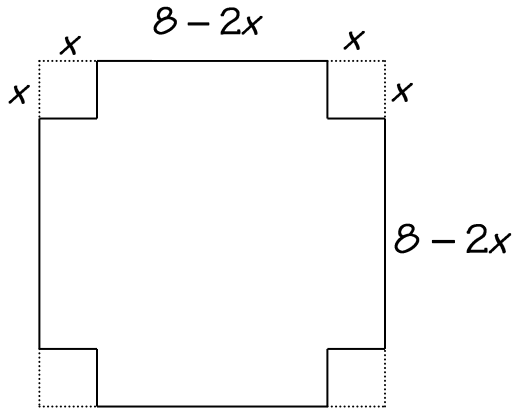


$V = 4 \text{ cm} \times 4 \text{ cm} \times 2 \text{ cm}$
$= \mathbf{32 \text{ cm}^3}$

Therefore, the boxes have different volumes, even though each box was created from identical 8-cm squares of cardboard. The next example will ask us to find the size of the cut-out corner that will produce the *maximum* volume. After all, given an 8-cm square of cardboard, we might as well get the most volume that we possibly can from the box.

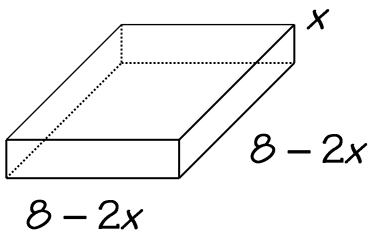
EXAMPLE 7: **An 8-cm square piece of cardboard is to be made into an open box by cutting squares from the corners and folding up the flaps. What size squares should be cut to achieve a box of maximum volume? What will the maximum volume be?**

Solution: Here's the plan of attack. We'll sketch the piece of cardboard with the corners removed, assuming that each square corner removed has dimensions of x cm by x cm. Then we'll write an expression that represents the length and width of the box which results from folding up the flaps. Since the height of the box is x cm, we'll be able to create a volume formula from the length, the width, and the height. Last, we'll graph the resulting volume function and estimate the value of x which produces the maximum volume.



The following are the dimensions of the box when the flaps are folded up:

length = $8 - 2x$
width = $8 - 2x$
height = x



And so the volume has the formula:

$$\begin{aligned} V &= lwh \\ &= (8-2x)(8-2x)x \\ &= (64-32x+4x^2)x \\ &= 4x^3-32x^2+64x \end{aligned}$$

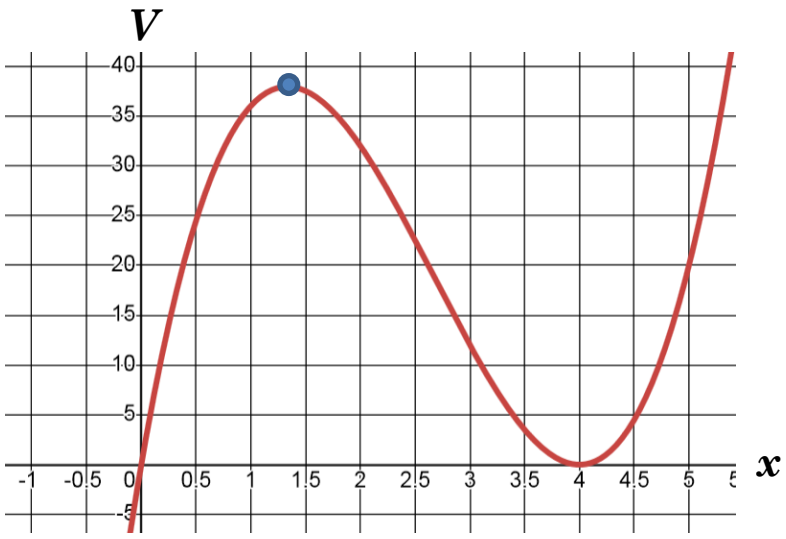
Therefore, the function we're trying to maximize is given by

$$V = 4x^3 - 32x^2 + 64x$$

Factoring produces

$$V = 4x(x-4)^2$$

which yields two x-intercepts, (0, 0) and (4, 0). Clearly we could plot lots of other points to get a decent graph, but let's just cut to the chase with the following graph:



Before we get to the goal of this problem, let's observe what the graph is telling us. It says that when $x = 0$, $V = 0$ (i.e., the graph passes through the origin). This makes sense because if we don't cut out any corners, we can't fold up the flaps, we get no box, and so obviously the volume is 0. By the same token, if we cut out flaps that are 4 cm long, there's no base left to the box, since it's only 8 cm to begin with. The graph indicates this situation at the point (4, 0).

Now to find the size of the squares to cut out and the maximum volume: Look at the top of the mountain — this point represents maximum volume. Going straight down to the x-axis, we estimate the x-coordinate to be about 1.3 cm. Now move left from the top of the mountain to the V-axis, and the volume appears to be about 38 cm^3. In summary,

> Cutting a 1.3cm square from each corner will produce a box with volume 38 cm^3.

Homework

15. Each of the following numbers is the length of the side of a piece of square cardboard. The cardboard square is to be made into an open box by cutting squares from the corners and folding up the flaps. What size square should be cut from each corner to achieve a box of maximum volume? Note that the solutions are approximations only. As long as your answers are in the ballpark (and you really understand what you're doing!), consider yourself correct.

    a. 2 b. 3 c. 4 d. 6

Review Problems

16. Let $y = 2x^3 - 4x^2 + 9$.

    a. Is it a function? Why?
    b. Why is it cubic?
    c. What's its domain?
    d. Find the y-intercept.
    e. What would you do to find the x-intercepts?

17. How does the graph of $g(x) = (x - 2)^3 + 7$ compare to the graph of $f(x) = x^3$?

18. Prove that the graph of $y = 2x^3 + 7x$ has origin symmetry.

19. Let $f(x) = 3x^3 + 4x - 7$ and $g(x) = -\pi x^3 + x^2 + 100$.

    a. As $x \to \infty$, $f(x) \to$ _____ b. As $x \to -\infty$, $f(x) \to$ _____

    c. As $x \to \infty$, $g(x) \to$ _____ d. As $x \to -\infty$, $g(x) \to$ _____

20. If one of the x-intercepts of the cubic graph $y = x^3 + 3x^2 - 61x - 63$ is (7, 0), find the other two intercepts.

21. Graph $y = -x^3 + 2x$. Be sure to discuss symmetry, intercepts, and domain. As $x \to \infty$, $y \to$ ___. As $x \to -\infty$, $y \to$ ___.

22. Graph $y = x^3 + 3x^2 - x - 3$. Hint: (1, 0) is one of the intercepts. Discuss symmetry, intercepts, and domain. As $x \to \infty$, $y \to$ ___. As $x \to -\infty$, $y \to$ ___.

23. A 5-cm square piece of aluminum is to be made into an open box by cutting squares from the corners and folding up the flaps. What size squares should be cut to achieve a box of maximum volume?

24. True/False:

    a. Both the domain and range of the function $y = x^3$ are $\mathbb{R}$.

    b. The slope of the curve $y = x^3$ at the origin is about 1.

    c. The curve $y = x^3$ never touches the 4th quadrant.

    d. For the function $y = x^3$, as $x \to -\infty$, $y \to 0$.

    e. The graphs of $f(x) = (-x)^3$ and $g(x) = -x^3$ are identical.

    f. The graph of $y = x^3 - 16x$ has three x-intercepts.

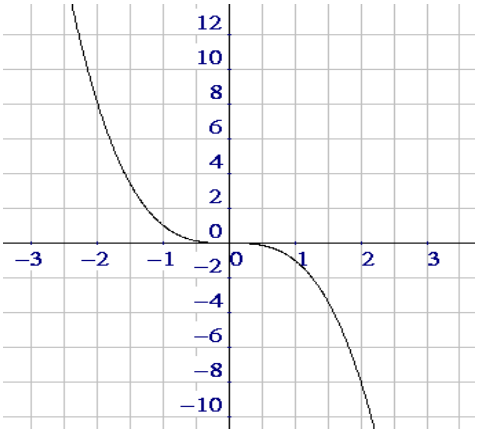
    g. The graph of $y = x^3 - 25x$ has its minimum point in Quadrant II.

    h. If you know one of the x-intercepts of a cubic function, long division will help you find the others.

Solutions

1. a. $1000,\ \pi^3,\ 7,\ -\sqrt{11}$ b. $4,\ \frac{1}{3},\ -5,\ \sqrt[3]{\pi},\ \sqrt[3]{-e}$

2.



3. The graph of f is x^3 shifted up 2 units.
The graph of g is x^3 shifted down 3 units.

4. The graph of f is x^3 shifted right 1 unit.
The graph of g is x^3 shifted left 4 units.

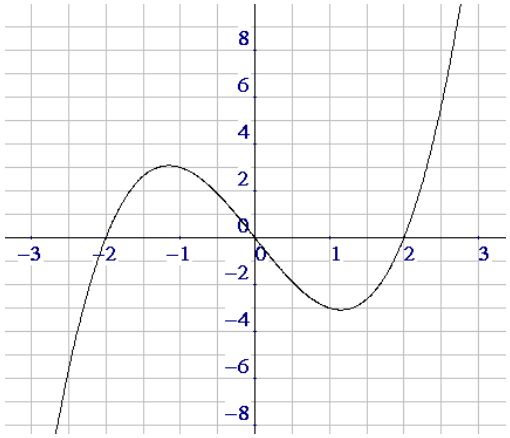
5. Make $x \geq 1000$ and y will be $\geq 1{,}000{,}000{,}000$.

6. I'd like to know your opinion.

7. a. $(x-5)(x-10) = 0 \Rightarrow x^2 - 15x + 50 = 0$
b. $x^2 - 5x - 36 = 0$
c. $x^2 + 13x + 22 = 0$
d. Only one solution is given, but we're required to provide a quadratic equation; so we must use the solution $x = 12$ twice:
$(x-12)(x-12) = 0$, or $x^2 - 24x + 144 = 0$
e. $x^2 - 49 = 0$
f. $x(x-5) = 0 \Rightarrow x^2 - 5x = 0$

Any of the quadratic equations here could be multiplied by any nonzero constant and still have the same solutions. I've chosen the simplest answers.

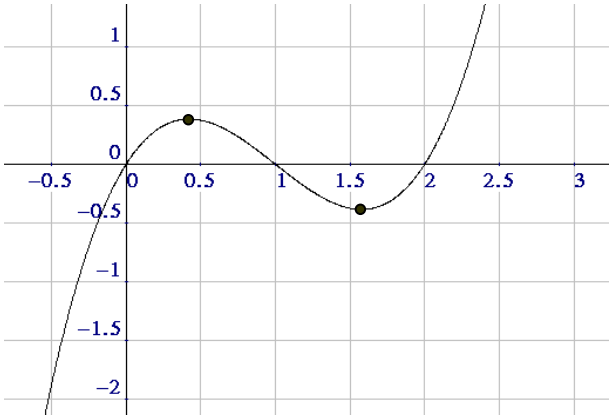
8.



9. Divide $x - 1$ into $x^3 - 4x^2 - 7x + 10$; the quotient is $x^2 - 3x - 10$. The complete factorization is $(x - 1)(x - 5)(x + 2)$, and therefore the x-intercepts are (1, 0), (5, 0) and (–2, 0).

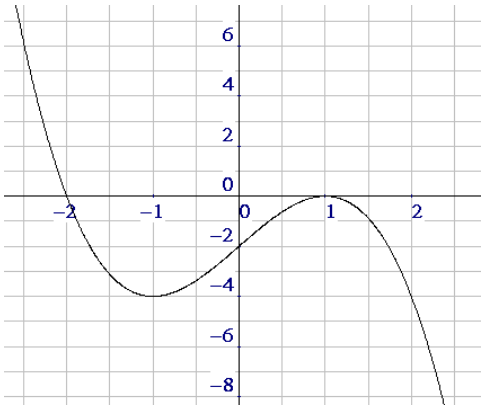
10. For the symmetries, use the litmus tests; e.g., to test for y-axis symmetry, substitute $-x$ for x and show that you do not get the same equation.

11.



A maximum point appears roughly at (.4, .4), and a minimum point near (1.6, –.4).

12.

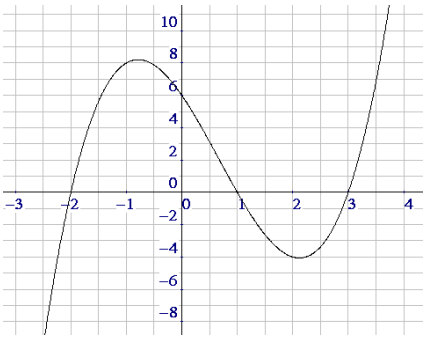


max at (1, 0)

min at (–1, –4)

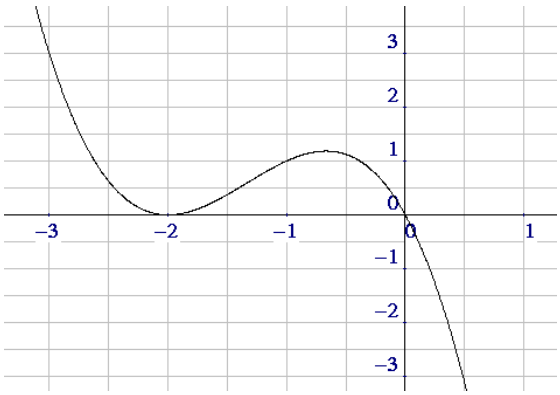
Limits: $-\infty$; ∞

13.



Make sure your four intercepts match those on the graph.

14.



Limits: $-\infty$; ∞

15. a. 0.3 b. 0.5 c. 0.7 d. 1

16. a. Yes; Given any input (an x), there's only <u>one</u> output (a y).
b. Because the highest exponent is 3.
c. $\mathbb{R}$, since there's no real number that could possibly cause any problems.
d. Setting $x = 0$ gives $y = 9$. Therefore, the y-intercept is $(0, 9)$.
e. Set $y = 0$, then solve the resulting equation for x.

17. Compared to f, the graph of g is 2 units to the right and 7 units up.

18. We replace x with $-x$ and y with $-y$ at the same time:

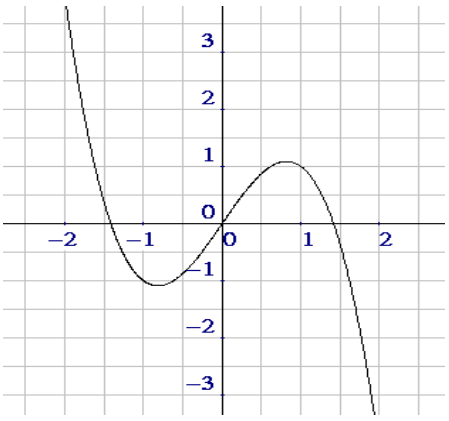
$$\begin{aligned} & -y = 2(-x)^3 + 7(-x) \\ \Rightarrow \quad & -y = 2(-x^3) + (-7x) \\ \Rightarrow \quad & -y = -2x^3 - 7x \\ \Rightarrow \quad & y = 2x^3 + 7x, \text{ the same as the original equation.} \end{aligned}$$

Therefore, the graph has origin symmetry.

19. a. ∞ b. $-\infty$ c. $-\infty$ d. ∞

20. $(-1, 0)$ and $(-9, 0)$

21.



Origin symmetry

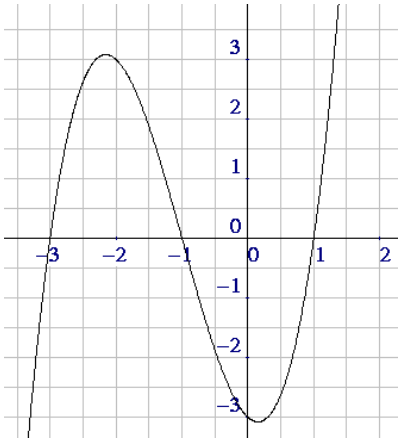
x-int: $(0, 0)$ and $(\pm\sqrt{2}, 0)$

y-int: $(0, 0)$

Domain = $\mathbb{R}$

Limits: $-\infty$; ∞

22.



No symmetry

x-int: (1, 0), (–1, 0), and (–3, 0)

y-int: (0, –3)

Domain = $\mathbb{R}$

Limits: ∞; $-\infty$

23. approximately 0.8

24. a. T b. F c. T d. F e. T f. T g. F h. T

Give a man a fish

and you feed him for a day.

Teach a man to fish

and you feed him for a lifetime.

Chinese Proverb

STATISTICS – THE MEAN AND THE MEDIAN

❑ *WHOLE NUMBERS*

The Mean



Monica and Jarod are arguing over who is doing better in their Algebra class. Monica boasts the highest test score and claims she is doing better. But Jarod says his scores are more consistent, so he's doing better. How can we help them figure out who is doing better?

One way (of many) to settle this dispute is to calculate the ***mean*** of each person's test scores, and then simply compare the means to see who's done better. The mean of a set of numbers is just a fancy word for what most people call the *average*. The ***mean*** is calculated by adding all the numbers and then dividing that sum by how many numbers were added.

For example, assume Monica's test scores in the class were 78, 92, 99, 63, and 93. To find the **mean** test score, we divide the sum of the five tests by 5:

$$\text{Monica's mean} = \frac{78+92+99+63+93}{5} = \frac{425}{5} = \mathbf{85}$$

For Jarod's scores of 94, 83, 88, 91, and 89, we get his mean:

$$\text{Jarod's mean} = \frac{94+83+88+91+89}{5} = \frac{445}{5} = \mathbf{89}$$

So instead of arguing about all 10 test scores, and trying to decide if Monica's score of 99 makes her scores better than Jarod's, or trying to

estimate the damage done by Monica's score of 63, or arguing about whose grades are more consistent, we simply compute each student's mean score, and let the math decide: Jarod wins.

In business and science, the mean is often not a fair summary of data, so we now study another measure of center, the *median*.

The Median



The highway median is the "center divider," the portion of the highway that is in the middle of the lanes. Assuming a normal 6-lane highway, there are three lanes on each side of the median. In the same way, the ***median*** of a set of numbers is the middle number in the list of numbers.

There are two critical issues here. The first is that in order for the median to make any sense at all and truly be the middle number, we must first put the numbers in order. It doesn't matter whether we sort them in ascending (small to big) or descending (big to small) order. The second issue will be covered in homework problem 14.

For instance, let's calculate the **median** of Monica's test scores in the previous example. Notice that if we take her five scores in the order presented in the problem, the middle score would be 99:

78 92 99 63 93 the **wrong** way ☹

But that's absurd — who could possibly claim that 99 (the biggest score!) is the middle score, just because it's in the middle of the list? So let's not forget to put the five numbers in order first; for no particular reason (other than tradition), we'll sort them in ascending order:

63 78 92 93 99 the **right** way ☺

Now which is the middle score? Do you see that the 92 has two scores below it and two scores above it? 92 is the middle score, and therefore, **the median is 92**.

To help you keep track of the meanings of *mean* and *median*, notice that the words "me**d**ian" and "mi**d**dle" each have "d" as their third letter.

Contrasting the Mean and the Median

Consider the following set of numbers:

18 47 89 103 143

Add the five numbers together and divide by 5 to get a mean of 80. Also, since the numbers are already in ascending order, the median is 89.

Now watch what happens when we change the 143 at the end of the data listed above to 8,573 and leave the rest of the numbers alone:

18 47 89 103 **8,573**

Calculate the new mean and you should get 1,766, way bigger than the original mean of 80. But now check this out: The median is still 89! So, even though the mean jumped considerably when we increased the last number, the median stayed exactly the same.

Conclusion: Extremely small or extremely large numbers may wreak havoc on the mean, but may not affect the median.

Homework

If you take Statistics, you'll find two notations for the ***mean***. The *population mean* is denoted by μ, the Greek letter mu (pronounced myoo). The *sample mean* is written $\bar{x}$ (read as *x*-bar).

1. Find the median of 37, 77, 58, 41, 67
2. Find the median of 53, 29, 3, 55, 98
3. Find the mean of 74, 68, 12, 54
4. Find the mean of 5, 27
5. Find the mean of 16, 41, 40, 70, 78
6. Find the median of 44, 30, 99

7. Find the median of 63, 72, 99

8. Find the mean of 17, 10, 32, 65

9. Find the median of 29, 75, 65

10. Find the mean of 52, 32, 93

11. Find the mean of 8, 18

12. Find the median of 48, 73, 71, 2, 9, 15, 99

13. Your friend is calculating the mean of 50 numbers, where the smallest number in the bunch is 17 and the largest is 99.

    a. What would you say if your friend calculated a mean of 123?
    b. What about a mean of 88?
    c. What about a mean of 15?

14. You may have noticed something about the concept of median and all the homework problems on median: Every problem contained an odd quantity of numbers, namely either 3, 5, or 7 numbers. What if there's an even number of numbers? For example, how do we find the median of the four numbers (notice that they're already in ascending order)

    23 30 38 97

    where there's no middle number? We'll calculate the number halfway between the two numbers closest to the middle. In other words, we'll find the *mean* of 30 and 38, which is

    $$\frac{30+38}{2} = \frac{68}{2} = 34$$

    Thus, our conclusion is that **the median is 34**, even though 34 was not one of the original numbers. This is alright because we remember that the main feature of the median is that there are as many numbers below it as there are above it. And sure enough, two numbers (23 & 30) are below the median 34 and two numbers (38 & 97) are above it.

    And don't forget to put the numbers in order before you calculate the median.

15. Calculate the median of each set of numbers:

a. 27 25 100 18

b. 6 22 21 37 13 2

c. 50 15 12 9 12 8

d. 17 45 100 0

e. 34 113 287 77 89 50

f. 6 8 4 7 7 3 4 2

❑ *FRACTIONS*

Review of Mean and Median

Assume we are given a set of numbers. The **mean** is found by dividing the sum of the numbers by the number of numbers.

The **median** is found by listing the numbers in order (ascending or descending), and then selecting the one in the middle. If there is an even number of numbers, the median is found by calculating the *mean* of the two numbers closest to the middle.

EXAMPLE 1: **Find the mean:** $\frac{3}{4}$ $\frac{1}{2}$ $\frac{5}{6}$

Solution: We add the three numbers and divide by 3:

Step 1:
$$\begin{aligned} \tfrac{3}{4} &= \tfrac{9}{12} \\ \tfrac{1}{2} &= \tfrac{6}{12} \\ +\ \tfrac{5}{6} &= \tfrac{10}{12} \\ \hline & \tfrac{25}{12} \end{aligned}$$

Step 2:
$$\begin{aligned} & \tfrac{25}{12} \div 3 \\ = & \tfrac{25}{12} \div \tfrac{3}{1} \\ = & \tfrac{25}{12} \times \tfrac{1}{3} \\ = & \boxed{\tfrac{25}{36}} \end{aligned}$$

Question: How wise would it have been had we converted the sum $\frac{25}{12}$ into the mixed number $2\frac{1}{12}$ before dividing by 3?

EXAMPLE 2: **Find the median:** $\frac{3}{4}$ $\frac{1}{2}$ $\frac{5}{6}$

Solution: We line the numbers up in ascending order, from smallest to largest (largest to smallest would work, too), and then select the number in the **middle**.

To discern the relative sizes of the fractions, convert them to equivalent fractions with a common denominator and then line them up:

$\frac{3}{4}$ $\frac{1}{2}$ $\frac{5}{6}$ (the given data)

$\Rightarrow$ $\frac{9}{12}$ $\frac{6}{12}$ $\frac{10}{12}$ (common denominator, with new numerators)

$\Rightarrow$ $\frac{6}{12}$ $\frac{9}{12}$ $\frac{10}{12}$ (sort in ascending order)

The middle number is $\frac{9}{12}$, and so the median is

$\boxed{\frac{3}{4}}$

Homework

16. Find the **mean**:

a. $\frac{2}{3}$ $\frac{1}{4}$ $\frac{1}{2}$

b. $\frac{9}{10}$ $\frac{5}{12}$

c. $\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ $\frac{1}{5}$

d. $\frac{1}{9}$ $\frac{1}{8}$

e. $\frac{5}{6}$ $\frac{2}{3}$ $\frac{7}{12}$

f. $2\frac{1}{2}$ $\frac{3}{4}$ $1\frac{5}{8}$

g. $3\frac{1}{4}$ $2\frac{2}{3}$

h. $1\frac{1}{2}$ $\frac{4}{5}$ $\frac{5}{6}$

i. $\frac{1}{2}$ $\frac{2}{3}$ $\frac{3}{4}$ $\frac{5}{6}$

17. Find the **median**:

a. $\frac{2}{3}$ $\frac{1}{4}$ $\frac{1}{2}$

b. $\frac{5}{6}$ $\frac{2}{3}$ $\frac{7}{12}$

c. $2\frac{1}{2}$ $\frac{3}{4}$ $1\frac{5}{8}$

d. $\frac{1}{7}$ $\frac{1}{5}$ $\frac{1}{9}$

e. $\frac{2}{3}$ $\frac{4}{5}$ $\frac{1}{2}$

f. $\frac{5}{8}$ $\frac{1}{2}$ $\frac{3}{4}$

g. $\frac{1}{5}$ $\frac{3}{10}$ $\frac{2}{11}$

h. $\frac{4}{5}$ $\frac{3}{4}$ $\frac{1}{2}$

i. $\frac{7}{15}$ $\frac{11}{30}$ $\frac{2}{5}$

❑ *DECIMALS*

Putting Decimals in Order

As you know, the key to finding the median of a set of numbers is the ability to put those numbers in numerical order. The method most students find easiest to understand is converting the decimals to money.

For example, suppose we need to list the following numbers in ascending order (smallest to biggest):

2.3 0.129 0.07 0.8 5

Translate each number into currency format:

\$2.30 \$0.129 \$0.07 \$0.80 \$5.00

[Note that the second number, \$0.129, is a tad short of \$0.13.] Now we put these money amounts in order from smallest to biggest:

\$0.07 \$0.129 \$0.80 \$2.30 \$5.00

Since \$0.80 is now the middle number, we conclude that the median of the original set of numbers is **0.8**.

Homework

18. Find the **mean** (NO rounding):

a. 2.3 8.9

b. 1.1 0.5 2.3

c. 0.02 0.2 2

d. 1.1 2.2 3.3 4.4

e. 0.23 0.44 1.1 0.092

f. 0.2 9.1 2.33 8 5.023

19. Find the **median**:

a. 0.1 0.11 0.01

b. 1.1 0.5 2.3

c. 0.02 0.2 2

d. 2.3 4.5 4.05 2.003 .999

e. 0.03 0.04 0.4 0.3 0.9

f. 0.01 0.002 1.0005 0.03 0.017

20. Find the **median**:

a. 2.3 0.02 1.77 0.001

b. 23.9 14.89 17.9 30

c. 0.07 0.8 0.006 2

d. 2.3 2 1.99 5

e. 3.4 5.6 1.89 3.77

f. 0.09 0.045 2 1.17

g. 23.5 12.99 3.567 0.07

h. 0.25 0.078 0.667 0.0099

i. 23.9 32.007 18.345

j. 0.23 0.32 0.4 0.003 0.09 0.1

Solutions

1. 58 **2**. 53 **3**. 52 **4**. 16 **5**. 49 **6**. 44

7. 72 **8**. 31 **9**. 65 **10**. 59 **11**. 13 **12**. 48

13. Only the 88 makes sense. Ask if you don't understand.

14. This was just an example, not a problem.

15. a. 26 b. 17 c. 12 d. 31 e. 83 f. 5



The new "zipper" **MEDIAN** on the Golden Gate Bridge

16. a. $\frac{17}{36}$ b. $\frac{79}{120}$ c. $\frac{77}{240}$ d. $\frac{17}{144}$ e. $\frac{25}{36}$ f. $1\frac{5}{8}$

g. $2\frac{23}{24}$ h. $1\frac{2}{45}$ i. $\frac{11}{16}$

17. a. $\frac{1}{2}$ b. $\frac{2}{3}$ c. $1\frac{5}{8}$ d. $\frac{1}{7}$ e. $\frac{2}{3}$ f. $\frac{5}{8}$

g. $\frac{1}{5}$ h. $\frac{3}{4}$ i. $\frac{2}{5}$

18. a. 5.6 b. 1.3 c. 0.74 d. 2.75 e. 0.4655 f. 4.9306

19. a. 0.1 b. 1.1 c. 0.2 d. 2.3 e. 0.3 f. 0.017

20. a. 0.895 b. 20.9 c. 0.435 d. 2.15 e. 3.585 f. 0.63
g. 8.2785 h. 0.164 i. 23.9 j. 0.165

"Life does not need to be changed.

Only your intent and actions do."

– *Swami Rama*

LOGIC

In everyday use, "Go see *Star Wars* tonight" might be considered a statement. But in logic and math, it's not a statement because a ***statement*** is an assertion that can clearly be classified as either *true* or *false* (even if we don't know which one).



Rodin's *The Thinker*

❑ *STATEMENTS AND IMPLICATIONS*

"*The quadratic formula works for all quadratic equations*" is a ***statement*** (and it's true). Another statement is "*2 + 2 = 5*" (but it's false). Even "*There are 100 billion stars in our galaxy*" is a statement. I have no idea whether it's true or false, but I'm sure that it's either true or false.

Most of the results in mathematics are actually a special hooking-together of two statements; we call these ***implications***. An implication is sometimes called an "***if-then***" statement. If p and q are statements, then an implication can be written

If *p*, then *q*

or, ***p* implies *q***

or, $\boldsymbol{p \Rightarrow q}$

The essence of computer programming lies in the *if-then* statement.

The statement ***p*** is called the ***hypothesis*** and the statement ***q*** is called the ***conclusion***.

For example, the fact that the square of an odd number is odd (for example, $5^2 = 25$) can be written

If n is odd, then n^2 is odd,

or n odd $\Rightarrow$ n^2 odd.

Hypothesis: n odd

Conclusion: n^2 odd

Homework

1. Is the following a statement? "There are exactly 7 people in New York City with 32,000 hairs on their heads."

2. Is the following a statement? "Walk tall!"

3. Is the following a statement? "Arvis said, 'Walk tall!'"

4. Goldbach stated many years ago, "Every **even** number starting at 4 is the sum of two **primes**." For example, 20 is an even number, and **20 = 17 + 3**, where 17 and 3 are prime. As of today, this conjecture has never been proven to be true or false — mathematicians simply don't know. Is Goldbach's Conjecture a statement?

5. State the hypothesis and conclusion of the statement "If x is any real number, then $x^2 \geq 0$."

6. State the Pythagorean Theorem in the "if-then" form.

7. "All dogs go to heaven" is an implication. Write it in the "if-then" form.

8. Consider the implication, "I'll get an A if I study hard enough." State the hypothesis and conclusion.

9. Convert each implication into the

   "If *hypothesis*, then *conclusion*" form:

   a. Whenever I eat pie, I get sick.
   b. The sidewalks get wet whenever it rains.
   c. I get A's if I study.
   d. The President must be at least 35 years of age.

❑ *NEGATION*

The ***negation*** of a statement is the "opposite" of the statement, and is denoted with a squiggle, ~, in front of the statement, although there are other notations. For example, let p be the statement "T is a right triangle." Then $\sim p$ is read "not p" and is the statement "T is not a right triangle."

A more subtle example: Let q = "All prime numbers are odd." Then $\sim q$ = "Not all prime numbers are odd," meaning, "There is at least one prime number which is not odd." [Can you name it?]

The bottom line for negation is that if p is true, then $\sim p$ is false — and if p is false, then $\sim p$ is true. Can you see that $\sim(\sim p)$ has the same truth value of p? What does the statement "Nobody doesn't like Sara Lee" mean?

p	$\sim p$
True	False
False	True

From the implication $p \Rightarrow q$ and the negation ~, we can create two variations of the implication, described in the following two sections.

Homework

10. Let p be the statement, "The 49ers lost the Super Bowl." Assume that the game cannot end in a tie.
    State $\sim p$**.**

11. Let p be the statement "$x > 0$". State $\sim p$**.**

12. Let p be the statement, "Every triangle is a polygon." State $\sim(\sim p)$**.**

13. Simplify: $\sim(\sim(\sim(\sim p)))$

❑ *CONVERSE*

The ***converse*** of the statement $p \Rightarrow q$ is $q \Rightarrow p$. We interchange the hypothesis and the conclusion. Consider the statement, "If you live in San Francisco, then you live in California." Its converse is "If you live in California, then you live in San Francisco." The original statement is true, but its converse can easily be false.

What's the converse of the statement "If a triangle has three equal sides, then each of its angles is 60°"? It's the statement, "If each angle of a triangle is 60°, then the triangle has three equal sides." In this example, both the statement and its converse are true.

In short, the converse of a true statement may — or may not — be true.

❑ *CONTRAPOSITIVE*

Consider the true statement above, "If you live in San Francisco, then you live in California." Though the converse of this statement is false, let's take the converse and negate each part. This is called constructing the ***contrapositive***. If we do this "switch and negate" maneuver, we get the statement, "If you don't live in California, then you don't live in San Francisco." This statement makes sense. Do you agree?

Therefore, the contrapositive of the statement $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$. It seems, at least using the above example, that the contrapositive of a true statement is true, and indeed this is always the case.

Statement	$p \Rightarrow q$	
Its Converse	$q \Rightarrow p$	Truth value has no relation to the truth of the original statement.
Its Contrapositive	$\sim q \Rightarrow \sim p$	Truth value must match that of the original statement.

Homework

14. Let p = "If it's dark, then it's scary."

    a. State the converse of p.

    b. State the contrapositive of p.

    c. Which of the two statements must have the same truth value as p?

15. Let q = "If angle A is a right angle, then A = 90°." State the converse. Is the converse true or false?

16. Let r = "If $x = 5$, then $x^2 = 25$." State the converse. Is the converse true or false?

17. State the converse of the Pythagorean Theorem. Is the converse true or false?

18. Consider the true statement from calculus: "If f is differentiable at x, then f is continuous at x." State the converse and the contrapositive of the statement. Which of the two statements must be true?

19. Consider the false statement from a branch of math called topology: "If $\varnothing$ is in T, then T is a topology." State the converse and the contrapositive of the statement. Which of the two statements must be false?

20. Consider the implication $r \Rightarrow s$. State the converse and the contrapositive of the implication. Which of the two statements must have the same truth value as the original implication?

❑ *MORE ON COMBINING STATEMENTS*

AND Another way to hook two statements together to make a new statement is to place the word *AND* between them. The meaning is the same as that in English. If both p and q are true, then the statement p AND q is true. But if either p or q (or both) is false, then the statement p AND q is false.

p	q	p AND q
True	True	True
True	False	False
False	True	False
False	False	False

The statement "2 + 2 = 4 AND The letter *a* is a vowel" is true.
The statement "2 + 2 = 4 AND The letter *m* is a vowel" is false.
The statement "2 + 2 = 5 AND The letter *z* is a vowel" is false.

OR Two statements can be linked by the word *OR* to create a new statement. In this case, the logical meaning of OR may not be consistent with the use of the word OR in English. In math, as long as <u>at least one</u> of the statements p or q is true, then the statement p OR q is true. In particular, if both p and q are true, then p OR q is true.

p	q	p OR q
True	True	True
True	False	True
False	True	True
False	False	False

Thus, the following are true statements (since at least one part is true):

2 + 2 = 4 OR Lincoln is our current president.
2 + 2 = 4 OR California is in the U.S.A.

But the following statement is false (since both parts are false):

The earth is cubical OR 10 + 10 = 29.

Note that our use of the word OR is <u>not</u> the way your mother meant the word OR when she said, "You can have ice cream OR a cookie."

The OR we use in logic and math – one or the other or both – is called the **inclusive OR**.

The OR that means one or the other – but <u>not</u> both – is called the **exclusive OR**.

Homework

21. a. What are the conditions for p and q that will make the statement p AND q true?

    b. What will make p AND q false?

22. a. What are the conditions for p and q that will make the statement p OR q true?

    b. What will make p OR q false?

23. T/F: a. "2 + 2 = 4 OR Lincoln was the 1st president."
    b. "2 + 2 = 4 OR $5^2 = 25$."
    c. "1 + 3 = 10 OR I'm 10 feet tall."

24. T/F: a. "2 + 2 = 4 AND Lincoln was the 1st president."
    b. "2 + 2 = 4 AND Washington was the 1st president."
    c. "California is in Idaho AND a circle is a square."

25. "$6 \leq 6$" is a true statement. Explain.

26. When is the statement p AND q AND r true?

27. When is the statement p OR q OR r true?

28. [Optional] Consider the logical connective, denoted by *XOR*, and defined by the truth table:

p	q	p XOR q
True	True	False
True	False	True
False	True	True
False	False	False

Give an example from real life that uses this connective.

❑ *DEFINITIONS, AXIOMS, AND THEOREMS*

There are really only two kinds of statements in all of mathematics: those you <u>don't</u> prove (the definitions and axioms), and those you <u>do</u> prove (the theorems).

Consider the following ***definition***: A *right angle* is an angle whose measure is 90°. [The Babylonians adored the number 360, and 90 is one of its many factors.] Is this definition right or wrong? It's not right or wrong — it just is. If you disagree with the definition, then you can either accept it so you'll pass your math course, or you can define it differently and perhaps create a new branch of mathematics!

An example of an ***axiom*** is the *commutative property* for addition of two numbers: $a + b = b + a$. We accept this as fact — we don't consider whether or not there's a proof. Why do we accept this axiom? Well, it seems to work for all the number systems we've seen in algebra — it seems "intuitively" clear. [Not necessarily the case in advanced math courses.]

A ***theorem***, on the other hand, is a statement that requires a **proof**. The proof of a theorem is based on definitions, axioms, and previously proved theorems. The *Quadratic Formula* is an example of a theorem, and can be expressed as an implication:

If $ax^2 + bx + c = 0$ (where $a \neq 0$), **then** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The classic theorem is the *Pythagorean Theorem*:

> **If** a and b are the legs of a right triangle and c is the hypotenuse, **then** $a^2 + b^2 = c^2$.

At the end of a proof of a theorem we sometimes write Q.E.D., which is Latin for *quod erat demonstrandum*, which loosely translates to "as was to be proved."

Homework

29. A *group* is a semigroup with identity. Is this definition right or wrong?

30. Prove the axiom: If a and b are numbers, then $ab = ba$.

31. The formula for *slope* $\left(m = \dfrac{\Delta y}{\Delta x}\right)$ is not a theorem. What is it, then?

32. a. Why is the Distance Formula in the plane considered a theorem?
    b. State the hypothesis and conclusion.
    c. What geometry theorem is the Distance Formula based upon?

33. State the hypothesis and conclusion of the Quadratic Formula.

❑ *EXAMPLES AND COUNTEREXAMPLES*

Consider the statement: If x is any number, then $x^2x^3 = x^5$. Here's an ***example*** of this.

> Let $x = 2$. Then,

x^2x^3 gives $2^2 \cdot 2^3 = 4 \cdot 8 = 32$; and

x^5 gives $2^5 = 32$. ✓

What have we proved? Not much — just that $2^2 \cdot 2^3 = 2^5$. But this in no way proves the more general statement, $x^2x^3 = x^5$. It is just an example. It gives some credence to the general statement, but nothing more.

Now consider the statement: If a and b are numbers, then $(a + b)^2 = a^2 + b^2$. Though this statement is true sometimes, it's considered to be a false statement. To prove that a statement is false, all we need is one ***counterexample***: an example that runs counter (contrary) to the statement.

Let $a = 3$ and $b = 4$. Then,

$$(3 + 4)^2 = 7^2 = 49,$$

whereas $3^2 + 4^2 = 9 + 16 = 25$. ☹

Therefore, $(a + b)^2 = a^2 + b^2$ is a false statement, and it took only one counterexample to prove that it's false.

Homework

34. Consider the statement "$x \geq 0$."
    a. Describe the statement in words.
    b. Give two different x values that make the statement true, and don't make both positive.

35. Consider the statement: $x - y = y - x$ for any real numbers. Is the statement true or false? Prove your answer.

36. Describe precisely when $(a + b)^2 = a^2 + b^2$ is a true statement.

37. A natural number with precisely two factors is called a *prime number*. The first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, . . .

    a. T/F: Every prime number is odd. Prove your answer.

    b. T/F: Every odd number is prime. Prove your answer.

38. Disprove the statement: $x^2 = 2x$ for any real number x. Hint: Don't let $x = 0$ or 2. Explain why I'm not allowing you to use 0 or 2.

Review Problems

39. a. Give an example of a true statement.
    b. Give an example of a false statement.
    c. Give an example of a statement whose truth is unknown.
    d. Give an example of a sentence which is not a statement.

40. What distinguishes a statement from a non-statement?

41. A *semigroup* is defined to be a set with an associative binary operation. Prove this statement.

42. Simplify the logic expression $\sim (\sim (\sim p))$.

43. A true theorem from group theory states: If H is a subgroup of G, then the order of H divides the order of G.
    a. State the hypothesis of this theorem.
    b. State the converse and the contrapositive of this theorem.
    c. Which of the two statements in part b must be true?

44. Let p = "2 + 3 = 5" and q = "Lincoln was the 2nd president."
    a. Is p OR q true or false? b. Is p AND q true or false?

45. a. A "lemma" is a math statement similar to a theorem. Is a lemma a statement we do, or do not, prove?
    b. A "postulate" is a math statement similar to an axiom. Is a postulate a statement we do, or do not, prove?

46. Consider the statement $(x + y)^3 = x^3 + y^3$ for any real numbers x and y.
    a. Give an example to give credibility to the statement.
    b. Now prove that the statement is false. What did you use to prove it's false?

47. Explain why "Ultraviolet waves have a higher frequency than infrared waves" is a *statement* in logic.

48. One of the axioms of a group is that there must be an identity element. Prove this statement.

49. Simplify the logic expression $\sim (\sim (\sim (\sim (\sim p))))$.

50. A true theorem from group theory states: If H is a subgroup of a cyclic group G, then H is cyclic.
    a. State the conclusion of this theorem.
    b. State the converse and the contrapositive of this theorem.
    c. Which of the two statements in part b must be true?

51. Let p = "2 + 3 = 9" and q = "Lincoln was the 2nd president."
    a. Is p OR q true or false? b. Is p AND q true or false?

52. Consider the statement $\sqrt{a^2 + b^2} = a + b$ for any real numbers a and b.
    a. Give an example to give some credibility to the statement.
    b. Now prove that the statement is false. What did you use to prove it's false?

53. True/False

a. $p \Rightarrow q$ is called an implication.

b. In $p \Rightarrow q$, q is called the hypothesis.

c. $\sim(\sim p)$ simplifies to $\sim p$.

d. The converse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.

e. The contrapositive of $\sim q \Rightarrow \sim p$ is $p \Rightarrow q$.

f. $x = 2$ is a counterexample to the conjecture $x^2 = 2x$.

g. If p is true, q is false, and r is true, then (p OR q) AND r is true.

h. Axioms and theorems are proved.

i. Definitions are never proved.

j. The converse of a true implication is always true.

k. The contrapositive of a false implication is always false.

Solutions

1. Yes — I don't know whether it's true or false, but it's certainly one or the other.

2. No — It's simply a command. There's no possible way the sentence can be evaluated as true or false.

3. Yes, this sentence is a statement. Even though it sounds like the previous problem, it makes the statement that Arvis said something. Either he did or he didn't, so the statement is either true or false.

4. Yes — as we're learning, a statement is a sentence with a truth value of either true or false, regardless of whether we mortals know which it is. By the way, 100 is the sum of what two primes?

5. Hypothesis: If x is any real number — Conclusion: $x^2 \geq 0$.

6. If a and b are the legs of a right triangle and c is the hypotenuse, then

$a^2 + b^2 = c^2$.

7. If you're a dog, then you'll go to heaven.

8. Hypothesis: If I study hard enough — Conclusion: I'll get an A

9. a. If I eat pie, then I get sick.
b. If it rains, then the sidewalks get wet.
c. If I study, then I get A's.
d. If one is the President, then one is at least 35 years of age.

10. The 49ers won the Super Bowl.

11. Hint: The answer is <u>not</u> "$x < 0$."

12. Every triangle is a polygon.

13. p

14. a. If it's scary, then it's dark b. If it's not scary, then it's not dark.
c. The contrapositive always has the same truth value as the original statement.

15. If A = 90°, then A is a right angle. The converse is true.

16. If $x^2 = 25$, then $x = 5$. The converse is false (since $x = \pm 5$).

17. If a, b, and c are the sides of a triangle and $a^2 + b^2 = c^2$, then the triangle is a right triangle. The converse is true.

18. Converse: If f is continuous at x, then f is differentiable at x.
Contrapositive: If f is not continuous at x, then f is not differentiable at x.
The contrapositive must be true if the original statement is true.

19. Converse: If T is a topology, then $\varnothing$ is in T.
Contrapositive: If T is not a topology, then $\varnothing$ is not in T.
The contrapositive will be false.

20. Converse: $s \Rightarrow r$ Contrapositive: $\sim s \Rightarrow \sim r$
The contrapositive always has the same truth value as the original statement.

21. a. Both p and q must be true.
b. If either or both of them are false.

22. a. Either p is true, or q is true, or both are true.
b. Only if both of them are false.

23. a. T b. T c. F

24. a. F b. T d. F

25. The statement says, "6 is less than OR equal to 6." Since 6 is equal to 6, and since an OR statement is true as long as one of the parts is true, the entire statement is true.

26. Only when all three parts are true.

27. As long as any one of them (or two of them or all of them) is true.

28. I'm curious what you think the truth table really means.

29. Neither — it's a definition.

30. Axioms aren't proven — they're accepted.

31. It's a definition. (It's not a theorem because it cannot be proven.)

32. a. Because it can be derived (proved).

b. Hypothesis: If (a, b) and (c, d) are points in the plane . . .
Conclusion: then the distance between them is $\sqrt{(a-c)^2+(b-d)^2}$

c. The Pythagorean Theorem

33. Hypothesis: Given the quadratic equation $ax^2 + bx + c = 0$ . . .
Conclusion: the solutions are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

34. a. "x is greater than or equal to zero."

b. If $x = 7$, the statement is true, and if $x = 0$, the statement is true.

35. False — to prove this, we take the *counterexample*: $7 - 2 = 5$, whereas $2 - 7 = -5$.

36. It has something to do with zero.

37. a. See the 2? b. Is 15 prime?

38. Let $x = 5$ and the statement fails. Choosing x as 0 or 2 is a lousy choice because the statement actually works for these numbers.

39. a. For example, "The *Harry Potter* books were written by a woman."

b. For example, "Every parabola has two x-intercepts."

c. For example, "It took exactly 197 geeks to write Windows."

d. For example, "Why did you break the window?"

40. A statement can be assigned a true/false value.

41. A definition cannot be proved.

42. $\sim p$

43. a. H is a subgroup of G.

b. *converse*: If the order of H divides the order of G, then H is a subgroup of G.

contrapositive: If the order of H does not divide the order of G, then H is not a subgroup of G.

c. The contrapositive must be true.

44. a. True b. False **45**. a. Prove b. Do not prove

46. a. Let $x = 1$ and $y = 0$ and the statement is true. b. One counterexample is to let $x = 2$ and $y = 3$.

47. It's a statement because it's an assertion that is either true or false.

48. It can't be proved; it's an axiom, which we assume to be true.

49. $\sim p$

50. a. H is cyclic.

b. *converse*: If H is cyclic, then H is a subgroup of a cyclic group G.

contrapositive: If H is not cyclic, then H is not a subgroup of a cyclic group G.

c. The contrapositive must be true.

51. a. false b. false

52. a. Letting $a = 5$ and $b = 0$, for instance, will make the statement true.

b. Letting $a = 4$ and $b = 3$ will make the statement false. We used a *counterexample* to prove the statement false.

53. a. T b. F c. F d. F e. T f. F

g. T h. F i. T j. F k. T

"The opposite of a correct statement is a false statement. But the opposite of a profound truth may well be another profound truth."

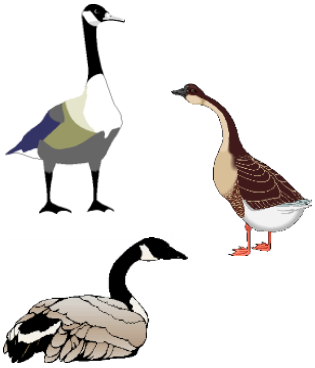


Niels Bohr, physicist (1885–1962)

SETS



A *bunch* of bananas, a *gaggle* of geese – each is an example of a ***set***, referred to as an *ensemble* in French. The idea of a set is so fundamental that it doesn't even have a definition, yet sets form the foundation of all mathematics.



❑ *BASIC SET DEFINITIONS*

A ***set*** is a well-defined collection of objects (but what's a collection?). Well-defined means that, for instance, "all whole numbers greater than 10" is a set, but "all tall people" is not a set (it's too vague).

Consider the sets

$$A = \{1, 3, 5\} \qquad B = \{1, 3, 5, 7, 10\}$$

The ***elements*** (or members) of set A are 1, 3, and 5, surrounded by (curly) braces. We say "3 is an element of A" and write "$\mathbf{3 \in A}$". We can also say that "12 is not an element of A", in which case we write "$\mathbf{12 \notin A}$".

In the sets A and B above, notice that all the elements of A are also elements of B. In a sense, A is contained in B. We say that "A is a ***subset*** of B" and write "$\mathbf{A \subseteq B}$". Can you see that B is not a subset of A? We write "$B \nsubseteq A$".

There's a very special set that contains *no* elements. It is called the ***null set*** (or the empty set), and is written ∅. Some teachers prefer to write { } for the null set.

Most books use the symbol $\mathbb{N}$ to denote the set of all **natural numbers**:

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

Also, we'll use the symbol $\mathbb{Z}$ to represent the set of ***integers***:

$$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Z is the first letter of ***Z***ahl, the German word for *number*.

Homework

1. Consider the collection of all registered students in this class. Is this a set? Why or why not?

Do you see that $\mathbb{N} \subseteq \mathbb{Z}$?

2. Consider the collection of all students at the college with a high GPA. Is this a set? Why or why not?

3. Fill in the blanks:
   a. 7 ____ { all primes } b. 13 ____ { all even numbers }

4. Fill in the blanks:
   a. 6 ____ {3, 6, 9} b. {6} ____ {3, 6, 9}

5. a. T/F: {students in this class} $\subseteq$ { DVC students}
   b. T/F: {people in this class} $\subseteq$ { DVC students}

6. T/F: {primes} $\subseteq$ {odd natural numbers}
   Hint: The first few primes are 2, 3, 5, 7, 11, 13, ...

7. Explain why $\{7, 8, 9\} \not\subseteq \{7, 9, 11, 13, 20\}$.

8. T/F: $\{\varnothing\} = \varnothing$. Be sure to explain your answer.

9. Which statement is true, $\{1, 9, 21\} \subseteq \mathbb{N}$ or $\mathbb{N} \subseteq \{1, 9, 21\}$?

10. Prove that $\mathbb{N} \subseteq \mathbb{Z}$. Now prove that $\mathbb{Z} \not\subseteq \mathbb{N}$.

11. Fill in the blank: If $A \subseteq B$ and $B \subseteq A$, then _____.

12. T/F: $\{1, 3\} \in \{\{a\}, \{x, y\}, \{1, 3\}\}$

❑ *IMPORTANT PROPERTIES OF SETS*

- The elements of a set may be explicitly listed, such as the set of positive even numbers: {2, 4, 6, 8, . . .}. The positive even numbers can also be written $\{n \in \mathbb{N} \mid n \text{ is even}\}$, where the vertical bar, $\mid$, is read *"such that."* [Some books write this with a colon instead of a vertical bar: $\{n \in \mathbb{N}: n \text{ is even}\}$]

- The order in which the elements of a set are listed doesn't matter. For example, {1, 2, 3} = {2, 3, 1}.

- Repeated elements in a set can be thrown out without changing the set. For instance, {a, b, a, c} = {a, b, c}.

- Is every element of the set {1, 2, 3} also in the set {1, 2, 3}? Yes, so {1, 2, 3} $\subseteq$ {1, 2, 3}. In general, for any set A,

  $$\mathbf{A \subseteq A}$$

- We now contend that $\varnothing \subseteq$ {1, 2, 3}. This is really strange, but look at it this way: If the null set were <u>not</u> a subset of {1, 2, 3}, then there would have to be something in the null set that <u>fails</u> to be in {1, 2, 3}. But this is impossible since there's nothing in the null set! Thus, it can <u>not</u> be the case that the null set is <u>not</u> a subset of {1, 2, 3}, and therefore it <u>is</u> a subset: $\varnothing \subseteq$ {1, 2, 3}. This may be hard to fathom, so you may have to accept it on faith. Just remember the following fact: If A is <u>any</u> set,

  $$\mathbf{\varnothing \subseteq A}$$

Homework

13. List the elements of $\{x \in \mathbb{N} \mid x \leq 5\}$.

14. List the elements of $\{x \in \mathbb{N} \mid 5 \leq x < 10\}$.

15. Write the set {2, 4, 6, 8, 10} in the form $\{x \in \mathbb{N} \mid x \text{ is } \ldots\}$.

16. Write the set {2, 3, 5, 7, 11, 13} in the form $\{x \in \mathbb{N} \mid x \text{ is } \ldots\}$.

17. T/F: {6, 7, 8, 7, 4} = {4, 7, 6, 8}. Explain.

18. T/F: $\{a, b, c, d, e\} = \{b, c, d, e\}$. Explain.

19. T/F: {1, 5, 9} ⊆ {1, 5, 9}

20. T/F: ∅ ⊆ ∅. Explain.

21. The number of elements in a set is called its ***cardinality***. For example, the cardinality of {2, 4, 8, 16, 32} is 5. What is the cardinality of ∅? What is the cardinality of {1, 3, 5, 7, . . ., 21}?

22. Let A be a set with cardinality 100. What is the cardinality of {A}? [This is tricky.]

23. What is the cardinality of ℕ? What is the cardinality of the set of even numbers? Do ℕ and the even numbers have the same cardinality?

❑ *COMBINING SETS*

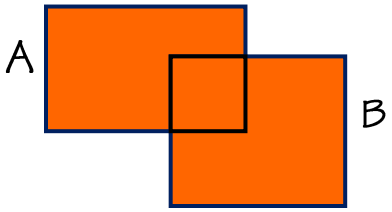
The ***union*** of two sets A and B is the new set obtained by lumping together all the elements that are in A or B (or both). The union of A and B is written $\mathbf{A} \cup \mathbf{B}$. For example,

$$\{1, 2, 7\} \cup \{2, x, \Delta\} = \{1, 2, 7, x, \Delta\}$$

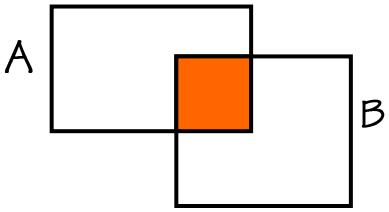
The ***intersection*** of sets A and B is the new set obtained by extracting only the elements that are in both A and B — that is, the elements that A and B have *in common*. The intersection of A and B is written **A ∩ B**. For example,

$$\{a, b, R\} \cap \{b, c, R\} = \{b, R\}$$

Union and intersection can be displayed as *Venn diagrams*:



A ∪ B is shaded



A ∩ B is shaded

The following are the official definitions of union and intersection:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

EXAMPLE 1:

A. $\{a, b, c\} \cup \{d, w, z\} = \{a, b, c, d, w, z\}$

B. $\{1, 3, 5, 7\} \cap \{2, 4, 6\} = \varnothing$

C. $\{m, n, p\} \cup \{m, n, o, p, q\} = \{m, n, o, p, q\}$

D. $\{7, 8, 9\} \cap \{7, 8, 9, 10\} = \{7, 8, 9\}$

E. $\{a, 1, 2, \pi\} \cup \varnothing = \{a, 1, 2, \pi\}$

F. $\{x, y, z\} \cap \varnothing = \varnothing$

Homework

24. Sketch a Venn diagram representing the statement: $A \subseteq B$.

25. Use a Venn diagram to prove:

    If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

26. Find $\{a, b, c\} \cup \{b, d, w, z\} \cup \{a, d, w, m, z\}$.

27. Find $\{1, 3, 5\} \cap \{3, 5, 7\} \cap \{5, 7, 9\}$.

28. Two sets A and B are said to be ***disjoint*** if $A \cap B = \varnothing$; for example, the set of even numbers and the set of odd numbers are disjoint. Sketch a Venn diagram of two disjoint sets.

29. Find a *counterexample* to the conjecture:

    $A \cup (B \cap C) = (A \cup B) \cap C$

    Hint: Find sets A, B, and C that make the statement false.

30. Use the sets $A = \{1, 2, 3\}$, $B = \{2, 3, 7\}$, and $C = \{3, 7, 9\}$ to give some evidence for a *distributive property* of sets:

    $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

    Does this example constitute a proof of a distributive property of sets?

❑ *FINDING ALL THE SUBSETS OF A SET*

EXAMPLE 2: **List all the subsets of {1, 2, 3}.**

Solution: Recall the Important Properties section a few pages back. There, we learned that the null set, $\varnothing$, is a subset of any set; thus $\varnothing$ is one of the subsets of {1, 2, 3}. Also, we discussed

the fact that any set is a subset of itself — thus, {1, 2, 3} is one of the subsets of {1, 2, 3}. That's two subsets so far — essentially the "smallest" and the "largest" subsets.

Next, we'll list the subsets containing exactly one element:

{1} {2} {3}

And finally, we'll list the subsets containing exactly two elements:

{1, 2} {1, 3} {2, 3}

Remembering the null set and the entire set, we obtain a total of eight subsets:

$$\varnothing,\ \{1\},\ \{2\},\ \{3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},\ \{1,2,3\}$$

EXAMPLE 3: **List all the subsets of {*a*, *b*, *c*, *d*}.**

Solution: We'll organize the subsets in order of *cardinality* (the number of elements in the subset):

0 elements:	$\varnothing$
1 element:	$\{a\}$ $\{b\}$ $\{c\}$ $\{d\}$
2 elements:	$\{a, b\}$ $\{a, c\}$ $\{a, d\}$ $\{b, c\}$ $\{b, d\}$ $\{c, d\}$
3 elements:	$\{a, b, c\}$ $\{a, b, d\}$ $\{a, c, d\}$ $\{b, c, d\}$
4 elements:	$\{a, b, c, d\}$

Count 'em all up — there should be **16 subsets** altogether.

Homework

31. a. List all the subsets of $\{a, b\}$.

    b. List all the subsets of $\{x\}$.

    c. List all the subsets of $\varnothing$.

32. How many subsets does the set $\{a, b, c, d, e, f, g, h, i, j\}$ have?

33. Suppose a set has cardinality n. How many subsets does the set have?

34. How many subsets does the set $\{\{1\}, \{1, 2\}, \{7, 9, 10\}\}$ have?

35. A set X has 128 subsets. Find the cardinality of X.

Review Problems

36. What's the distinction between a set and a non-set?

37. a. T/F: $\{5\} = 5$
    b. $A \cup A =$ _____
    c. $A \cap A =$ _____

38. We know that $\mathbb{N}$ has cardinality ∞. What is the cardinality of $\{\mathbb{N}\}$?

39. T/F: $\{1, 3\} \in \{0, 1, 2, 3, 4, 5\}$

40. Explicitly list the elements of the set $A = \{x \in \mathbb{N} \mid x \text{ is even } \textbf{and } x < 7\}$.

41. T/F: If $A \subseteq B$ and $C \subseteq B$, then $A \cup C = B$.

42. For any set A,

$A \cap \varnothing =$ _____ and $A \cup \varnothing =$ _____.

43. If $A = \{4, 5, 9\}$, find a set B such that $A \cap B = \{5, 9\}$.

44. How many subsets does the set $\{5, 12, 3, a, 7, x, \frac{3}{8}\}$ have?

45. a. How many subsets does $\varnothing$ have?

b. How many subsets does $\{\varnothing\}$ have?

46. Find the cardinality of a set which has 512 subsets.

47. T/F: $\{5, 6\} = \{6, 5\}$

48. The set C has cardinality 99. What is the cardinality of $\{C\}$?

49. T/F: $\{1, 3\} \subseteq \{0, 1, 2, 3, 4, 5\}$

50. Explicitly list the elements of the set $A = \{x \in \mathbb{N} \mid x \text{ is odd and } x < 10\}$.

51. T/F: If $A \subseteq B$ and $B \subseteq C$, then $C \subseteq A$.

52. For any set A,

$A \cap A =$ _____ and $A \cup A =$ _____.

53. If $A = \{4, 5, 9\}$, find a set B such that $A \cup B = \{1, 4, 5, 9\}$.

54. How many subsets does the set $\{5, 12, 3, a, 7, y, z, w\}$ have?

55. a. How many subsets does $\varnothing$ have?

b. How many subsets does $\{\varnothing, \{\varnothing\}\}$ have?

56. Find the cardinality of a set which has 1024 subsets.

57. T/F: For any sets A and B, $A \cup B = A \cap B$. Prove your answer.

58. True/False:

a. $\{1\} \in \{ 0, 2, \{1\}, \pi \}$

b. If Y is any set, $Y \subseteq Y$.

c. If Z is any set, $\varnothing \in Z$.

d. $\varnothing \subseteq \varnothing$.

e. $\{a, b, c\} \cup \{b, c, d\} = \{c\}$.

f. $\{1, 2\} \cap \{3, 9\} = \varnothing$.

g. If A is any set, $A \cup \varnothing = \varnothing$.

h. $\{1, 2\}$ and $\{3, \pi\}$ are disjoint.

i. If B is any set, B and $\varnothing$ are disjoint.

j. A set has 5 elements. It follows that the set has 25 subsets.

k. A set has 2048 subsets. It follows that the set has 11 elements.

l. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

m. If $A \subseteq B$, then $A \cup B = B$.

n. If $A \subseteq B$, then $A \cap B = A$.

❑ *TO ∞ AND BEYOND*

Let A be any set. We will denote the ***cardinality*** of A (the number of elements in A) by the notation $\boldsymbol{n(A)}$. We also define $\boldsymbol{\mathcal{P}(A)}$, called the ***power set*** of A, to be the set of all subsets of A.

For example, let $A = \{1, 5\}$. Then

$$n(A) = 2 \text{ and } \mathcal{P}(A) = \{ \varnothing, \{1\}, \{5\}, \{1, 5\} \}$$

Can you see that $n(\mathcal{P}(A)) = 4$?

A. State the definition of $\mathcal{P}(A)$ (fill in the blank):

$$\mathcal{P}(A) = \{ X \mid _____ \}$$

B. If A is a set such that $n(A) = 12$, calculate $n(\mathcal{P}(A))$.

C. Suppose B is a set such that $n(\mathcal{P}(B)) = 256$. Calculate $n(B)$.

Solutions

1. Yes; well-defined. **2**. No; exactly what is a high GPA?

3. $\in$, $\notin$ **4**. $\in$, $\subseteq$ **5**. a. T b. F **6**. F

7. 8 is an element of the first set, but it's not an element of the second set.

8. F — one set has 1 element in it, while the other set has 0 elements in it.

9. The first **10**. Every natural number is an integer, so $\mathbb{N} \subseteq \mathbb{Z}$. But $-3 \in \mathbb{Z}$, yet $-3 \notin \mathbb{N}$; thus $\mathbb{Z} \not\subseteq \mathbb{N}$.

11. A = B **12**. T **13**. {1, 2, 3, 4, 5} **14**. {5, 6, 7, 8, 9}

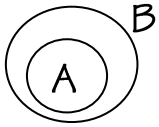
15. $\{x \in \mathbb{N} \mid x \text{ is even and } x \leq 10\}$ **16**. $\{x \in \mathbb{N} \mid x \text{ is a prime number} \leq 13\}$

17. T — duplicates can be discarded, and the order doesn't matter.

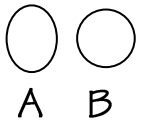
18. F **19**. T **20**. T — the null set is a subset of <u>every</u> set, including itself. **21**. 0; 11 **22**. 1

23. ∞; ∞; What do you think?

24.



25. Draw A inside B and B inside C. Is it clear that A is inside C, and therefore that A ⊆ C?

26. {*a*, *b*, *c*, *d*, *w*, *z*, *m*} **27**. {5} **28**. 

29. Let me know what sets you came up with.

30. Work out each side by doing what's in parentheses first. An example does not prove anything.

31. a. {*a*, *b*}, {*a*}, {*b*}, ∅ b. {*x*}, ∅ c. ∅

32. 1024 **33**. What do you think? **34**. 8 **35**. 7

36. A set is well-defined; it's always possible to tell whether something is in the set or not.

37. a. False b. A c. A **38**. 1 **39**. False **40**. A = {2, 4, 6}

41. False **42**. ∅; A **43**. B = {5, 7, 9}, for example

44. 128 **45**. a. 1 b. 2 **46**. 9

47. a. T **48**. 1 **49**. T

50. A = { 1, 3, 5, 7, 9} **51**. F **52**. A; A **53**. B = {1}

54. 256 **55**. a. 1 b. 4 **56**. 10

57. False; for example, let A = {1, 2} and B = {3}. Then
A ∪ B = {1, 2, 3}, while A ∩ B = ∅.

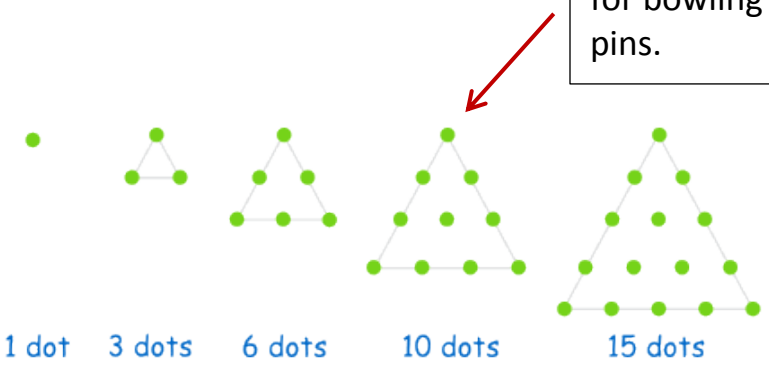
58. a. T b. T c. F d. T e. F f. T g. F h. T
i. T j. F k. T l. T m. T n. T

"Our progress as a nation can be no swifter than our progress in education."

– *John Fitzgerald Kennedy*

SEQUENCES

❑ *INTRODUCTION*



The setup for bowling pins.

A ***sequence*** is a listing of numbers in a particular order. (Actually, a sequence can be a listing of almost anything you can imagine, like baseball games, but for the first couple of Algebra courses we'll assume sequences of numbers.) For example: 1, 3, 8, 5 is a **finite** sequence of four terms, and is different from the finite sequence 3, 5, 1, 8 (even though each sequence consists of the same four numbers).

A sequence can also be **infinite**; an example would be

0, 2, 4, 6, 8, 10, 12, . . . ,

which represents all the **even** whole numbers.

The three dots "**. . .**" is an English punctuation symbol called an *ellipsis*, and denotes an *omission* of something — or in this case it means "and so on, forever."

Since order is the key idea behind a sequence, a sequence has a first term, a second term, a third term, and so on. For instance, the sequence (notice the ellipsis)

5, 8, 11, 14, . . ., 29

is a finite sequence consisting of nine terms:

5, 8, 11, 14, 17, 20, 23, 26, 29

The second term is 8, the fifth term is 17, and the ninth term is 29.

Our goal is to look at the first few terms of a given sequence, and then try to determine the value of a term farther out in the sequence.

Homework

1. Determine whether the sequence is finite or infinite:

   a. 2, 3, 4, 5, . . .

   b. 5, 7, 9, . . . 1001

   c. 7, 7, 7, 7, . . .

   d. π, π^2, π^3, π^4

2. a. What is the 7th term of the sequence 4, 8, 3, 0, 0, 9, 13, –1, π?

   b. Are the sequences 1, 2, 3, 4 and 4, 3, 2, 1 the same sequence?

3. In the dots diagram on Page 1, find the number of dots in each of the next five collections of dots.

❑ *Examples of Sequences*

EXAMPLE 1: **Consider the infinite sequence**

4, 8, 12, 16, 20, . . .

The 1st term is 4, the 2nd term is 8, and so on. What's the 10th term?

Solution: Here's an approach: Find the pattern among the existing terms, extend that pattern, and then predict the 10th term. Each term of the sequence appears to be four more than the preceding term. Continuing this pattern, it appears that the first 10 terms of the sequence are

4, 8, 12, 16, 20, 24, 28, 32, 36, 40.

Hence, **the 10th term is 40**, and we've accomplished our goal.



EXAMPLE 2: **Consider the sequence in Example 1 again: 4, 8, 12, 16, 20, . . . This time find the 900th term.**

Solution: Would you like to list 900 terms to see what the 900th term is? I didn't think so! Let's attack this problem from a different perspective. Here's the idea in a nutshell: *Instead of seeing how each term comes from the ones before it, let's see how each term comes from its position in the sequence.* We'll let n represent the position.

n	1	2	3	4	5
Term	4	8	12	16	20

Let's determine each term as a *function* of its position, *n*.

This may not be obvious at first, but notice that each term is 4 times its position. For example, the term 12 is 4 times its position of 3. The term 20 is 4 times its position of 5. As a formula, we can write

$$n\text{th term} = 4n$$

Answering the original question is now a snap. The 900th term is given by the expression 4(900), which is **3600**, and we're done.

EXAMPLE 3: **Find the 1000th term of the sequence 4, 7, 10, 13, 16, . . .**

Solution: As in Example 2, let's construct a chart showing each term of the sequence as a function of its position:

n	1	2	3	4	5
Term	4	7	10	13	16

A little analysis (a phrase that means "it's not easy!") shows that each term is 1 more than 3 times its position. For example, the term 16 is 1 more than 3 times its position of 5. In general,

$$n\text{th term} = 3n + 1$$

Therefore, the 1000th term is $3(1000) + 1 =$ **3001**.

EXAMPLE 4: **Find the 1000th term of 1, 4, 9, 16, 25, 36, . . .**

Solution: Listing our data in a nice chart gives:

n	1	2	3	4	5	6
Term	1	4	9	16	25	36

This example doesn't have the same kind of formula that's in the previous examples, since the terms are not growing at a constant rate — they're getting big very quickly. So let's look at it in a somewhat different way.

Notice that the term 25 is the square of its position $n = 5$. In fact, every term of the sequence is the square of its position (you verify this, for if there's just one place where our theory fails, then we have the wrong formula). We can write this as

$$n\text{th term} = n^2$$

And now it's just arithmetic. The 1000th term is $1000^2 =$ **1,000,000**.

EXAMPLE 5: **Find the 500th term of 2, 5, 10, 17, 26, 37, . . .**

Solution: Let's set up a chart.

n	1	2	3	4	5	6
Term	2	5	10	17	26	37

The trick here is to notice that each term of this sequence is one more than the square of the position. For instance, the term 26 is 1 more than the square of its position of 5. Thus, our formula here needs to be

$$n\text{th term} = n^2 + 1$$

The 500th term is therefore $500^2 + 1 =$ **250,001**.

EXAMPLE 6: **Find the 25th term of 2, 4, 8, 16, 32, 64, 128, . . .**

Solution:

n	1	2	3	4	5	6	7
Term	2	4	8	16	32	64	128

This one is also kind of obscure. But notice that each term is twice the previous term. This leads us to the realization that each term of the sequence can be found by raising 2 to the power of its position. For instance, the 5th term is $2^5 = 32$. Check out the first term: $2^1 = 2$. Also, the 7th term, which is 128, is indeed 2^7. Our summary is therefore

$$n\text{th term} = 2^n$$

The 25th term must be $2^{25} =$ **33,554,432**.

EXAMPLE 7: **Consider the infinite sequence**

–1, 1, 5, 13, 29, 61, 125, . . .

Find the 20th term.

Solution:

n	1	2	3	4	5	6	7
Term	–1	1	5	13	29	61	125

Think powers of 2 (as in the previous example). But this time notice that each of the terms of the sequence is 3 less than 2 raised to the power of its position. As an example, the 6th term (61) is 3 less than 2^6 (64). In general,

$$n\text{th term} = 2^n - 3.$$

In conclusion, the 20th term is $2^{20} - 3 =$ **1,048,573**.

EXAMPLE 8: **Find the 125th term of 1, 8, 27, 64, 125, 216, . . .**

Solution: Don't be upset if you don't see a pattern here. I wouldn't be able to see it either if I hadn't made up the problem myself. This is the sequence of perfect cubes:

$$1^3, 2^3, 3^3, 4^3, 5^3, 6^3, \ldots$$

Thus, the formula for the nth term is

$$n\text{th term} = n^3$$

We therefore have a 125th term of $125^3 =$ **1,953,125**.

EXAMPLE 9: **Find the 60th term of 0, 7, 26, 63, 124, 215, . . .**

Solution: Comparing this sequence with the sequence of perfect cubes in Example 8, we see that each term is 1 less than a perfect cube, so the formula must be

$$n\text{th term} = n^3 - 1$$

Letting $n = 60$, we calculate the 60th term to be **215,999**.

Homework

4. Find the 2000th term of 7, 12, 17, 22, 27, . . .
5. Find the 100th term of –1, 2, 7, 14, 23, 34, . . .
6. Find the 25th term of 5, 7, 11, 19, 35, 67, . . .
7. Find the 57th term of –1, 6, 25, 62, 123, . . .
8. Find the 7354th term of 6, 9, 14, 21, 30, 41, 54, . . .
9. Find the 3000th term of 13, 17, 21, 25, 29, . . .
10. Find the 123rd term of 4, 11, 30, 67, 128, . . .

"Life does not need to be changed. Only your intent and actions do."

Swami Rama

11. Find the 30th term of $-3, -1, 3, 11, 27, 59, \ldots$
12. Find the 62nd term of $4, 7, 10, 13, 16, \ldots$
13. Find the 81st term of $4, 7, 12, 19, 28, 39, \ldots$
14. Find the 71st term of $-2, 5, 24, 61, 122, 213, \ldots$
15. Find the 87th term of $0, 7, 26, 63, 124, 215, \ldots$
16. Find the 67th term of $-3, 0, 5, 12, 21, 32, \ldots$
17. Find the 64th term of $2, 5, 10, 17, 26, 37, \ldots$

Review Problems

18. Find the 32nd term of $5, 7, 11, 19, 35, 67, 131, \ldots$
19. Find the 200th term of $10, 13, 16, 19, \ldots$
20. Find the 100th term of $4, 7, 12, 19, 28, \ldots$
21. Find the 30th term of $1, 3, 7, 15, 31, 63, \ldots$
22. Find the 100th term of $3, 10, 29, 66, 127, \ldots$

23. True/False:
    a. The sequence 1, 2, 3 is the same sequence as 3, 2, 1.
    b. It's possible that the terms of a sequence are all the same.
    c. The 2000th term of $20, 23, 26, 29, \ldots$ is 6017.
    d. The 100th term of $11, 14, 19, 26, 35, \ldots$ is 10,000.
    e. The 100th term of $-9, -2, 17, 54, 115, \ldots$ is 999,990.
    f. The 30th term of $10, 12, 16, 24, 40, 72, \ldots$ is 1,073,741,832.

Solutions

1. a. Infinite b. Finite c. Infinite d. Finite

2. a. 13 b. No **3**. 21, 28, 36, 45, 55

4. 10,002 **5**. 9,998 **6**. 33,554,435

7. 185,191 **8**. 54,081,321 **9**. 12,009 **10**. 1,860,870

11. 1,073,741,819 **12**. 187 **13**. 6564 **14**. 357,908

15. 658,502 **16**. 4485 **17**. 4097 **18**. 4,294,967,299

19. 607 **20**. 10,003 **21**. 1,073,741,823

22. 1,000,002

23. a. F b. T c. T d. F e. T f. T

❑ *To ∞ and Beyond*

"I think and think for months and years. Ninety-nine times, the conclusion is false. The hundredth time I am right."

– Albert Einstein

SERIES



A mathematical ***series*** is a fancy term for **sum**, or summation. A **finite** series might be the summation

$$1 + 2 + 5 + 6 + 10 \qquad \text{(which has a } \textbf{sum} \text{ of 24)}$$

An **infinite** series might be something like this:

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\dots$$

Believe it or not, this *infinite series* actually has a sum of $\frac{1}{2}$.

where the three dots (called an *ellipsis*) mean to go on and on forever (infinitely many terms being added).

Sometimes, if there's a nice pattern, a series can be written in "**sigma**" notation. The Greek capital letter sigma, Σ, is used to represent the summation (since sigma and sum both begin with the letter ***s***). The following examples should explain how this notation should be read and calculated.

EXAMPLE 1: **Evaluate:** $\displaystyle\sum_{k=2}^{5}(2k+1)$

Solution: The expression that determines the numbers we will add together is $2k + 1$. Below the sigma sign is the starting value of the index variable k, in this case 2; above the sigma sign is the ending value of k, in this case 5. So k starts at 2 and ends at 5, but what does k do in between? We agree that it goes up by one — in other words, k will go 2, 3, 4, and then 5. Check it out, remembering that the sigma sign, Σ, means ADD:

$$\sum_{k=2}^{5}(2k+1) = \underset{(k=2)}{(2\cdot\mathbf{2}+1)} + \underset{(k=3)}{(2\cdot\mathbf{3}+1)} + \underset{(k=4)}{(2\cdot\mathbf{4}+1)} + \underset{(k=5)}{(2\cdot\mathbf{5}+1)}$$

$$= \quad 5 \quad + \quad 7 \quad + \quad 9 \quad + \quad 11$$

$$= \boxed{\mathbf{32}}$$

EXAMPLE 2: **Evaluate:** $\displaystyle\sum_{n=0}^{3}(n^2-n)$

Solution: Does it matter that Example 1 used the index variable k and this example uses the index variable n? Not at all — the variable used makes no difference in the final answer; it's just a placeholder. The values of n will be 0, 1, 2, and 3. For each value of n we evaluate the expression $n^2 - n$. Then we add up the results.

$$\sum_{n=0}^{3}(n^2-n) = \underset{(n=0)}{(\mathbf{0}^2-\mathbf{0})} + \underset{(n=1)}{(\mathbf{1}^2-\mathbf{1})} + \underset{(n=2)}{(\mathbf{2}^2-\mathbf{2})} + \underset{(n=3)}{(\mathbf{3}^2-\mathbf{3})}$$

$$= \quad 0 \quad + \quad 0 \quad + \quad 2 \quad + \quad 6$$

$$= \boxed{\mathbf{8}}$$

Homework

Evaluate each series (calculate each sum):

1. $\displaystyle\sum_{k=3}^{5}(7k-1)$
2. $\displaystyle\sum_{n=2}^{5}(n^2+n)$
3. $\displaystyle\sum_{k=1}^{6}\frac{1}{2^k}$
4. $\displaystyle\sum_{j=0}^{4}3^j$
5. $\displaystyle\sum_{n=16}^{16}\frac{1}{2}\sqrt{n}$
6. $\displaystyle\sum_{k=3}^{5}\frac{1}{k}$

7. $\sum_{m=0}^{2} \frac{1}{m+1}$ 8. $\sum_{t=-1}^{4} 2t$ 9. $\sum_{j=-2}^{2} j^3$

For an extra challenge, write each summation in Σ-notation; use k as the index variable, and start at $k = 1$:

10. $10+11+12+13$ 11. $4^3 + 5^3 + 6^3$

12. $1+\sqrt{2}+\sqrt{3}+2$ 13. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$

14. $1+\sqrt[3]{2}$ 15. $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$

Review Problems

16. $\sum_{n=2}^{5} (n^2+n) =$ 17. $\sum_{k=1}^{6} \frac{1}{2^k} =$ 18. $\sum_{j=0}^{4} 3^j =$

19. $\sum_{n=16}^{16} \frac{1}{2}\sqrt{n} =$ 20. $\sum_{k=1}^{10} k =$ 21. $\sum_{n=0}^{4} 2^{n-2} =$

22. True/False:

a. $\sum_{i=2}^{5} i^2 = 54$ b. $\sum_{j=0}^{3} 2^j = 16$

c. $\sum_{k=-1}^{1} k^3 = 3$ d. $\sum_{L=0}^{4} \sqrt{L} = 3+\sqrt{2}+\sqrt{3}$

23. [Only if you've studied *Combinations*]:

$$\sum_{k=0}^{4} \binom{4}{k} a^{4-k} b^{k} \;=$$

This can also be written $\sum_{k=0}^{4} C(4,k)\, a^{4-k} b^{k}$.

❑ *To ∞ and Beyond*

A. Evaluate: $\sum_{k=1}^{99} \left(\frac{1}{k} - \frac{1}{k+1} \right)$

B. Evaluate: $\sum_{k=1}^{\infty} \frac{1}{2^k}$

C. Evaluate: $\sum_{k=1}^{\infty} \frac{1}{k}$

[Research]

D. Evaluate: $\sum_{n=1}^{4} \frac{(-1)^{n+1}}{n^3}$

[A decimal answer will be fine.]

E. Write in Σ-notation. Use k for the index variable, starting with $k = 0$.

$$\frac{1}{2} + \frac{3}{3} + \frac{5}{6} + \frac{7}{11} + \frac{9}{18} + \frac{11}{27}$$

Solutions

1. 81 **2.** 68 **3.** $\frac{63}{64}$ **4.** 121

5. 2 **6.** $\frac{47}{60}$ **7.** $\frac{11}{6}$ **8.** 18

9. 0 **10.** $\sum_{k=1}^{4}(k+9)$ **11.** $\sum_{k=1}^{3}(k+3)^3$ **12.** $\sum_{k=1}^{4}\sqrt{k}$

13. $\sum_{k=1}^{5}\frac{1}{k}$ **14.** $\sum_{k=1}^{2}\sqrt[3]{k}$ **15.** $\sum_{k=1}^{4}\frac{1}{2^{k+1}}$

16. 68 **17.** $\frac{63}{64}$ **18.** 121

19. 2 **20.** 55 **21.** $\frac{31}{4}$

22. a. T b. F c. F d. T

23. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, which can also be written $(a + b)^4$

Check out: **Math is Fun - Infinite Series**

“The highest activity a human being can attain is learning for understanding, because to understand is to be free.”

– Baruch Spinoza

THE COUNTING PRINCIPLE

A certain **computer password** must contain exactly 4 characters:



1st character: any uppercase letter of the English alphabet

2nd character: any uppercase letter different from the 1st character

3rd character: any digit

4th character: any digit different from the 3rd character

How many distinct passwords can be formed? Stay tuned; we'll find out.

❑ *THE COUNTING PRINCIPLE*

EXAMPLE 1:

A. Joey has 4 belts and 2 caps. How many ways can he choose a belt and a cap? Let's list them all:

b1, c1	b1, c2	b2, c1	b2, c2
b3, c1	b3, c2	b4, c1	b4, c2

We see **8 ways** to choose his belt and cap. What's the shortcut? Just multiply 4 by 2, and we have 8 ways.

B. A new design for a license plate has 3 slots, consisting of a letter, a digit from 0 to 9, and another letter <u>different</u> from the first letter. How many license plates can be formed?

There are **26** possibilities for the first slot (since we're using English).

There are **10** possibilities for the second slot (there are 10 digits from 0 to 9).

There are 25 possibilities for the third slot (the letter in the third slot must differ from the letter in the first slot — that leaves just 25 possibilities).

Therefore, there are $26 \times 10 \times 25 =$

6,500 possible license plates

Homework

1. Anita has 3 skirts, 4 blouses, and 2 caps. How many different outfits can Anita create from these items?

2. James can choose 1 of 4 flavors of ice cream, 1 of 3 kinds of nuts, and 1 of 5 toppings. How many different ice cream sundaes can James create?



3. A license plate has 4 slots. The first slot is a vowel (AEIOU), the second slot is a vowel different from the first, the third slot is any digit, and the fourth slot is any letter different from the first two. How many different license plates can be formed?

4. Maria can choose 1 of 3 CPUs for her new computer, 1 of 4 different hard drives, 1 of 5 different monitors, and 1 of 7 different mice. How many different ways can Maria choose the

components for her computer? In other words, how many different computer "configurations" can she choose from?

5. A license plate has three slots. The first slot is any consonant (a letter that's not a vowel). The second slot is a digit from 0 to 9. The third slot is a digit from 0 to 9 that must be different from the digit in the second slot. How many distinct license plates can be formed?

6. At a pizza restaurant, a customer can choose 1 of 3 types of crust, 1 of 5 kinds of sauce, and 1 of 12 toppings. How many different pizza combinations can a customer choose?

7. How many different ways can 5 students be arranged in a line for a photograph?

8. Two dice are rolled at the same time. How many different outcomes are possible?

9. An ice cream shop offers 4 different types of cones and 10 different flavors of ice cream. If a customer orders an ice cream cone with one scoop, how many different combinations of cone and ice cream flavor are possible?

10. Now we solve the problem stated in the Introduction:

    A computer password must contain exactly 4 characters —

    1st character: any uppercase letter of the alphabet

    2nd character: any uppercase letter different from the 1st character

    3rd character: any digit

    4th character: any digit different from the 3rd character

    How many distinct passwords can be formed?

11. Joe has 3 pairs of shoes, 5 pairs of pants, 4 shirts, and 8 pairs of sunglasses. How many different "outfits" can Joe create?

12. True/False:

    If a family of five ducks is trying to line up in all possible orders, but the papa duck must be in the first slot, then there are 24 ways the family of ducks can line up.

13. True/False:

    Suppose a 3-slot license plate must have a digit in the first slot, a vowel in the second slot, and any digit different from the first slot in the third slot. In that case, 500 distinct license plates are possible.

Solutions

1. 24 outfits

2. 60 sundaes

3. 4,800 license plates

4. 420 configurations

5. 1,890 license plates

6. 180 pizzas

7. 120 arrangements

8. 36 outcomes

9. 40 cones

10. 58,500 passwords

11. 480 outfits

12. T

13. F

"The whole purpose of education is to turn mirrors into windows."

Sydney J. Harris

FACTORIALS



The odds of winning the original California lottery depended on the number of ways that **6** numbers could be chosen from a set of **51** numbers. Listing them would be ludicrous (since, as you'll discover in the chapter Combinations, there are over **18** million ways!). This chapter will develop the concept of the ***factorial*** function, leading us to a not-too-difficult formula for such a calculation.

❑ *GETTING ALL OUR DUCKS IN A ROW*



We start by posing the question:

How many ways are there to line up 4 ducks in a straight line?

We'll call the ducks *a*, *b*, *c*, and *d*. Let's start listing all the ways, organized in columns to keep track of what we've already listed.

abcd	*bacd*	*cadb*	*dabc*
abdc	*badc*	*cadb*	*dacb*
acbd	*bcad*	*cbad*	*dbac*
acdb	*bcda*	*cbda*	*dbca*
adbc	*bdac*	*cdab*	*dcab*
adcb	*bdca*	*cdba*	*dcba*

Make sure that we haven't omitted any, and also make sure we haven't included duplicates. It appears that we have **24** ways to line up the 4 ducks. That wasn't too bad, but now consider this: *How many ways are there to line up 10 ducks in a straight line*? My advice to you is don't bother — at 1 second per duck lineup, working 24 hours per day, it would take exactly 42 days to complete the list. What we need is a formula, based on some new insight, to describe what it really means to place 4 things in a row. We'll discover one in the following section.

Homework

1. How many ways are there to line up 1 ostrich in a straight line?
2. How many ways are there to line up 2 people in a straight line?
3. How many ways are there to line up 3 books in a straight line?
4. How many ways are there to line up 0 ducks in a straight line?

❑ *FACTORIAL*

Now we re-solve the duck question posed above:

How many ways are there to line up 4 ducks in a straight line?

using the chapter The Counting Principle.

How many ways are there to choose a duck to be first in line? There are **4** ways (since there are 4 ducks from which to choose).

Now, how many choices are there for the second position? There are **3**, because only 3 ducks remain.

For the third position, there are **2** choices because 2 ducks remain, and for the fourth position, there's only **1** choice because only 1 duck remains. Thus, there are **4 × 3 × 2 × 1** ways of lining up the ducks,

which comes to **24** — the same answer we obtained when we listed them by brute force in the chapter The Counting Principle.

"Because computations like **4 × 3 × 2 × 1** occur frequently in mathematics, they are represented by a special notation. We say "four ***factorial***" and use an exclamation point to write "four factorial" as "**4!**". Thus,

$$\mathbf{4!} = 4 \times 3 \times 2 \times 1 = \mathbf{24}$$

Also, in the first section of this chapter, I claimed that it would take 42 days to line up 10 ducks. Let's see why: First, we know there are "10 factorial" ways to line up the 10 ducks. That's

$$\mathbf{10!} = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{3{,}628{,}800}$$



ways to line up the 10 ducks. At 1 second per duck lineup, that's a total time of 3,628,800 seconds, which equals 60,480 minutes, which is 1,008 hours, which converts to exactly **42 days**. But we don't need 42 days to list them all to find out how many ways there are to line up the ducks — we just calculate **10!** and we have the number of ways.

Now we're ready for the official definition of factorial:

If n is a natural number (1, 2, 3, . . .), then

n factorial is defined to be

$$n! = n(n-1)(n-2)(n-3)\cdots(3)(2)(1)$$

EXAMPLE 1:

A. $3! = 3 \cdot 2 \cdot 1 = \mathbf{6}$

B. $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{5{,}040}$

C. $\dfrac{10!}{7! \times 3!} = \dfrac{3{,}628{,}800}{5{,}040 \times 6} = \dfrac{3{,}628{,}800}{30{,}240} = \mathbf{120}$

D. $\dfrac{200!}{198!}$ This is a tough one. Even with a calculator, multiplying 200 by 199 by 198 by 197 right on down the line would take a really long time. Or, we could use the "factorial" button on the calculator, but we'll likely get an ERROR message. So, what to do? Check this out:

$$\frac{200!}{198!} = \frac{200 \times 199 \times 198 \times 197 \times 196 \times \cdots \times 3 \times 2 \times 1}{198 \times 197 \times 196 \times \cdots \times 3 \times 2 \times 1}$$

See what cancels? All the factors from 198 down to 1.

$$= \frac{200 \times 199 \times \cancel{198 \times 197 \times 196 \times \ldots \times 3 \times 2 \times 1}}{\cancel{198 \times 197 \times 196 \times \ldots \times 3 \times 2 \times 1}}$$

All that remains is 200 × 199, which equals **39,800**. Cool?

E. $\dfrac{75!}{72! \times 3!} = \dfrac{75 \times 74 \times 73 \times 72 \times 71 \times 70 \times \cdots \times 3 \times 2 \times 1}{[72 \times 71 \times 70 \times \cdots \times 3 \times 2 \times 1] \times [3 \times 2 \times 1]}$

$$= \frac{75 \times 74 \times 73 \times \cancel{72 \times 71 \times 70 \times \cdots \times 3 \times 2 \times 1}}{[\cancel{72 \times 71 \times 70 \times \cdots \times 3 \times 2 \times 1}] \times [3 \times 2 \times 1]}$$

$$= \frac{75 \times 74 \times 73}{3 \times 2 \times 1}$$

$= 25 \times 37 \times 73$ [75/3 = 25 and 74/2 = 37]

$= \mathbf{67{,}525}$

F. $2! = 2 \times 1 = \mathbf{2}$

G. $1! = \mathbf{1}$

H. $0! = \mathbf{1}$

This result doesn't follow from the definition of factorial. We simply define **0!** to equal **1**. It's more of a philosophical question: How many ways are there to line up no ducks? Do you buy the notion that there's **1** way to do that?

But there's also a more concrete reason why we define 0! = 1. It makes future formulas (permutations and combinations) work perfectly.

Homework

5. Calculate: a. $(7 + 2)!$ b. $8! - 7!$ c. $2^{3!}$

6. Calculate: a. $\frac{97!}{95!}$ b. $\frac{100!}{95! \times 5!}$ c. $\frac{347!}{(0!)(347!)}$

7. You have to line up 20 kindergartners at the door in all possible ways. It takes one second for each possible lineup. How many years will it take you to accomplish this feat (assuming they cooperate)?

8. Does the factorial operation "respect addition"? That is, is the following statement true?

$$(a + b)! = a! + b!$$

FACTORIALS WITH VARIABLES: N! (N-1)! (N – 2)! (N + 3)!

Review Problems

Factorials

9. Calculate: $\frac{200!}{5! \times 195!}$

10. Calculate: $0! + 1! + 2! + 3! + 4! + 5!$

11. Analyze the following conjecture: $(ab)! = a!\, b!$

12. True/False:

    a. If a family of five ducks is trying to line up in all possible orders, but the papa duck must be in the first slot, then there are 24 ways the family of ducks can line up.

    b. There are 8! ways of lining up 8 children in a straight line.

    c. $\frac{1{,}000!}{500!} = 2$

    d. $0! = 0$

    e. $(m + n)! \neq m! + n!$ [where m and n are any whole numbers]

Solutions

1. 1 **2**. 2 **3**. 6

4. It's somewhat of a philosophical answer, but let's agree that there's 1 way.

5. a. 362,880 b. 35,280 c. 64

6. a. 9,312 b. 75,287,520 c. 1

7. About 77,000,000,000 years (77 billion years — vastly longer than the age of the universe).

8. Choose $a = 2$ and $b = 3$. Then

$(a + b)! = (2 + 3)! = 5! = 120$

$a! + b! = 2! + 3! = 2 + 6 = 8$ They're not even close.

Conclusion: $(a + b)! \neq a! + b!$

9. 2,535,650,040 **10**. 154

11. Let $a = 3$ and $b = 2$.
Then $(ab)! = (3 \cdot 2)! = 6! = 120$,
while $a!\, b! = 3!\, 2! = 6 \times 2 = 12$.
The conjecture is false.

12. a. T b. T c. F d. F e. T

❑ *TO ∞ AND BEYOND*

[Research] Find the sum of the series consisting of the reciprocals of the factorials:

$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

which is the infinite sum $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$ out to infinity.

"What the mind of man can conceive and believe, it can achieve."

Napoleon Hill

COMBINATIONS

Now we discuss the lottery problem mentioned in the Introduction of the chapter *Factorials*.



There are 51 numbers — you choose 6 of them. The order in which you choose the numbers does not make any difference, as long as they're the same 6 numbers selected on the Big Spin. So, to figure the probability of getting all 6 numbers right, we have to ask:

How many ways are there to select 6 numbers from a set of 51 numbers?

Let's put our toes in the water by solving a simpler problem first.

❑ *COMBINATIONS*

EXAMPLE 1: **How many ways are there to choose 3 letters from the set of 5 letters {a, b, c, d, e}?**

Solution: Remember, the order doesn't count, so the selection "bce," for instance, is the same as "ecb." A little trial and error yields the following list:

abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde

Count them; there are **10 ways** to select 3 letters from the 5.

Can we solve the lottery problem (choosing 6 out of 51) using the same method we just used in Example 1? Sure we could, requiring a boxful of paper and months of writing. There's got to be a formula for this!

Our formula will use the notation $\binom{n}{k}$, read "***n* choose *k***." It's the number of ways we can choose ***k*** objects from a set of ***n*** objects. The formula, using our friend the factorial, is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The "*n* choose *k*" Formula

$\binom{n}{k}$ is also called "a ***combination*** of ***n*** things taken ***k*** at a time."

Another common notation for $\binom{n}{k}$ is $C(n,k)$.

Let's redo Example 1 using our new combination formula.

EXAMPLE 2: **How many ways are there to choose a subcommittee of 3 people from a committee of 5 people?**

Solution: We need to calculate "***5 choose 3***."

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!\times 2!} = \frac{120}{6\times 2} = \frac{120}{12} = \mathbf{10}$$

Is this the answer we obtained in Example 1?

We now have the tools and the talent (I stole this phrase from *Ghostbusters*) to solve the lottery problem. However, we need one more calculation example demonstrating the power of reducing a fraction before we grab our calculators.

EXAMPLE 3: **Calculate:** $\binom{10}{7}$.

Solution: This is a **combination of 10 things taken 7 at a time** (and order doesn't matter).

$$\binom{10}{7} = \frac{10!}{7!(10-7)!}$$

$$= \frac{10!}{7!\,3!}$$

$$= \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{[7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1]\times[3\cdot 2\cdot 1]}$$

$$= \frac{10\cdot 9\cdot 8\cdot \cancel{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}}{[\cancel{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}]\times[3\cdot 2\cdot 1]}$$

$$= \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}$$

$$= \mathbf{120}$$

EXAMPLE 4: **The Lottery Question: How many ways can 6 numbers be chosen from a set of 51 numbers?**

Solution: We apply the combination formula again:

$$\binom{51}{6} = \frac{51!}{6!\,(51-6)!}$$

$$= \frac{51!}{6!\times 45!}$$

$$= \frac{51\times 50\times 49\times 48\times 47\times 46\times 45\times 44\times 43\times\cdots\times 3\times 2\times 1}{[6\times 5\times 4\times 3\times 2\times 1]\times[45\times 44\times 43\times\cdots\times 3\times 2\times 1]}$$

$$= \frac{51\times 50\times 49\times 48\times 47\times 46\times \cancel{45\times 44\times 43\times\cdots\times 3\times 2\times 1}}{[6\times 5\times 4\times 3\times 2\times 1]\times[\cancel{45\times 44\times 43\times\cdots\times 3\times 2\times 1}]}$$

$$= \frac{12{,}966{,}811{,}200}{720} = \mathbf{18{,}009{,}460}$$

Don't panic if your calculator goes into scientific notation in the numerator. It will all work out in the end.

Since there are over 18 million ways to select 6 out of 51 numbers, you have about **1 chance in 18 million** to win the lottery. And yes, you could win the lottery — guaranteed — by buying 18,009,460 different lottery tickets. What are the fallacies in this reasoning?

Our final example involves the combination formula using a variable.

EXAMPLE 5: **How many ways are there to select 2 objects out of n objects? (We assume $n \geq 2$.)**

Solution: The combination formula gives us

$$\binom{n}{2} = \frac{n!}{2! \times (n-2)!} = \frac{n(n-1)(n-2)(n-3)\cdots(2)(1)}{[2\times 1] \times [(n-2)(n-3)\cdots(2)(1)]}$$

$$= \frac{n(n-1)\,\cancel{(n-2)(n-3)\cdots(2)(1)}}{2\times \cancel{(n-2)(n-3)\cdots(2)(1)}} = \frac{n(n-1)}{2} = \boldsymbol{\frac{n^2-n}{2}}$$

Homework

1. Calculate: a. $\binom{7}{3}$ b. $\binom{10}{9}$ c. $\binom{12}{6}$ d. $\binom{20}{12}$

2. Calculate: a. $\binom{100}{98}$ b. $\binom{200}{1}$ c. $\binom{132}{0}$ d. $\binom{750}{750}$

3. Calculate: a. $\binom{n}{0}$ b. $\binom{n}{1}$ c. $\binom{n}{2}$ d. $\binom{n}{3}$

4. Calculate: a. $\binom{n}{n}$ b. $\binom{n}{n-1}$ c. $\binom{n}{n-2}$ d. $\binom{n}{n-3}$

5. a. Show that $\binom{1{,}000}{250} = \binom{1{,}000}{750}$.

   b. [Hard] Now try to prove that $\binom{n}{k} = \binom{n}{n-k}$.

6. Calculate:

   a. $\binom{2}{0}+\binom{2}{1}+\binom{2}{2}$

   b. $\binom{3}{0}+\binom{3}{1}+\binom{3}{2}+\binom{3}{3}$

   c. $\binom{4}{0}+\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4}$

   d. $\binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5}$

   e. $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}+\binom{n}{n}$

7. How many ways are there to choose 5 numbers in a lottery containing the numbers 1 through 100?

Review Problems

8. How many ways are there to choose 12 letters from the first 15 letters of the alphabet, where order does not matter?

9. In a lottery with 75 numbers, how many ways can you choose 3 numbers?

10. How many ways can you select 3 objects from n objects?

11. Calculate: a. $\binom{197}{197}$ b. $\binom{299}{0}$ c. $\binom{1024}{1}$

12. Prove that $\binom{1000}{123} = \binom{1000}{877}$.

13. True/False:

    a. $\binom{n}{0} = \binom{n}{n}$

    b. $\binom{p}{q} = \binom{p}{p-q}$

    c. There are over 20 million ways to choose 6 numbers from a set of 51 numbers.

    d. There are $\frac{n^2-n}{2}$ ways to choose 2 people out of n people.

Solutions

1. a. 35 b. 10 c. 924 d. 125,970

2. a. 4,950 b. 200 c. 1 d. 1

3. a. 1 b. n c. $\frac{n^2-n}{2}$ d. $\frac{n^3-3n^2+2n}{6}$

4. a. 1 b. n c. $\frac{n^2-n}{2}$ d. $\frac{n^3-3n^2+2n}{6}$

5. a. $\binom{1000}{250} = \frac{1000!}{250!\times(1000-250)!} = \frac{1000!}{250!\times 750!}$. Also, $\binom{1000}{750} = \frac{1000!}{750!\times(1000-750)!} = \frac{1000!}{750!\times 250!}$. Since these last two expressions are equivalent (no matter their final numerical values), we've proved the assertion.

b. Like I'm going to give this one away!

6. a. 4 b. 8 c. 16 d. 32
e. I'm curious what you got.

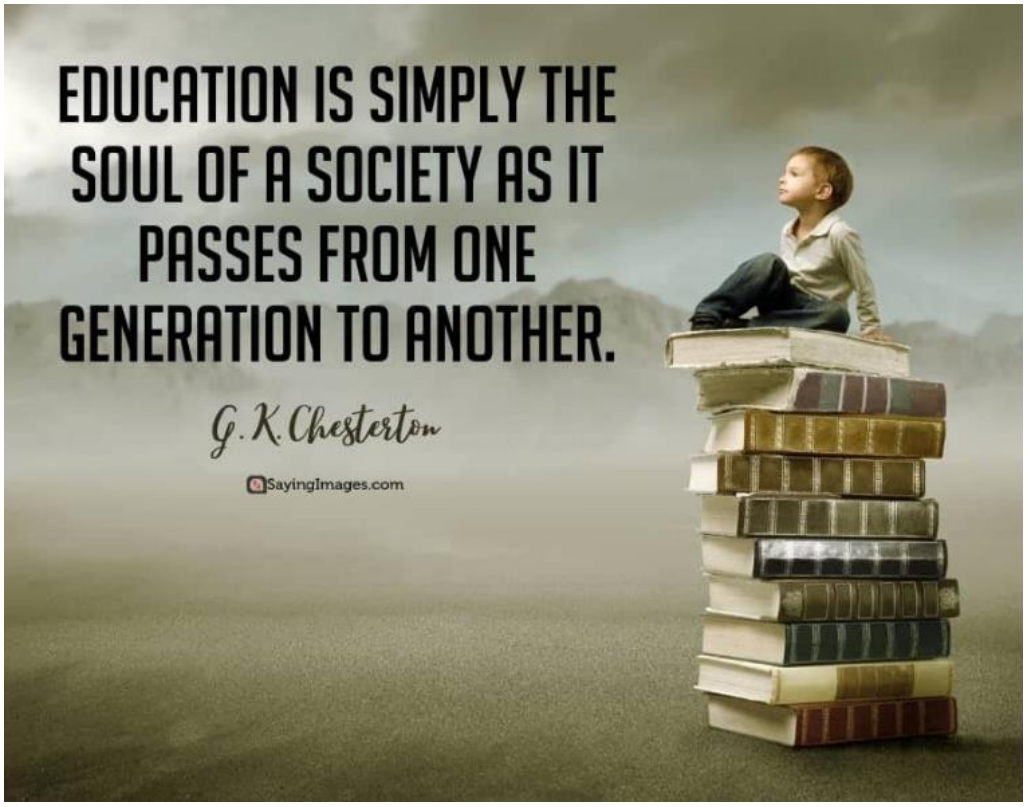
7. 75,287,520

8. 455 **9**. 67,525 **10**. $\frac{1}{6}n^3-\frac{1}{2}n^2+\frac{1}{3}n$

11. a. 1 b. 1 c. 1,024

12. Convince yourself.

13. a. T b. T c. F d. T


EDUCATION IS SIMPLY THE SOUL OF A SOCIETY AS IT PASSES FROM ONE GENERATION TO ANOTHER.
G. K. Chesterton
SayingImages.com

THE BINOMIAL THEOREM – PASCAL'S TRIANGLE

Imagine expanding $(a + b)^{100}$ using regular algebra. This chapter will show us a neat shortcut for solving this type of problem. The concepts involved in raising a binomial to a high power have applications in probability, from the flipping of coins to quality control in manufacturing light bulbs.



❑ *EXPANDING AND SEEING PATTERNS*

The goal of the Binomial Theorem is to expand powers of binomials, things like $(a + b)^{100}$, without actually multiplying anything out. Let's start with some expansions we already know how to do using brute-force algebra.

$(a + b)^0 = 1$ (anything [but 0] to the zero power is 1)

$(a + b)^1 = a + b$ (anything to the 1st power is itself)

$(a + b)^2 = a^2 + 2ab + b^2$ (just multiply $a + b$ by $a + b$)

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ (multiply $a^2 + 2ab + b^2$ by $a + b$)

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ (etc.)

$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$ (etc.)

Homework

1. Verify the calculations above for each expansion:

   $(a + b)^2$ $(a + b)^3$ $(a + b)^4$ $(a + b)^5$

2. How many terms are there in each expansion?

   a. $(a + b)^0$ b. $(a + b)^1$ c. $(a + b)^2$

   d. $(a + b)^3$ e. $(a + b)^4$ f. $(a + b)^5$

3. a. How many terms are there in the expansion of $(a + b)^{100}$?

   b. How many terms are there in the expansion of $(a + b)^n$?

4. a. Notice the term $10a^3b^2$ in the expansion of $(a + b)^5$. What is the *sum* of the exponents on the a and the b?

   b. In the same expansion, the 5th term is $5ab^4$. What is the *sum* of the exponents on the a and the b?

5. a. See the first term in the expansion of $(a + b)^4$? It's a^4. What is the *sum* of the exponents on the a and the b? [Hint: Maybe there's no explicit b in the term a^4, but what power of b could be placed next to the a^4 so that it remains a^4? That is, $a^4b^{?} = a^4$.]

   b. What is the *sum* of the exponents for <u>any</u> term in the expansion of $(a + b)^4$?

6. What is the *sum* of the exponents for <u>any</u> term in the expansion of $(a + b)^n$?

7. If the expansion of $(a + b)^k$ has a term of the form $ca^{44}b^{33}$, where c is a constant, what is the value of exponent k?

❑ *OBSERVATIONS ON THE EXPANSIONS*

1. The **number of terms** in each expansion is <u>one more than the exponent</u> on the binomial. For example, the expansion of $(a + b)^5$ has **6** terms, and the expansion of $(a + b)^n$ has $\boldsymbol{n + 1}$ terms.

2. Within any given term, the **sum of the exponents** always equals the exponent on the binomial. For instance, looking at the expansion of $(a + b)^5$ from page 1:

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

If we "patch up" the expansion of $(a + b)^5$ to include both the a and the b in every term, we can write the previous expansion as

$$(a + b)^5 = a^5\boldsymbol{b^0} + 5a^4\boldsymbol{b^1} + 10a^3b^2 + 10a^2b^3 + 5\boldsymbol{a^1}b^4 + \boldsymbol{a^0}b^5$$

We notice that in any of the 6 terms, the <u>sum</u> of the exponents is always **5**. This always works: In every term in the expansion of $(a + b)^n$, the sum of the exponents is always $\boldsymbol{n}$.

3. The exponents on the a go down while the exponents on the b go up. Again, look at the form in the previous observation:

$$(a + b)^5 = a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + a^0b^5$$

Notice that the exponents on the a go down: 5, 4, 3, 2, 1, 0, while the exponents on the b go up: 0, 1, 2, 3, 4, 5.

<u>EXAMPLE 1:</u> **Apply the three observations above to the expansion of $(a + b)^{100}$.**

<u>Solution:</u> First, there will be **101** terms in the expansion. Second, in any term the sum of the exponents will be **100**. For instance, the term with a^{40} in it will also have b^{60} in it. Third, the exponents on the a will go 100, 99, 98, . . ., 2, 1, 0 while the exponents on the b will go 0, 1, 2, . . . , 98, 99, 100. In short, using boxes for the unknown coefficients (the front numbers), the expansion of $(a + b)^{100}$ looks like this:

$$\square a^{100} + \square a^{99}b + \square a^{98}b^2 + \cdots + \square a^2b^{98} + \square ab^{99} + \square b^{100}$$

❑ *PASCAL'S TRIANGLE*

Now let's focus on the ***coefficients*** in the expansions we did on the first page (remember that the term a^4, for instance, is really $\mathbf{1}a^4$):

$(a+b)^0$ 1

$(a+b)^1$ 1 1

$(a+b)^2$ 1 2 1

$(a+b)^3$ 1 3 3 1

$(a+b)^4$ 1 4 6 4 1

$(a+b)^5$ 1 5 10 10 5 1



Our goal is to extend this triangle of coefficients further by looking at the rows of the current triangle. Then we'll know the coefficients for even higher powers of $a + b$. Check out the **6** in the triangle; notice that it's the sum of the **3** and the **3** above it. Look at the first **10**; it's the sum of the **4** and the **6** above it. This pattern generates ***Pascal's Triangle***. Let's extend the triangle even further:

1 ← **Row 0**

1 1 ← **Row 1**

1 2 1 ← **Row 2**

1 3 3 1 ← **Row 3**

1 4 6 4 1 ← **Row 4**

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1 ← **Row 8**

1 9 36 84 126 126 84 36 9 1 ← **Row 9**

Each entry of *PASCAL'S TRIANGLE* (starting in Row 2) is the sum of the two entries above it.

Homework

8. Consider the expansion of $(a + b)^{23}$. How many terms are there in the expansion? In the term containing a^{15}, what is the exponent on the b?

9. There are 102 terms in the expansion of $(x + y)^n$. What is n?

10. Find "Row 10" of Pascal's Triangle.

❑ *An Example of the Binomial Theorem*

Let's use the three observations mentioned earlier, along with Pascal's Triangle, to deduce the expansion, without using algebra, of

$$(a + b)^7$$

1. There will be **8** terms in the expansion.

2. The sum of the exponents in each term will be **7**.

3. The exponents on the a will go from 7 down to 0, while those on the b will go from 0 up to 7.

   Also, the relevant row of Pascal's Triangle is

   1 7 21 35 35 21 7 1

   There are two ways we can know this. First, it's the only row in Pascal's Triangle with 8 entries in it, and we need 8 coefficients for the 8 terms of the expansion. Moreover, the second number in the row, the 7, matches the exponent in the expression $(a + b)^7$.

Let's try it: $(a+b)^7$

$$= \underline{1}a^7b^0 + \underline{7}a^6b^1 + \underline{21}a^5b^2 + \underline{35}a^4b^3 + \underline{35}a^3b^4 + \underline{21}a^2b^5 + \underline{7}a^1b^6 + \underline{1}a^0b^7$$

The underlined coefficients come from Row 7 of Pascal's Triangle.

$$= \boldsymbol{a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7}$$

Is the Binomial Theorem worth knowing? Try expanding $(a+b)^7$ without it.

Homework

11. Find the first two terms and the last two terms of $(a+b)^{100}$.

12. Even though we know how to expand $(a+b)^2$ from basic algebra, show that the Binomial Theorem produces the same result.

13. Expand $(m+z)^8$.

14. Expand $(x+y)^{10}$.

Review Problems

15. How many terms are there in the expansion of $(a+b)^{200}$?

16. What is the sum of the exponents for any term in the expansion of $(a+b)^{123}$?

17. Consider the expansion of $(a + b)^{250}$.

    a. How many terms are there in the expansion?
    b. In the term containing a^{100}, what is the exponent on the b?
    c. What is the first term of the expansion?
    d. What is the second term of the expansion?
    e. What is the second-to-last term of the expansion?
    f. What is the last term of the expansion?

18. Expand $(u + w)^9$.

Solutions

1. They're boring, but do them!

2. a. 1 b. 2 c. 3 d. 4 e. 5 f. 6

3. a. 101 b. $n + 1$ **4**. a. $3 + 2 = 5$ b. 5

5. a. 4 b. 4 **6**. n

7. $k = 77$ **8**. 24; 8 **9**. $n = 101$

10. 1 10 45 120 210 252 210 120 45 10 1

11. $a^{100} + 100a^{99}b + \cdots + 100ab^{99} + b^{100}$

12. To expand $(a + b)^2$ by the Binomial Theorem, we know that the relevant row of Pascal's Triangle is "**1 2 1**". We also know that the exponents on the a go from 2 down to 0, while the exponents on the b go from 0 up to 2. Putting it all together:

$(a+b)^2 = \mathbf{1}a^2b^0 + \mathbf{2}a^1b^1 + \mathbf{1}a^0b^2 = a^2 + 2ab + b^2$, as it should be.

13. $m^8 + 8m^7z + 28m^6z^2 + 56m^5z^3 + 70m^4z^4 + 56m^3z^5 + 28m^2z^6 + 8mz^7 + z^8$

14. $x^{10} + 10x^9y + 45x^8y^2 + 120x^7y^3 + 210x^6y^4 + 252x^5y^5 + 210x^4y^6 + 120x^3y^7 + 45x^2y^8 + 10xy^9 + y^{10}$

15. 201 **16**. 123

17. a. 251 b. 150 c. a^{250} d. $250a^{249}b$

e. $250ab^{249}$ f. b^{250}

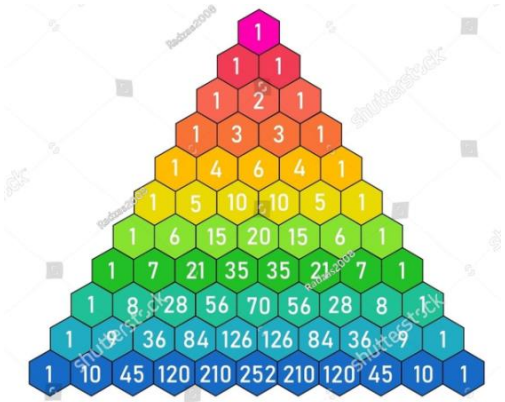
18. $u^9 + 9u^8w + 36u^7w^2 + 84u^6w^3 + 126u^5w^4 + 126u^4w^5 + 84u^3w^6 + 36u^2w^7 + 9uw^8 + w^9$

"Cherish your visions and your dreams, as they are the children of your soul, the blueprints of your ultimate achievements."

– Napoleon Hill

A New View of Pascal's Triangle

Notice that to write a certain row of Pascal's Triangle, we need the row above it, which of course requires the row above that one, etc. This can be cumbersome; we need a way of finding <u>any</u> row of Pascal's Triangle directly — without having to figure out all the rows that precede it.



Consider the sequence of *combinations*, from "6 choose 0" to "6 choose 6":

$$\binom{6}{0}, \binom{6}{1}, \binom{6}{2}, \binom{6}{3}, \binom{6}{4}, \binom{6}{5}, \binom{6}{6}$$

Evaluating each one of these (in other words, you verify!) gives the sequence of numbers:

1 6 15 20 15 6 1

which is none other than the 6th row of Pascal's Triangle (if we count the top row as the 0th row). So, to expand $(a + b)^6$, rather than using the appropriate row of Pascal's Triangle, we can use the sequence of combinations instead. Are you ready?

$$(a+b)^6$$

$$= \binom{6}{0}a^6b^0 + \binom{6}{1}a^5b^1 + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}a^1b^5 + \binom{6}{6}a^0b^6$$

$$= \boldsymbol{a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6}$$

Similarly, to start the expansion of $(a+b)^{100}$, we would write

$$\binom{100}{0}a^{100}b^0 + \binom{100}{1}a^{99}b^1 + \binom{100}{2}a^{98}b^2 + \ldots$$

$$= \boldsymbol{a^{100} + 100a^{98}b + 4{,}950a^{98}b^2 + \ldots}$$

The interesting thing here is that we've begun the expansion of $(a+b)^{100}$ <u>without</u> generating all the preceding rows of Pascal's Triangle.

We now tie everything in this chapter into a single formula which will calculate $(a+b)^n$ <u>without</u> any reference to Pascal's Triangle.

<u>The Binomial Theorem</u>

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$$

Homework

1. Use the Binomial Theorem to prove: $(x + y)^2 = x^2 + 2xy + y^2$.

2. Find the first four terms of the expansion of $(a + b)^{20}$.

3. Use the Binomial Theorem to expand $(x + y)^5$.

Review Problems

4. Consider the expansion of $(x + y)^{99}$. How many terms are there in the expansion? In the term containing y^{33}, what is the exponent on the x?

5. There are 250 terms in the expansion of $(a + b)^n$. What is n?

6. Find the 11th row of Pascal's Triangle (the one which starts 1 11 . . .)

7. Find the last two terms of $(a + b)^{66}$.

8. Use Pascal's Triangle to expand $(a + b)^8$.

9. Use the Binomial Theorem to prove: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

10. Find the first five terms of the expansion of $(a + b)^{30}$.

11. Use the Binomial Theorem to expand $(a + b)^7$.

12. True/False:

    a. There's a power of $a + b$ that's equal to 1.
    b. The expansion of $(x + y)^{123}$ contains 123 terms.
    c. The first term of the expansion of $(a + b)^{55}$ is $55a^{55}$.

d. In every term of the expansion of $(p + q)^m$, the sum of the exponents is m.

e. The expansion of $(a + b)^{300}$ begins with $a^{300} + 300a^{299}b$.

f. Every natural number (1, 2, 3, . . .) occurs in at least one place in Pascal's Triangle.

g. If there are 1000 terms in the expansion of $(a + b)^n$, then $n = 1001$.

h. Given any element in Pascal's Triangle, there are numbers n and k such that $\binom{n}{k}$ is that element.

i. The expansion of $(a + b)^{50}$ begins with $a^{50} + 50a^{49}b + 1225a^{48}b^2$.

❑ *To ∞ and Beyond*

A. Use the Binomial Theorem to prove that

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

B. Recall how each row of Pascal's Triangle is generated: Each element is the sum of the two numbers above it. Prove this fact using combinations:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Solutions

1. They're boring, but do them!

2. a. 24 b. $a^{23}; b^{23}$ c. 8

3. $n = 101$

4. 1 10 45 120 210 252 210 120 45 10 1

5. $a^{100} + 100a^{99}b + \cdots + 100ab^{99} + b^{100}$

6. $x^{10} + 10x^9y + 45x^8y^2 + 120x^7y^3 + 210x^6y^4 + 252x^5y^5 + 210x^4y^6$
$+120x^3y^7 + 45x^2y^8 + 10xy^9 + y^{10}$

7. $(x+y)^2 = \binom{2}{0}x^2y^0 + \binom{2}{1}x^1y^1 + \binom{2}{2}x^0y^2$
$= 1x^2 \cdot 1 + 2xy + 1 \cdot 1y^2$
$= x^2 + 2xy + y^2$

8. $a^{20} + 20a^{19}b + 190a^{18}b^2 + 1140a^{17}b^3 + \cdots$

9. $(x+y)^5 = \binom{5}{0}x^5y^0 + \binom{5}{1}x^4y^1 + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}x^0y^5$
$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

10. Hint: Let $a = 1$ and $b = 1$ in the Binomial Theorem, and then simplify each side of the equation.

11. Hint: Start with the sum of the combinations on the right. Expand each one into its definition using factorials. Find an LCD and add the fractions together. Simplify the numerator and you should end up with "n choose k," the left side of the equation.

12. 100; 66 **13**. 249

14. 1 11 55 165 330 462 462 330 165 55 11 1

15. $\ldots + 66ab^{65} + b^{66}$

16. $a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$

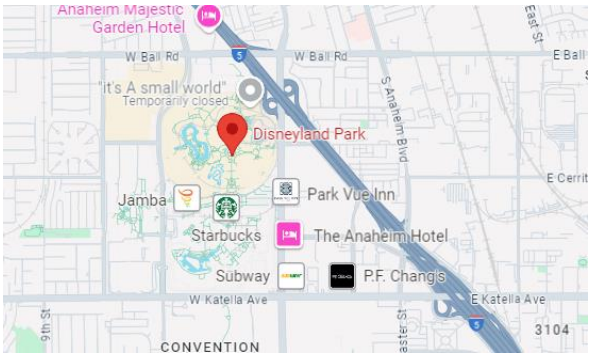
17. $(a+b)^3 = \binom{3}{0}a^3b^0 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}a^0b^3$
$= a^3 + 3a^2b + 3ab^2 + b^3$

18. $a^{30} + 30a^{29}b + 435a^{28}b^2 + 4060a^{27}b^3 + 27{,}405a^{26}b^4$

19. $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

20. a. T b. F c. F d. T e. T f. T g. F h. T i. T

DISTANCE ON THE LINE AND IN THE PLANE



One important factor when visiting Disneyland is the **distance** between your home and the theme park. It may determine your mode of transportation, the travel time, and thus the total cost of the trip.

You might want to review the chapter *Absolute Value, the Basics* to help you understand this chapter.

❑ *DISTANCE ON A LINE*

Our plan now is to create a formula that will give us the ***distance*** between two points on a line. Consider the two points 10 and 17 on a line. Is it pretty clear that the distance between them is 7? If it's not really obvious, you can simply subtract the smaller number from the larger one:

Distance is never negative.

$$\textbf{\textit{larger} – \textit{smaller}} = 17 - 10 = \mathbf{7} \checkmark$$

This formula (*larger* – *smaller*) works perfectly for any two numbers on a line:

The distance between –7 and 5 = 5 – (–7) = 5 + 7 = **12**.
[Note that 5 is larger than –7.]

The distance between –9 and –20 = –9 – (–20) = –9 + 20 = **11**.
[Note that –9 is larger than –20.]

The distance between 12 and 12 = 12 – 12 = **0**.
[Note that it doesn't matter which you choose as the larger number.]

NOTE: Subtracting in the wrong order (*smaller – larger*) is catastrophic. For example, if we try to find the distance between 10 and 17 like this:

$$10 - 17 = -7$$

we get a negative distance — quite forbidden!

Our way out of this mess is to come up with a *formula* for the distance between any two points on a line. In other words, we need a formula for the distance between the numbers ***a*** and ***b*** on a line, when we may not know which one of them, a or b, is the larger one. [See the NOTE above.]

Here's the secret: Use ***absolute value***. Then, if we were to "accidentally" subtract in the wrong direction — and end up with a negative distance (which DOESN'T EXIST) — the absolute value will automatically convert that negative number into a positive number.

Moreover, if we properly subtract *larger – smaller*, we'll get a positive number, whose absolute value will be just that number. And if the two numbers are the same, the absolute value of 0 is still 0. No matter the sizes of the two numbers, subtracting them (regardless of which one's bigger, or even if they're the same) — and then applying the absolute value to the difference — works every time.

So, in short, if ***a*** and ***b*** are any two numbers on the number line, we don't even have to worry about which one is bigger. We calculate the distance between them by using the following formula:

$$d = |a - b|$$

The distance between two points on the line is the absolute value of their difference.

Four Examples:

A. The distance between 7 and 3 is $|7-3| = |4| = \mathbf{4}$

B. The distance between 10 and 25 is $|10-25| = |-15| = \mathbf{15}$

C. The distance between π and π is $|\pi-\pi| = |0| = \mathbf{0}$

D. The distance between −17 and −5 is

$$|-17-(-5)| = |-17+5| = |-12| = \mathbf{12}$$

Homework

1. Find the **distance** between the given pair of points on the number line, using the absolute-value formula, and show each step:

a. 7 and 2	b. −2 and 9	c. −3 and −3
d. 99 and −99	e. −5 and −13	f. −20 and −4

❑ *DISTANCE IN THE PLANE [USING A TRIANGLE]*

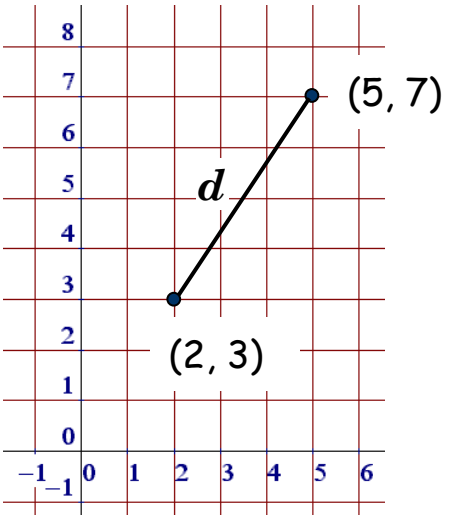
Assuming that you know how to plot points in the plane and remember the Pythagorean Theorem, we can tackle the question:

*How do we find the **distance** between two points in the **plane**?*

If the Earth were flat, it would be like asking how far apart two cities are if we know the latitude and longitude of each city.

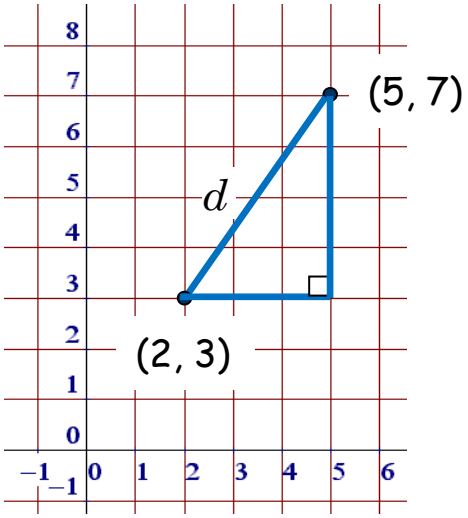
EXAMPLE 1: **Find the distance between the points (2, 3) and (5, 7) in the plane.**

Solution: Let's draw a picture and see what we can see. We'll plot the two given points and connect them with a straight line segment. The distance between the two points, which we'll call ***d***, is simply the length of that line segment.



How far is it between the two points (2, 3) and (5, 7)? Equivalently, what is the length (*d*) of the line segment connecting the two points?

Now what do we do? Well, here comes the interesting part. If we're creative enough, we might see that the segment connecting the two points can be thought of as the *hypotenuse* of a right triangle — as long as we sketch in a pair of legs to create such a triangle. Let's do that:



We've created a right triangle whose legs have lengths 3 and 4, and whose hypotenuse has a length equal to the distance between the two given points.

Sure enough, we've

constructed a right triangle where d is the length of the hypotenuse. If we can determine the lengths of the legs, then we can use the Pythagorean Theorem to find the length of the hypotenuse. By counting squares along the base of the triangle, we see that one leg is 3. Similarly, the other leg (the height) is 4. Since the square of the hypotenuse is equal to the sum of the squares of the legs, we can write the equation

$$\begin{aligned} & d^2 = 3^2 + 4^2 && \text{(Pythagorean Theorem: } \text{hyp}^2 = \text{leg}^2 + \text{leg}^2\text{)} \\ \Rightarrow\ & d^2 = 9 + 16 && \text{(square the legs)} \\ \Rightarrow\ & d^2 = 25 && \text{(add)} \\ \Rightarrow\ & d = 5 && \text{(since } \sqrt{25} = 5\text{)} \end{aligned}$$

You may notice that $d = -5$ also satisfies the equation $d^2 = 25$, since $(-5)^2 = 25$; but does a negative value of d make sense? No, because distance can never be negative; so we conclude that

The distance between the two points is **5**

Homework

2. By plotting the two given points in the plane and using the Pythagorean Theorem, find the **distance** between the points:

   a. (1, 1) and (4, 5)

   b. (2, –3) and (6, –6)

   c. (–3, 5) and (2, –7)

   d. (–4, –5) and (1, 7)

   e. (–5, 0) and (1, 8)

   f. the origin and (6, 8)

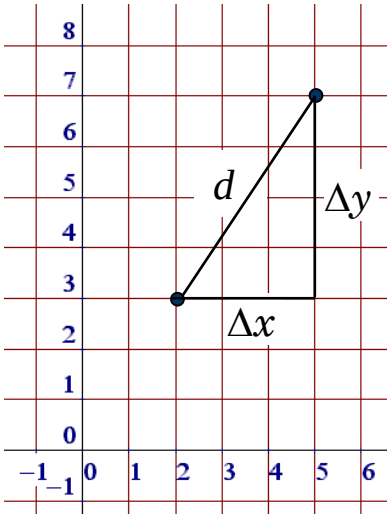
   g. the origin and (–5, –12)

   h. (2, 5) and (2, –1)

   i. (–3, 4) and (2, 4)

   j. (π, 99) and (π, 99)

❑ DISTANCE IN THE PLANE [USING A FORMULA]



To find the distance between two points in the plane, we've learned to create a right triangle so that the hypotenuse is the distance between the points.

Notice that the bottom leg is the change in x (Δx), while the vertical leg is the change in y (Δy). Thus, by employing the Pythagorean Theorem, we know that $d^2 = (\Delta x)^2 + (\Delta y)^2$. Solving this equation for d (and ignoring the negative square root), results in our Distance Formula:

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

EXAMPLE 2: **Find the distance between the points (2, –8) and (12, –14) in the plane.**

Solution: Using the formula we just created (and with no reference to a graph), we can calculate the following:

$$\Delta x = 2 - 12 = -10$$

$$\Delta y = -8 - (-14) = -8 + 14 = 6$$

So the distance between the points is

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Note: If we calculate Δx or Δy by subtracting in the reverse order, it doesn't matter in the distance formula, since these changes are being squared anyway.

$$\Rightarrow \quad d = \sqrt{(-10)^2 + (6)^2}$$

$$\Rightarrow \quad d = \sqrt{100 + 36}$$

$$\Rightarrow \quad d = \sqrt{136}$$

$$\Rightarrow \quad d = \boxed{2\sqrt{34}}$$

❑ *THE CLASSIC DISTANCE FORMULA*

An alternative formula, commonly found in books, is to realize that Δx is the difference of the x-values, and Δy is the difference of the y-values, so that the distance between the points (x_1, y_1) and (x_2, y_2) is

$$\boxed{d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}$$

Homework

3. Use either the Distance Formula with the Δx and Δy in it or the classic Distance Formula to find the **distance** between the given pairs of points:

a. (1, 1) and (4, 6)

b. (–2, –3) and (6, –6)

c. (–3, 5) and (0, –7)

d. (–4, 5) and (1, 7)

e. (–9, 0) and (1, 8)

f. the origin and (6, 11)

g. the origin and (–4, –10)

h. (2, 5) and (2, –1)

i. (–3, 4) and (2, 4)

j. $(\pi, \sqrt{2})$ and $(\pi, \sqrt{2})$

❑ *TO ∞ AND BEYOND*

Find the **distance** in 3-space between the points (1, –5, 8) and (–4, 9, 9).

Solutions

1. a. $|7-2| = |5| = 5$ b. $|-2-9| = |-11| = 11$

c. $|-3-(-3)| = |-3+3| = |0| = 0$

d. $|99-(-99)| = |99+99| = |198| = 198$

e. 8 f. 16

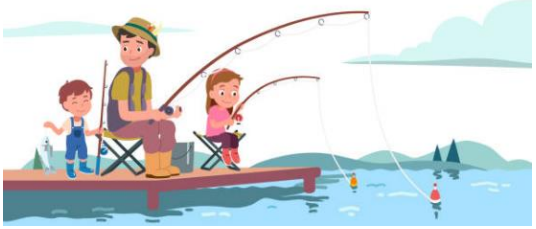
2. a. 5 b. 5 c. 13 d. 13 e. 10

f. 10 g. 13 h. 6 i. 5 j. 0

3. a. $\sqrt{34}$ b. $\sqrt{73}$ c. $3\sqrt{17}$ d. $\sqrt{29}$ e. $2\sqrt{41}$

f. $\sqrt{157}$ g. $2\sqrt{29}$ h. 6 i. 5 j. 0

"Give a man a fish
and you feed him for a day.
Teach a man to fish
and you feed him for a lifetime."



Chinese Proverb

THE DISTRIBUTIVE PROPERTY

We're becoming quite skillful at solving equations. But like a mutating virus, new varieties of equations are always confronting us. For example, how would we solve the equation

$$8(n - 1) = -(3n + 7)\ ?$$

This equation is distinctive for two reasons: the parentheses, and the fact that the variable *n* is locked up in two different sets of parentheses – on different sides of the equation, no less. If we could just open up the parentheses, we'd have access to the variable and thus be able to isolate it. In this chapter, we will learn how to unlock the parentheses so that we can solve the above equation.



The Distributive Property is the key to unlocking the variable when it is locked in parentheses.

❑ *STRETCHING AND SQUISHING*

To begin, recall that the single term 4w can be stretched out to four terms like this:

$$4w = w + w + w + w \qquad \text{[just as } 4 \times 5 = 5 + 5 + 5 + 5\text{]}$$

In the following examples we "stretch" out the multiplication and then "squish" the result by *combining like terms*:

$2 \cdot (x + y) = x + y + x + y = \underline{2x + 2y}$

$3 \times (a - b) = a - b + a - b + a - b = \underline{3a - 3b}$

$4(w + 5) = w + 5 + w + 5 + w + 5 + w + 5 = \underline{4w + 20}$

$3(2 - x) = 2 - x + 2 - x + 2 - x = \underline{6 - 3x}$, or $-3x + 6$

$2(m + n - q) = m + n - q + m + n - q = \underline{2m + 2n - 2q}$

$3(cd) = cd + cd + cd = \underline{3cd}$

$2(5n) = 5n + 5n = \underline{10n}$

$3(x \cdot 3) = 3(3x) = 3x + 3x + 3x = \underline{9x}$

$4(ab + 7) = ab + 7 + ab + 7 + ab + 7 + ab + 7 = \underline{4ab + 28}$

Homework

1. Use the method of the examples above (stretch-and-squish) to simplify each expression, i.e., remove parentheses:

a. $2 \cdot (x - 3)$	b. $3 \times (a + 5)$	c. $4(w + z)$
d. $2(AQ)$	e. $3(4 \cdot a)$	f. $4(uz)$
g. $2(x + 7)$	h. $3(1 - x)$	i. $4(t + 10)$
j. $2(a + b - c)$	k. $3(xy - 12)$	l. $5(x + 7)$

❑ *The Distributive Property*

You're in charge of food for the XYZ Widget, Inc. company picnic. You decide to serve box lunches, where each box contains

2 sandwiches
plus 5 cookies
plus 1 apple



You need to make <u>100</u> box lunches. That will require a total of

200 sandwiches
plus 500 cookies
plus 100 apples

How were these totals obtained? It should be pretty clear that we multiplied 100 by the number of each food item in one box lunch. That is, the quantity 100 was multiplied by 2 sandwiches, then by 5 cookies, and then by 1 apple:

100 (box lunches) = **100** (2 sandwiches + 5 cookies + 1 apple)

= 200 sandwiches + 500 cookies + 100 apples

In a sense, 100 has been "distributed" to all three food items in the box lunch. In fact, if we let

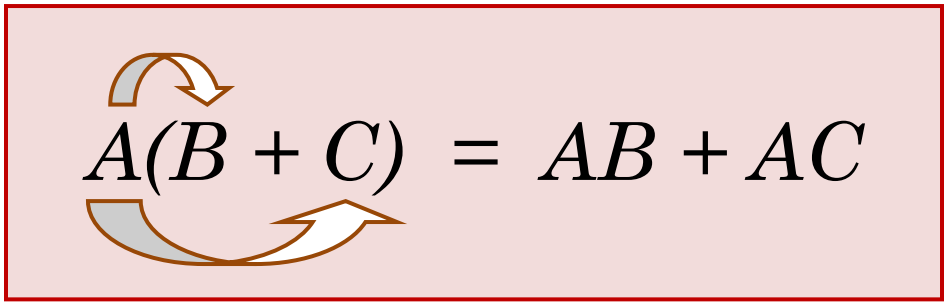
s = sandwiches
c = cookies
a = apples

then we can write

$$100\underbrace{(2s+5c+a)}_{\text{One box lunch}} = 200s+500c+100a$$

This statement is an example of the Distributive Property, which combines multiplication and addition into one rule, and is probably **THE MOST IMPORTANT STATEMENT IN ALL OF ALGEBRA.**

We can summarize the Distributive Property in the following way:



Note #1: If we have the expression $A(B - C)$, we can distribute as follows: $A(B - C) = A(B + (-C)) = AB + A(-C) = AB - AC$. In other words, just as you might expect,

$$A(B-C) = AB-AC$$

Note #2: There can be more than two terms in the parentheses, and we can still distribute; for example,

$$\boldsymbol{a}(b + c - d) = \boldsymbol{a}b + \boldsymbol{a}c - \boldsymbol{a}d$$

Note #3: First recall the concept of *terms*: "abc" is one term, while "$a + b + c$" is three terms. To apply the distributive property, there must be at least two terms in the parentheses. We can apply the distributive property to the expression $x(y + z)$, since there are two terms in the parentheses (and the answer is $xy + xz$). But we cannot apply the distributive property to the expression $x(y \cdot z)$ because there is only one term in the parentheses. [By the way, $x(y \cdot z)$ simplifies to xyz.] Basically, if there's no addition or subtraction in the parentheses, there can be no addition or subtraction in the answer.



Summary:

$3(4 + x)$ — Distribute? Yes: $3(4 + x) = 12 + 3x$

$3(4 \cdot x)$ — Distribute? No: $3(4 \cdot x) = 12x$

EXAMPLE 1: **Apply the distributive property to each expression:**

A. $\mathbf{7}(5 + 3)$

$= \mathbf{7} \cdot 5 + \mathbf{7} \cdot 3 = 35 + 21 = \mathbf{56}$

This problem could also be worked by doing the addition in the parentheses first, via the Order of Operations:

$7(5 + 3) = 7(8) = 56$ — this problem should convince you that the distributive property really works.

B. $2(x + 9)$

$= 2x + 2(9) = \mathbf{2x + 18}$

C. $4(3n + 5)$

$= 4(3n) + 4(5) = \mathbf{12n + 20}$

D. $7(5a - 3)$

$= 7(5a) - 7(3) = \mathbf{35a - 21}$

E. $7(a \cdot b)$ = Whoa! Hold it right there.

Are there at least two terms in the parentheses? NO, so we may <u>not</u> apply the distributive property. [See the Caution sign above] All we can say is that $7(a \cdot b) = \mathbf{7ab}$.

F. $-3(7x + 8)$

$= -3(7x) - 3(8) = \mathbf{-21x - 24}$

G. $-10(7 - 6A)$

$= -10(7) - 10(-6A) = -70 + 60A = \mathbf{60A - 70}$

H. $(19u - 20v)$

$= 1(19u - 20v) = 1(19u) - 1(20v) = \mathbf{19u - 20v}$

In other words, when a quantity sits in a pair of parentheses, with nothing out in front, you may simply remove the parentheses.

I. $-(7x + 8)$

$= -1(7x + 8) = -1(7x) - 1(8) = \mathbf{-7x - 8}$

Notice that we can skip the use of the –1 and just distribute the minus sign in front of the parentheses to each term inside the parentheses.

J. $-(-3n - 12)$
$= -1(-3n - 12) = -1(-3n) - 1(-12) =$ **$3n + 12$**

K. $-(13a - 99)$
$= -1(13a - 99) = -1(13a) - 1(-99) =$ **$-13a + 99$**

L. $3(2x + 3y - 10z)$
$= 3(2x) + 3(3y) - 3(10z) =$ **$6x + 9y - 30z$**

Homework

2. Use the distributive property (if appropriate) to simplify each expression:

a. $2(x + 5)$	b. $3(y - 1)$	c. $5(2 + n)$
d. $6(2 \cdot m)$	e. $10(4 \times z)$	f. $7(T - 7)$
g. $8(2x + 3)$	h. $12(a + b)$	i. $20(a \cdot b)$

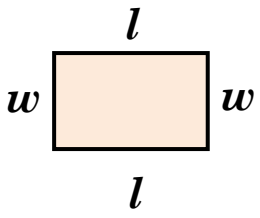
3. Use the distributive property (if appropriate) to simplify each expression:

a. $3(7x - 9)$	b. $3(a + 10)$	c. $-(2q + 8)$
d. $-2(-3n + 1)$	e. $-4(2 - y)$	f. $-(-x - 5)$
g. $7(5 - 3w)$	h. $-(3x - 9)$	i. $-5(3T - 8)$
j. $-3(-3 - 3n)$	k. $4(4 + 4u)$	l. $-(2v - 7)$
m. $20(x - 3)$	n. $-10(10 - 3n)$	o. $-(18 - 3a)$
p. $5(x \cdot 3)$	q. $-7(w \times z)$	r. $2(x - 3y + 7)$
s. $3(a \times b)$	t. $-2(x \cdot y \cdot z)$	u. $-(3x - 2y - 4z)$

❑ *CONNECTIONS*

1. The Distributive Property and the Perimeter of a Rectangle

Here's a more concrete way to approach the distributive property. Recall the formula for the perimeter of a rectangle:



$$P = 2l + 2w$$

In other words, we agreed that the sum of all four sides of the rectangle was simply 2 lengths plus 2 widths. But shouldn't we get the same result if we add one length and one width and then double that result? That is, couldn't the perimeter formula have been written $P = 2(l + w)$? Yes, it could. This leads us to conclude that

$$2(l + w) = 2l + 2w$$

which is simply another example of the distributive property at work.

2. The Distributive Property and Combining Like Terms

Here's another connection, this time between the distributive property from this chapter and the combining of like terms from another chapter.

Suppose we are asked to simplify the following expression by combining like terms:

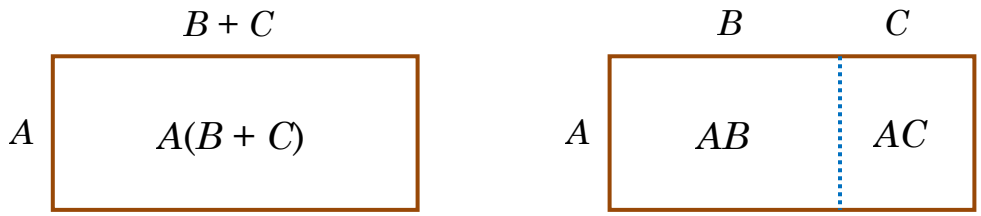
$$10x + 7x$$

We could use the common-sense notion that 10 of something plus 7 of the same somethings should result in 17 of those somethings: $10x + 7x = \underline{17x}$. But we can use the distributive property (in reverse) to get the same result:

$$10x + 7x = x(10 + 7) = x(17) = \underline{17x}$$

3. The Distributive Property and the Area of a Rectangle

We conclude this section will still another view of the distributive property:



Rectangle #1 Rectangle #2

The area of Rectangle #1 is its width times its length:

$A(B + C)$

The area of Rectangle #2 can be thought of as the sum of the areas of the two smaller rectangles contained within it:

$AB + AC$

However, Rectangle #1 and Rectangle #2 are the same size (since they have the same dimensions), so their <u>areas must be the same</u>. And thus we see that

$A(B + C) = AB + AC,$

which is none other than the distributive property again.

❑ *Distributing & Combining Like Terms in the Same Problem*

<u>EXAMPLE 2:</u> **Simplify each expression completely by applying the distributive property and then combining like terms:**

A. $2x + 3(7x - 9) = 2x + 21x - 27 = \mathbf{23x - 27}$

B. $3a - 7(a - 1) = 3a - 7a + 7 = \mathbf{-4a + 7}$

C. $5b - 8 + 2(3 - b) = 5b - 8 + 6 - 2b = \mathbf{3b - 2}$

D. $(7w + 9) - (3w + 6) = 7w + 9 - 3w - 6 = \mathbf{4w + 3}$

E. $-3(x - 1) + 3(2x + 4) = -3x + 3 + 6x + 12 = \mathbf{3x + 15}$

F. $4(-3n + 7) - 4(5n - 9) = -12n + 28 - 20n + 36 = \mathbf{-32n + 64}$

Homework

4. Apply the distributive property to remove the parentheses and then simplify your answer by combining like terms:

a. $6 + 3(x - 1)$

b. $3(2m + 3) + 5(1 - m)$

c. $2(x - 3) - (2x - 3)$

d. $-(3n - 1) - (1 - 3n)$

e. $-4(4t + 1) - 3(3t - 2)$

f. $3(a - b) + 4(b - a)$

g. $3(2y + 1) - 2(y - 1)$

h. $-(1 - x) - (x - 1)$

i. $3(4x - 1) - (12x - 1)$

j. $5 - 4(x - 3)$

k. $10 + 2(3x - 1) - x - 1$

l. $2(x + 3) - 2(x - 5)$

Review Problems

5. Simplify each expression:

a. $-2(3x - 4y)$ b. $7(7a + 8c - 1)$ c. $-4(3 - 4u + h - k)$

d. $-(4x - y)$ e. $-(9 + 12w)$ f. $-(-e - t - p)$

g. $8(9 \cdot y)$ h. $-2(-8 \times z)$ i. $-(ab)$

j. $17(xy + z)$ k. $-10(a \cdot b \cdot c)$ l. $4(3uw)$

m. $-2(-2x - 2y)$ n. $25(2 \times u \times w)$ o. $-2(g \cdot h)$

p. $7(3x \cdot 3)$ q. $7(3x - 3)$ r. $7(3x + 3)$

s. $-5(4n + 6)$ t. $-5(4n \cdot 6)$ u. $-5(4n \times 6)$

6. Simplify each expression:

a. $3(x + 4) + 5(2x - 1)$ b. $-2(n - 1) - 4(1 - n)$

c. $3(2y + 1) - 2(y - 1)$ d. $-(1 - x) - (x - 1)$

e. $3(4a + 1) - (10a - 2)$ f. $5 - 3(w - 4)$

g. $10 + 2(3h - 1) - 2h - 1$ h. $2(k + 4) - 2(k - 5)$

Solutions

1. a. $2 \cdot (x - 3) = x - 3 + x - 3 = 2x - 6$ b. $3a + 15$

c. $4w + 4z$ d. $2(AQ) = AQ + AQ = 2AQ$

e. $12a$ f. $4uz$ g. $2x + 14$

h. $3 - 3x$ i. $4t + 40$ j. $2a + 2b - 2c$

k. $3xy - 36$ l. $5x + 35$

2. a. $2x + 10$ b. $3y - 3$ c. $10 + 5n$, or $5n + 10$

d. $12m$ e. $40z$ f. $7T - 49$

g. $16x + 24$ h. $12a + 12b$ i. $20ab$

3. a. $21x - 27$ b. $3a + 30$ c. $-2q - 8$ d. $6n - 2$

e. $-8 + 4y$ f. $x + 5$ g. $35 - 21w$ h. $-3x + 9$

i. $-15T + 40$ j. $9 + 9n$ k. $16 + 16u$ l. $-2v + 7$

m. $20x - 60$ n. $-100 + 30n$, or $30n - 100$

o. $-18 + 3a$, or $3a - 18$ p. $15x$ q. $-7wz$

r. $2x - 6y + 14$ s. $3ab$ t. $-2xyz$

u. $-3x + 2y + 4z$, or $2y + 4z - 3x$

4. a. $6 + 3x - 3 = 3x + 3$ b. $6m + 9 + 5 - 5m = m + 14$

c. $2x - 6 - 2x + 3 = -3$ d. $-3n + 1 - 1 + 3n = 0$

e. $-25t + 2$ f. $b - a$, or $-a + b$

g. $4y + 5$ h. 0

i. -2 j. $-4x + 17$, or $17 - 4x$

k. $5x + 7$ l. 16

5. a. $-6x + 8y$ b. $49a + 56c - 7$ c. $-12 + 16u - 4h + 4k$

d. $-4x + y$ e. $-9 - 12w$ f. $e + t + p$

g. $72y$ h. $16z$ i. $-ab$

j. $17xy + 17z$ k. $-10abc$ l. $12uw$

m. $4x + 4y$ n. $50uw$ o. $-2gh$

p. $63x$ q. $21x - 21$ r. $21x + 21$

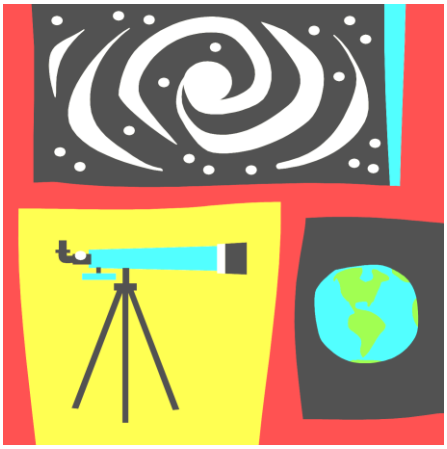
s. $-20n - 30$ t. $-120n$ u. $-120n$

6. a. $3(x + 4) + 5(2x - 1) = 3x + 12 + 10x - 5 =$ **$13x + 7$**

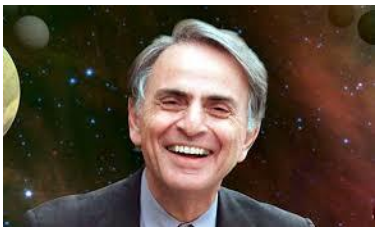
b. $-2(n-1)-4(1-n) = -2n+2-4+4n = \mathbf{2n-2}$

c. $4y+5$ d. 0 e. $2a+5$ f. $-3w+17$ g. $4h+7$ h. 18

"For most of human history we have searched for our place in the cosmos. Who are we? What are we? We find that we inhabit an insignificant planet of a hum-drum star lost in a galaxy tucked away in some forgotten corner of a universe in which there are far more galaxies than people. We make our world significant by the courage of our questions, and by the depth of our answers."



—Carl Sagan (1934–1996)



CH X – MULTIPLYING BINOMIALS

❑ *THE DOUBLE DISTRIBUTIVE PROPERTY*

What we need now is a way to multiply <u>two binomials</u> together, a skill so critical for success in Algebra. For example, how do we simplify the product $(3x + 1)(2x - 11)$? The **double distributive property** says, in a nutshell,

Multiply <u>each</u> term in the first binomial
by <u>each</u> term in the second binomial.

The following example demonstrates the general concept of the double distributive property.

<u>EXAMPLE 1:</u> **Use the double distributive property to multiply out the product $(a + b)(c + d)$.**

<u>Solution:</u> We carry out the multiplication in four steps:

i) Multiply the a by the c: ac
ii) Multiply the a by the d: ad
iii) Multiply the b by the c: bc
iv) Multiply the b by the d: bd

$(a + b)(c + d)$

a and b are dogs; c and d are cats. Each dog wants to chase each cat. That makes four possible pairings.

Notice that each term in the first set of parentheses is multiplied by each term in the second set of parentheses. In other words, we first distribute the a and then distribute the b — hence the phrase *double distribute*.

Now add the four terms together:

$$ac + ad + bc + bd$$

EXAMPLE 2: **Multiply out (simplify): $(x + 7)(x + 5)$**

Solution: We apply the four steps to double distribute:

i) Multiply the first x by the second x: x^2

ii) Multiply the first x by the 5: $5x$

iii) Multiply the 7 by the second x: $7x$

iv) Multiply the 7 by the 5: 35

Add the four terms together: $x^2 + 5x + 7x + 35$, and then combine like terms

$$\boxed{x^2 + 12x + 35}$$

EXAMPLE 3: **Simplify the given expression:**

A. $(2n + 1)(n - 8)$

$= 2n^2 - 16n + n - 8$ (double distribute)

$= \mathbf{2n^2 - 15n - 8}$ (combine like terms)

B. $(7a - 3)(4a - 5)$

$= 28a^2 - 35a - 12a + 15$ (double distribute)

$= \mathbf{28a^2 - 47a + 15}$ (combine like terms)

C. $(6k - 7)(6k + 7)$

$= 36k^2 + 42k - 42k - 49$ (double distribute)

$= \mathbf{36k^2 - 49}$ (combine like terms)

D. $(11y + 10)(11y - 10)$

$= 11y^2 - 110y + 110y - 100$

$= \mathbf{121y^2 - 100}$

E. $(2x + 9)^2$ The square of a quantity is the product of the quantity with itself:

$= (2x + 9)(2x + 9)$ (since $N^2 = N \cdot N$)

$= 4x^2 + 18x + 18x + 81$ (double distribute)

$= \mathbf{4x^2 + 36x + 81}$ (combine like terms)

Homework

1. Simplify each expression by double distributing and combining like terms:

a. $(x + y)(w + z)$ b. $(c + d)(a - b)$ c. $(x + 2)(y + 3)$

d. $(x + 3)(x + 4)$ e. $(n - 4)(n - 1)$ f. $(a + 3)(a - 7)$

g. $(y + 9)(y - 9)$ h. $(u - 3)(u + 3)$ i. $(t - 20)(t - 19)$

j. $(z + 3)(z + 3)$ k. $(v - 4)(v - 4)$ l. $(N + 1)(N - 1)$

2. Simplify:

a. $(3a + 7)(a - 9)$ b. $(2n - 3)(n + 4)$ c. $(3n - 8)(n - 1)$

d. $(5x + 7)(5x + 6)$ e. $(7w + 2)(7w - 2)$ f. $(x + 12)(x - 12)$

g. $(2y + 1)(2y + 1)$ h. $(7x + 3)(6x - 7)$ i. $(q + 7)(3q - 7)$

j. $(3n + 1)(3n + 1)$ k. $(3x - 7)(6x + 5)$ l. $(u - 7)(u - 7)$

3. Simplify:

a. $(y + 4)^2$ b. $(z - 9)^2$ c. $(3x + 5)^2$

d. $(2a - 1)^2$ e. $(n + 12)^2$ f. $(6t - 7)^2$

g. $(q - 15)^2$ h. $(5b + 3)^2$ i. $(7u - 1)^2$

j. $(2x + 1)^2$ k. $(3h - 12)^2$ l. $(5y - 5)^2$

4. Simplify:

a. $(a + b)(c - d)$	b. $(2x - 3)(2x + 3)$	c. $(3n - 1)^2$
d. $(3t + 1)(2t - 3)$	e. $(2x + 4)(3x - 6)$	f. $(n + 1)(n - 1)$
g. $(7a - 10)(6a - 10)$	h. $(10c + 7)^2$	i. $(L + 4)^2$
j. $(7x - 3)(3x + 7)$	k. $(13n - 7)(13n + 7)$	l. $(12d - 20)^2$

5. Prove that $(a + b)^2 \neq a^2 + b^2$ in two ways:
   i) Plug in numbers to provide a counterexample.
   ii) Simplify $(a + b)^2$ the correct way.

6. Find a counterexample to the proposition that $(x + y)^3 = x^3 + y^3$.

❑ *OTHER VIEWS OF THE DOUBLE DISTRIBUTIVE PROPERTY*

Using the Double Distributive Property for Simple Arithmetic

Let's multiply 34 by 75. Obviously, the double distributive property is overkill for an arithmetic problem like this, but this should convince you that the property is indeed valid.

	(34)(75)	
=	(30 + 4)(70 + 5)	(expand each number)
=	30(70) + 30(5) + 4(70) + 4(5)	(double distribute)
=	2100 + 150 + 280 + 20	(multiply before add)
=	**2550**, which is, of course, exactly what 34 times 75 is.	

A Geometric Picture of the Double Distributive Property

A picture of $(a + b)(c + d)$:

	c	d
a	ac	ad
b	bc	bd

Looking at the entire shaded rectangle, its dimensions are $a + b$ by $c + d$. Since the area of a rectangle is the product of its length and its width, the area of the entire rectangle is $(a + b)(c + d)$.

On the other hand, the areas of the four smaller rectangles have been labeled using the "length × width" formula. And you might note that the area of the entire rectangle is equal to the sum of the areas of the four rectangles that comprise it.

Therefore,

$$\underset{\text{the large rectangle}}{(a+b)(c+d)} = \underset{\text{the 4 small rectangles}}{ac+ad+bc+bd}$$

Yet Another View of Double Distributing

Let's multiply $a + b$ times $c + d$ the way it might be done in elementary school:

$$\begin{array}{r} a+b \\ \times\ c+d \\ \hline ad\ \ bd \\ ac\ \ bc \\ \hline \boldsymbol{ac+ad+bc+bd} \end{array}$$

Multiply *d* by *b*, and then multiply *d* by *a*. Now go down a row and shift over. Multiply *c* by *b* and then multiply *c* by *a*. Draw a line and add.

Review Problems

7. Simplify:

a. $(x + 9)(x + 8)$	b. $(y - 1)(y - 8)$	c. $(2z + 5)(2z - 5)$
d. $(N + 10)(N - 10)$	e. $(x - 9)^2$	f. $(a + 5)^2$
g. $(t + 9)(t - 5)$	h. $(a - 22)(a + 1)$	i. $(a - 11)(a + 2)$
j. $(2x + 1)(x - 5)$	k. $(3x + 8)(2x - 5)$	l. $(6x + 5)(x - 3)$
m. $(6a + 17)(a - 1)$	n. $(R + 12)(R - 12)$	o. $(5n - 3)^2$
p. $(1 - a)(2 - a)$	q. $(7w + 5)^2$	r. $(3a - 1)(3a - 2)$
s. $(9a - 1)(a - 2)$	t. $(x + 18)(x - 2)$	u. $(x + 36)(x + 1)$
v. $(5c - 1)(6c - 1)$	w. $(8a + 1)(2a - 1)$	x. $(6q + 5)^2$
y. $(3 + n)(3 - n)$	z. $(16n - 9)(2n - 3)$	

8. Prove that $(u + w)^4 \neq u^4 + w^4$.

Solutions

1.

a. $xw + xz + wy + yz$	b. $ac - bc + ad - bd$	c. $xy + 3x + 2y + 6$
d. $x^2 + 7x + 12$	e. $n^2 - 5n + 4$	f. $a^2 - 4a - 21$
g. $y^2 - 81$	h. $u^2 - 9$	i. $t^2 - 39t + 380$
j. $z^2 + 6z + 9$	k. $v^2 - 8v + 16$	l. $N^2 - 1$

2.

a. $3a^2 - 20a - 63$	b. $2n^2 + 5n - 12$	c. $3n^2 - 11n + 8$
d. $25x^2 + 65x + 42$	e. $49w^2 - 4$	f. $x^2 - 144$

g. $4y^2 + 4y + 1$ h. $42x^2 - 31x - 21$ i. $3q^2 + 14q - 49$

j. $9n^2 + 6n + 1$ k. $18x^2 - 27x - 35$ l. $u^2 - 14u + 49$

3. a. $y^2 + 8y + 16$ b. $z^2 - 18z + 81$ c. $9x^2 + 30x + 25$

d. $4a^2 - 4a + 1$ e. $n^2 + 24n + 144$ f. $36t^2 - 84t + 49$

g. $q^2 - 30q + 225$ h. $25b^2 + 30b + 9$ i. $49u^2 - 14u + 1$

j. $4x^2 + 4x + 1$ k. $9h^2 - 72h + 144$ l. $25y^2 - 50y + 25$

4. a. $ac - ad + bc - bd$ b. $4x^2 - 9$ c. $9n^2 - 6n + 1$

d. $6t^2 - 7t - 3$ e. $6x^2 - 24$ f. $n^2 - 1$

g. $42a^2 - 130a + 100$ h. $100c^2 + 140c + 49$ i. $L^2 + 8L + 16$

j. $21x^2 + 40x - 21$ k. $169n^2 - 49$ l. $144d^2 - 480d + 400$

5. i) By letting $a = 3$ and $b = 4$, for instance, we get:

$(a + b)^2 = (\mathbf{3} + \mathbf{4})^2 = 7^2 = 49$, whereas

$a^2 + b^2 = \mathbf{3}^2 + \mathbf{4}^2 = 9 + 16 = 25.$

Clearly, $(a + b)^2 \neq a^2 + b^2$ (except in rare circumstances)

ii) $(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$

6. Choosing, for example, $x = 1$ and $y = 2$, we would get the following results:

$$(x + y)^3 = (\mathbf{1} + \mathbf{2})^3 = 3^3 = 27;$$

on the other hand, $x^3 + y^3 = \mathbf{1}^3 + \mathbf{2}^3 = 1 + 8 = 9.$

7. a. $x^2 + 17x + 72$ b. $y^2 - 9y + 8$ c. $4z^2 - 25$

d. $N^2 - 100$ e. $x^2 - 18x + 81$ f. $a^2 + 10a + 25$

g. $t^2 + 4t - 45$ h. $a^2 - 21a - 22$ i. $a^2 - 9a - 22$

j. $2x^2 - 9x - 5$ k. $6x^2 + x - 40$ l. $6x^2 - 13x - 15$

m.	$6a^2 + 11a - 17$	n.	$R^2 - 144$	o.	$25n^2 - 30n + 9$
p.	$a^2 - 3a + 2$	q.	$49w^2 + 70w + 25$	r.	$9a^2 - 9a + 2$
s.	$9a^2 - 19a + 2$	t.	$x^2 + 16x - 36$	u.	$x^2 + 37x + 36$
v.	$30c^2 - 11c + 1$	w.	$16a^2 - 6a - 1$	x.	$36q^2 + 60q + 25$
y.	$9 - n^2$, or $-n^2 + 9$	z.	$32n^2 - 66n + 27$		

8. Choosing, for example, $u = 2$ and $w = 3$,

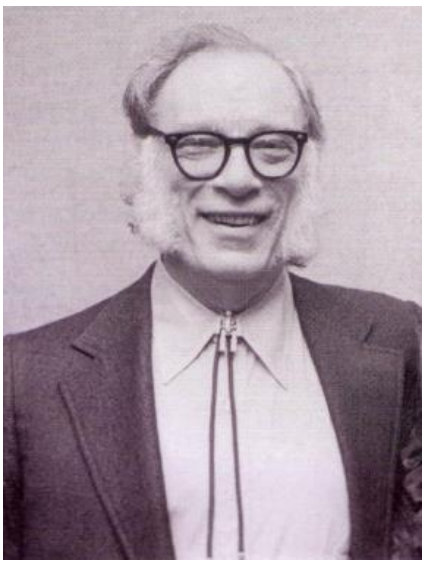
$(2 + 3)^4 = 5^4 = 625$, whereas $2^4 + 3^4 = 16 + 81 = 97$.

❑ *TO ∞ AND BEYOND*

A. Simplify: $(x + 2)(x^2 - 3x - 10)$

B. Simplify: $(2x + 5)^3$

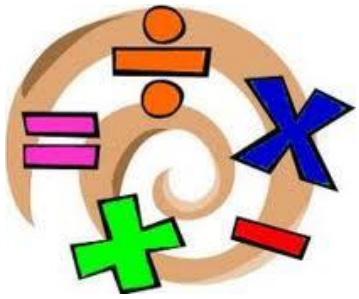
"If knowledge can create problems, it is not through ignorance that we can solve them."



– Isaac Asimov (1920 - 1992)

THE DISTRIBUTIVE PROPERTY AND EQUATIONS

❑ *INTRODUCTION*



You should now have all the tools you need to solve linear equations containing parentheses. These tools are as follows:

- *The Distributive Property*: $3(x + 7) = \mathbf{3x + 21}$
- *Combining Like Terms*: $4w + 7 - 3w - 13 = \mathbf{w - 6}$
- *Solving Equations*:

$$3n - 7 = n + 23$$
$$\Rightarrow \quad 3n - n = 23 + 7$$
$$\Rightarrow \quad 2n = 30$$
$$\Rightarrow \quad \mathbf{n = 15}$$

❑ *EXAMPLES*

EXAMPLE 1: **Solve for n: $8(n - 1) = -(3n + 7)$**

Solution: The variable n is locked up inside the parentheses, but the distributive property comes to our rescue:

	$8(n - 1) = -(3n + 7)$	(the original equation)
$\Rightarrow$	$8n - 8 = -3n - 7$	(distribute)
$\Rightarrow$	$8n + \mathbf{3n} - 8 = -3n + \mathbf{3n} - 7$	(add $3n$ to each side)
$\Rightarrow$	$11n - 8 = -7$	(simplify)
$\Rightarrow$	$11n - 8 + \mathbf{8} = -7 + \mathbf{8}$	(add 8 to each side)

$\Rightarrow \quad 11n = 1$ (simplify)

$\Rightarrow \quad \boxed{n = \frac{1}{11}}$ (divide each side by 11)

Homework

1. Solve each equation:

a. $2(2y - 3) = -8(2y - 9)$

b. $-6(-7b - 3) = -2(3b - 8)$

c. $-4(-2d + 9) = -5(2d - 9)$

d. $-4(-8j + 3) = 7(7j + 6)$

e. $3(3y - 5) = -(-y + 5)$

f. $-7(4t + 5) = -9(-5t + 3)$

g. $3(-2x + 8) = -3(-3x + 7)$

h. $7(2n + 3) = -6(3n + 7)$

EXAMPLE 2: **The Ultimate Challenge — Solve for x:**

$2(3x - 7) - 5(1 - 3x) = -(-4x + 1) + (x + 7)$

Solution: The steps are

1) Distribute
2) Combine like terms
3) Solve the simplified equation

$2(3x - 7) - 5(1 - 3x) = -(-4x + 1) + (x + 7)$

$\Rightarrow \quad 6x - 14 - 5 + 15x = 4x - 1 + x + 7$ (distribute)

$\Rightarrow \quad 21x - 19 = 5x + 6$ (combine like terms)

$\Rightarrow \quad 21x - \mathbf{5x} - 19 = 5x - \mathbf{5x} + 6$ (subtract $5x$ from each side)

$\Rightarrow \quad 16x - 19 = 6$ (simplify)

$\Rightarrow \quad 16x - 19 \mathbf{+ 19} = 6 \mathbf{+ 19}$ (add 19 to each side)

$\Rightarrow \quad 16x = 25$ (simplify)

$\Rightarrow \quad \frac{16x}{\mathbf{16}} = \frac{25}{\mathbf{16}}$ (divide each side by 16)

$\Rightarrow \quad \boxed{x = \frac{25}{16}}$ (simplify)

Homework

2. Solve each equation:

a. $-6(7u - 7) + 8(1 - 7u) = 4(4u + 2) + 5(8 - 9u)$

b. $7(-b - 9) - 2(-8 + 7b) = 8(-5b + 5) + 3(-5b + 3)$

c. $-6(4n + 5) - 10(-10 + 9n) = 7(-n) - (-7n + 4)$

d. $-10(-3g) + 3(2g) = (-8g - 5) + 8(-g + 5)$

e. $9(4 - k) - 8(-9k + 1) = 3(6 + k) - 3(8 - 5k)$

f. $8(5w - 1) - 6(-w - 5) = 5(8 - 4w) + 6(9 + 8w)$

Now CHECK your solution.

g. $-2(-4 - 2y) - 3(-9 + 8y) = 10(-3y - 8) - 2(-7 + 6y)$

h. $-4(a - 6) + (-5a - 3) = 6(2a + 1) - (5a + 4)$

Solutions

4

1. a. $y = \frac{39}{10}$ b. $b = -\frac{1}{24}$ c. $d = \frac{9}{2}$ d. $j = -\frac{54}{17}$

e. $y = \frac{5}{4}$ f. $t = -\frac{8}{73}$ g. $x = 3$ h. $n = -\frac{63}{32}$

2. a. $u = \frac{2}{69}$ b. $b = \frac{48}{17}$ c. $n = \frac{37}{57}$ d. $g = \frac{35}{52}$

e. $k = -\frac{34}{45}$

f. $w = 4$

Check:

Place $w = 4$ into the original equation, and then work the arithmetic on each side of the equation:

$8(5w - 1) - 6(-w - 5)$	$5(8 - 4w) + 6(9 + 8w)$
$8(5(4) - 1) - 6(-4 - 5)$	$5(8 - 4(4)) + 6(9 + 8(4))$
$8(20 - 1) - 6(-9)$	$5(8 - 16) + 6(9 + 32)$
$8(19) + 54$	$5(-8) + 6(41)$
$152 + 54$	$-40 + 246$
206	**206** ✓

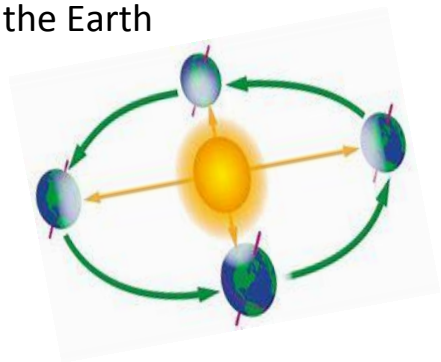
g. $y = -\frac{101}{22}$ h. $a = \frac{19}{16}$

"*Opportunity* is missed by most people because it is dressed in overalls and looks like work."

– **Thomas Edison**

THE ELLIPSE

An ***ellipse*** is oval-shaped and may be described as a circle that never quite made it. It's the shape of orbits. For example, the orbit of the Earth around the Sun is an ellipse, and the orbits of many satellites (including our Moon) revolving around the Earth are elliptical. The elliptical shape is also used in certain gears, and it is the key behind a medical procedure called *lithotripsy* (Greek for stone-crushing) — the use of shock waves to break up kidney stones.





❑ *FINDING INTERCEPTS*

In this section, we recall the procedures for finding intercepts; the practice will help us with the following ellipse problems.

EXAMPLE 1: **Find all the intercepts of the graph**

$$\mathbf{49x^2 + 9y^2 = 441}$$

Solution: To find any x-intercepts, we set $y = 0$ and solve for x:

$$49x^2 + 9y^2 = 441 \quad \Rightarrow \quad 49x^2 + 9(\mathbf{0})^2 = 441$$

$$\Rightarrow \quad 49x^2 = 441 \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad x = \pm 3$$

Thus, the x-intercepts are **(3, 0)** and **(–3, 0)**.

To find any y-intercepts, we set $x = 0$ and solve for y:

$$49x^2 + 9y^2 = 441 \Rightarrow 49(\mathbf{0})^2 + 9y^2 = 441$$
$$\Rightarrow \quad 9y^2 = 441 \Rightarrow y^2 = 49 \Rightarrow y = \pm 7$$

Therefore, the y-intercepts are **(0, 7)** and **(0, –7)**.

Homework

1. Find all the **intercepts** of each graph:

a. $3x + 5y = 30$	b. $y = 4x + 20$	c. $x = 3$
d. $y = -4$	e. $y = x^2$	f. $y = x^2 - x - 6$
g. $x^2 + y^2 = 121$	h. $x^2 + y^2 = 50$	i. $9x^2 + y^2 = 9$
j. $x^2 + 36y^2 = 36$	k. $81x^2 + 25y^2 = 2025$	l. $x^2 + 4y^2 = 16$
m. $x^2 - y^2 = 100$	n. $y^2 - x^2 = 121$	o. $y = x^2 + x + 1$

❑ *GRAPHING ELLIPSES*

EXAMPLE 2: **Graph the ellipse:** $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$

Solution: Let's begin by finding all the intercepts of the ellipse. These points, and a few others, will convince us that the graph is indeed elliptical.

***x*-intercepts:** Set $y = 0$ to get

$$\frac{x^2}{4} + \frac{\mathbf{0}^2}{25} = 1 \Rightarrow \frac{x^2}{4} = 1 \Rightarrow x^2 = 4 \Rightarrow x = \pm\sqrt{4} = \pm 2$$

and so the x-intercepts are **(2, 0)** and **(–2, 0)**.

***y*-intercepts:** Set $x = 0$ to get

$$\frac{\mathbf{0}^2}{4} + \frac{y^2}{25} = 1 \;\Rightarrow\; \frac{y^2}{25} = 1 \;\Rightarrow\; y^2 = 25 \;\Rightarrow\; y = \pm\sqrt{25} = \pm 5$$

and thus the y-intercepts are **(0, 5)** and **(0, –5)**.

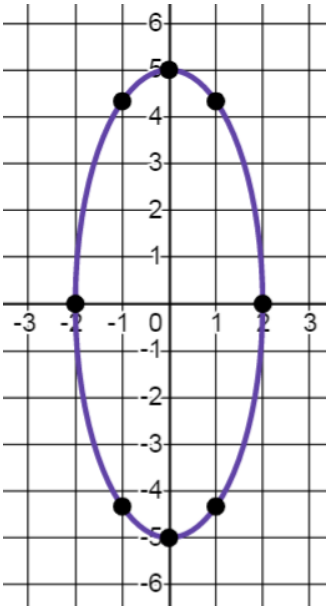
Additional points: Let $x = 1$. Then

$$\frac{\mathbf{1}^2}{4} + \frac{y^2}{25} = 1 \;\Rightarrow\; \frac{1}{4} + \frac{y^2}{25} = 1 \;\Rightarrow\; \frac{y^2}{25} = \frac{3}{4} \;\Rightarrow\; y^2 = \frac{75}{4}$$

$$\Rightarrow\; y = \pm\sqrt{\frac{75}{4}} \;\Rightarrow\; y = \pm\sqrt{18.75} \approx \pm 4.33$$

This yields the points **(1, 4.33)** and **(1, –4.33)**.

Note that if we let $x = -1$, we obtain the same y-values. Thus, two additional points are **(–1, 4.33)** and **(–1, –4.33)**. [Can you explain why?] We now have a total of eight points we can plot. When we do, we get the graph of the ellipse:



Note: The **center** of the ellipse

$$\frac{\boldsymbol{x}^2}{\mathbf{4}} + \frac{\boldsymbol{y}^2}{\mathbf{25}} = \mathbf{1}$$

is the **origin**.

A Final Note:

The graph shows that an x-value of 3, for example, should not be allowed in the formula (since there's no point on the

ellipse whose x-coordinate is 3). But how can we verify this fact before we graph anything? We'll let $x = 3$ in the ellipse equation and see what happens:

$$\frac{x^2}{4}+\frac{y^2}{25} = 1 \quad \overset{x=3}{\Rightarrow} \quad \frac{\mathbf{3}^2}{4}+\frac{y^2}{25} = 1 \quad \Rightarrow \quad \frac{9}{4}+\frac{y^2}{25} = 1$$

$$\Rightarrow \quad \frac{y^2}{25} = 1-\frac{9}{4} \quad \Rightarrow \quad \frac{y^2}{25} = -\frac{5}{4} \quad \Rightarrow \quad y^2 = -\frac{125}{4}$$

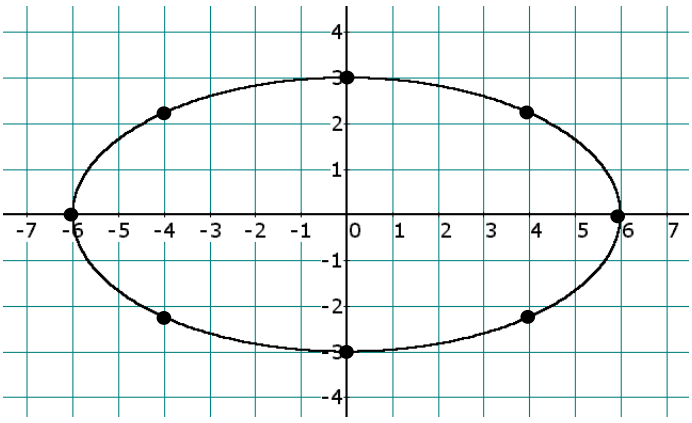
$$\Rightarrow \quad y = \pm\sqrt{-\frac{125}{4}}\text{, which are } \underline{\text{not}} \text{ real numbers.}$$

Since letting $x = 3$ resulted in no real-number solution for y, we conclude that no point on the graph will have an x-coordinate of 3, and thus 3 is not an allowable x-value; it's not in the ellipse's domain.

EXAMPLE 3: **Graph the ellipse:** $\dfrac{\boldsymbol{x^2}}{\mathbf{36}}+\dfrac{\boldsymbol{y^2}}{\mathbf{9}} = \mathbf{1}$

Solution: Since this equation is very similar to the previous one, its graph is likely an ellipse with center at the origin. You can calculate the intercepts to be **(6, 0)**, **(–6, 0)**, **(0, 3)**, and **(0, –3)**.

For extra "resolution," we can find four more points by letting $x = \pm 4$, and then calculating y to be $\pm\sqrt{5}$. We thus get four points with the approximate values (**4, 2.24**), (**4, –2.24**), (**–4, 2.24**), and **(–4, –2.24)**. [Be sure you can use a calculator to determine the decimal approximations.] Here's the graph:



EXAMPLE 4: **Graph the ellipse: $16x^2 + y^2 = 16$**

Solution: Is this an ellipse? It doesn't have the same form as the equations in Examples 2 and 3. Let's find the intercepts, then a few other points to construct the graph.

By letting $x = 0$, we get y-values of ± 4. By letting $y = 0$, we get x-values of ± 1. Therefore, the intercepts are

(1, 0), **(–1, 0)**, **(0, 4)**, and **(0, –4)**

Now we'll let $x = 0.5$. This will give us

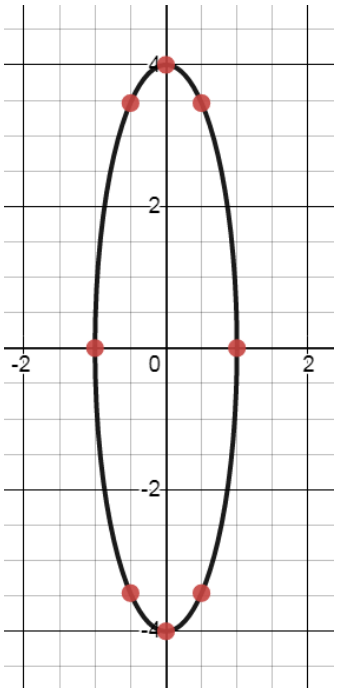
$$16(\mathbf{0.5})^2 + y^2 = 16 \Rightarrow 16(0.25) + y^2 = 16$$

$$\Rightarrow \quad 4 + y^2 = 16 \Rightarrow y^2 = 12 \Rightarrow y = \pm\sqrt{12} \Rightarrow y \approx \pm 3.46$$

Similarly, letting $x = -0.5$ will give the same y-values: ± 3.46. We now have four additional points to graph:

(0.5, 3.46), **(0.5, –3.46)**, **(–0.5, 3.46)**, and **(–0.5, –3.46)**

Let's plot all eight of our points — sure enough, we get another ellipse:



If we take the given ellipse

$$16x^2 + y^2 = 16$$

and divide each side by 16, we get

$$\frac{16x^2 + y^2}{16} = \frac{16}{16}$$

$$\frac{x^2}{1} + \frac{y^2}{16} = 1$$

which is the same form as the two previous examples.

Homework

2. Prove that no point with an x-coordinate of 12 will lie on the ellipse $\frac{x^2}{100}+\frac{y^2}{49}=1$.

3. Consider the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$.

   a. Find all four intercepts.

   b. Find two more points on the ellipse by letting $x = 2$.

   c. Find two more points on the ellipse by letting $x = -3$.

   d. Sketch the ellipse.

4. Find all four intercepts of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Graph each ellipse by plotting the four intercepts and four additional points:

5. $\frac{x^2}{25}+\frac{y^2}{9}=1$

6. $x^2+\frac{y^2}{49}=1$

7. $x^2+9y^2=9$

8. $\frac{x^2}{81}+\frac{y^2}{9}=1$

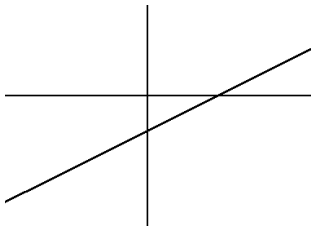
9. $\frac{x^2}{49}+\frac{y^2}{100}=1$

10. $25x^2+4y^2=100$

❑ *Lines, Parabolas, Circles, and Ellipses*

If you've covered lines, parabolas, circles, and ellipses in your course, it's the right time to consider the question: Given an equation of one of

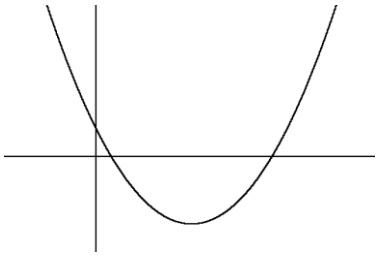
these four graphs, how can we tell (without graphing) which shape it will be?



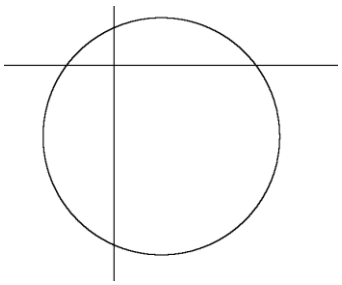
The answer depends primarily on how many of the two variables x and y are squared in the equation. In a ***line***, neither variable is squared; for example,

$3x - 4y = 10$ $\quad x = -3$ $\quad y = 7$

are the equations of lines.

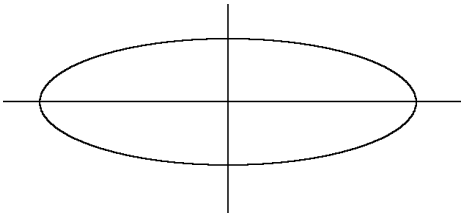


If exactly one variable is squared, then it's a ***parabola***. In a beginning course, a typical parabola might have the equation $y = 2x^2 - 5x + 3$, whose graph opens up. It's also possible that the y is squared, like $x = y^2 + 3$, whose graph is a sideways parabola.



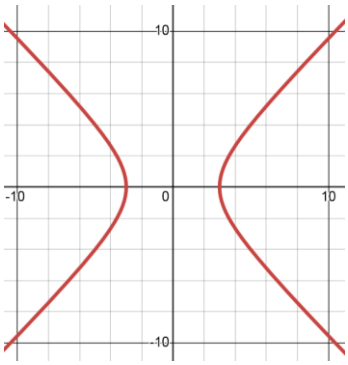
An equation with both variables squared, such as $x^2 + y^2 = 25$, has a graph which is a ***circle*** whose center is the origin and whose radius is 5. The equation $(x - 2)^2 + (y + 7)^2 = 20$ also is a circle, since it's pretty clear that, if expanded, both variables are being squared. In each circle, the coefficients of the two squared terms are the same, namely 1.

As for the ***ellipse***, a typical equation was $9x^2 + 4y^2 = 36$, or perhaps something like $\frac{x^2}{25} + \frac{y^2}{4} = 1$. Both variables are squared in these equations, so why aren't they circles? Here's the answer: In a circle, the coefficients of the squared terms are always the same; in an ellipse, they're <u>different</u> but <u>have the same sign</u>. In the first ellipse formula above, the coefficients are 9 and 4 (both positive); in the second ellipse, the coefficients are $\frac{1}{25}$ and $\frac{1}{4}$ (again, both positive). Note that the



equation $7x^2 + 7y^2 = 30$ is actually a circle because both variables are squared and the coefficients are the same. In fact, if you divide each side of the equation by 7, it might be clearer that it's indeed a circle.

We close this chapter by looking at the equation $x^2 - y^2 = 9$. It's not a line because both variables are squared. It's not a parabola for precisely the same reason. It's not a circle since the coefficients (1 and –1) are different. And it's not an ellipse because the coefficients don't have the same sign. In other words, it's likely a new graph to you and is called a ***hyperbola*** (hy-PER-bo-la).



Homework

11. For each formula, determine whether its graph is a line, a parabola, a circle, or an ellipse.

a. $3x - 4y = 22$

b. $x = -9$

c. $y - 7 = 0$

d. $y = -x^2 + 13x$

e. $x^2 + y^2 = 7$

f. $2x^2 + 3y^2 = 23$

g. $(x - 2)^2 + y^2 = 9$

h. $\frac{x^2}{5} + \frac{y^2}{6} = 1$

i. $\frac{x^2}{23} + \frac{y^2}{2} = 1$

j. $17x^2 + 22y^2 = 44$

k. $3x + y^2 - 6 = 0$

l. $(x + 1)^2 + (y - 2)^2 = 29$

m. $y = 17x - 19$

n. $x^2 - y^2 = 12$

Review Problems

12. Consider the ellipse $\frac{x^2}{9}+\frac{y^2}{100} = 1$.

    a. Prove that $x = 0$ is allowed in the formula.

    b. Prove that $x = 4$ is <u>not</u> allowed in the formula.

13. Find all the intercepts of $25x^2 + 4y^2 = 100$.

14. True/False:

    a. The ellipse $25x^2 + 4y^2 = 100$ has four intercepts.

    b. The graph of $9x^2 - 16y^2 = 144$ is an ellipse.

    c. The graph of $10x^2 = y^2 + 100$ is an ellipse.

15. Matching:

_____ $9x^2 - 18y + 7x + 9y^2 + 1 = 0$	A. line
_____ $18x - \pi y = 171.45$	B. parabola
_____ $y = \sqrt{2}\,x^2 - 200x + \frac{17}{2}$	C. circle
_____ $\frac{x^2}{17}+\frac{y^2}{5} = 1$	D. ellipse
_____ $y = -9$	E. hyperbola

16. Graph the ellipse by plotting the four intercepts and four additional points: $\frac{x^2}{9}+\frac{y^2}{25} = 1$

17. Graph the ellipse by plotting the four intercepts and four additional points: $9x^2 + 16y^2 = 576$

Solutions

1. a. Setting $y = 0$ gives $x = 10$. Setting $x = 0$ gives $y = 6$. The intercepts are (10, 0) and (0, 6).

b. (–5, 0) and (0, 20)

c. (3, 0)

d. (0, –4)

e. (0, 0) is both the x- and y-intercept.

f. Setting $y = 0$ yields the quadratic equation $x^2 - x - 6 = 0$, which factors into $(x - 3)(x + 2) = 0$, whose solutions are $x = 3, -2$. The x-intercepts are therefore (3, 0) and (–2, 0). The y-intercept is (0, –6).

g. There are four intercepts: (±11, 0) and (0, ±11).

h. $(\pm 5\sqrt{2}, 0)$ and $(0, \pm 5\sqrt{2})$

i. (±1, 0) and (0, ±3)

j. (±6, 0) and (0, ±1)

k. (±5, 0) and (0, ±9)

l. (±4, 0) and (0, ±2)

m. Setting $y = 0$ yields the quadratic equation $x^2 = 100$, with solutions $x = \pm 10$. But setting $x = 0$ gives $-y^2 = 100$, or $y^2 = -100$, which has no solutions in $\mathbb{R}$. The intercepts are (±10, 0).

n. (0, ±11)

o. (0, 1)

Remember: In this book, every intercept is written as an ordered pair.

2. Let $x = 12$ in the ellipse equation, and you should end up with y^2 equals a negative number, resulting in no solutions in $\mathbb{R}$.

3. a. (4, 0) (–4, 0) (0, 2) (0, –2)

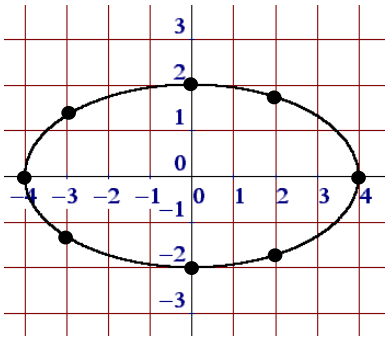
b. Letting $x = 2$ gives

$$\frac{2^2}{16}+\frac{y^2}{4} = 1 \quad \Rightarrow \quad \frac{1}{4}+\frac{y^2}{4} = 1 \quad \Rightarrow \quad \frac{y^2}{4} = \frac{3}{4}$$

$\Rightarrow \quad y^2 = 3 \quad \Rightarrow \quad y = \pm\sqrt{3}$, which give us the two approximate points (2, 1.732) and (2, –1.732).

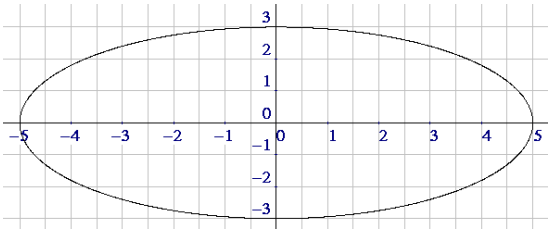
c. Letting $x = -3$ gives the points (–3, 1.323) and (–3, –1.323).

d.

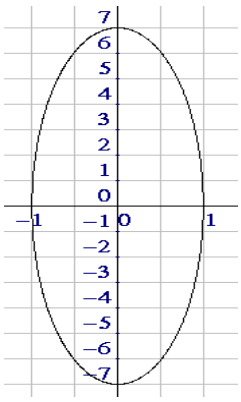


4. $(a, 0)$ $(-a, 0)$ $(0, b)$ $(0, -b)$

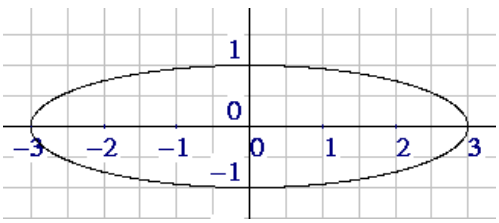
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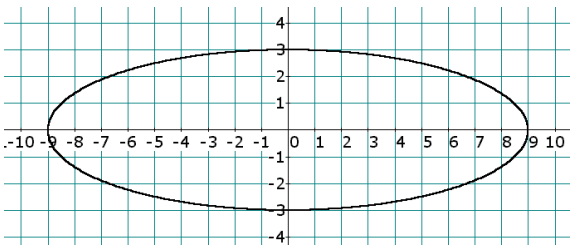
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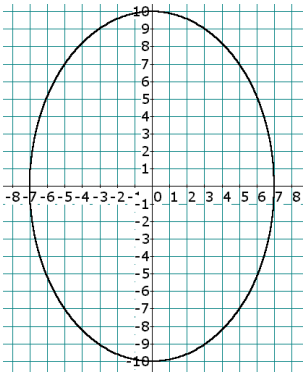
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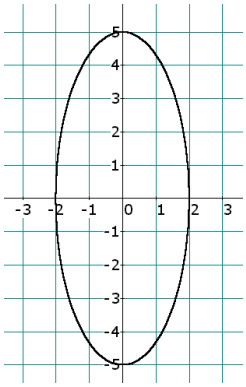
8.



9.



10.



11. a. Neither variable is squared, so it's a line.

b. It's a vertical line.

c. It's a horizontal line.

d. One variable is squared; the other isn't. It's a parabola.

e. Both variables are squared, each with the same coefficient (1), so it's a circle.

f. Both variables are squared, but with different, positive coefficients; therefore, it's an ellipse.

g. Both variables are squared, each with the same coefficient (1). It's a circle.

h. Both variables are squared, and the coefficients are 1/5 and 1/6; thus, it's an ellipse.

i. Both variables are squared, and the coefficients are 1/23 and 1/2; it follows that it's an ellipse.

j. Both variables are squared, and the coefficients are 17 and 22; so it's an ellipse.

k. One variable is squared; the other isn't. It's a parabola.

l. Circle

m. Line

n. Both variables are squared, but their coefficients have opposite signs. So it is neither a line, a parabola, a circle, nor an ellipse; it's a hyperbola.

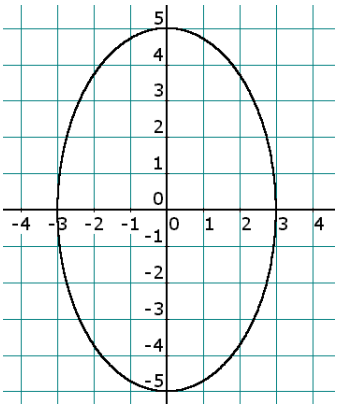
12. a. Letting $x = 0$ in the ellipse equation results in $y = \pm 10$, which of course are real numbers. Therefore, 0 is allowed in the formula.

b. Letting $x = 4$ in the equation results in $y^2 = -\frac{700}{9}$, which has no solution in the real numbers. Thus, 4 is not allowed in the formula.

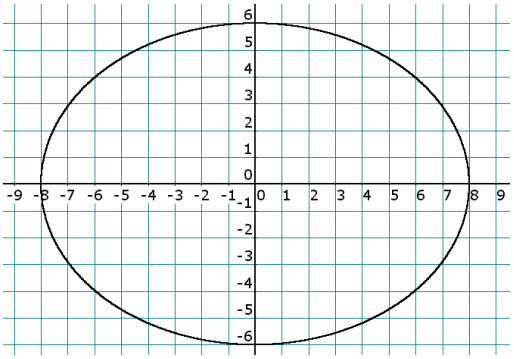
13. $(\pm 2, 0)$ (0 ± 5)

14. a. T b. F c. F **15**. C, A, B, D, A

16.



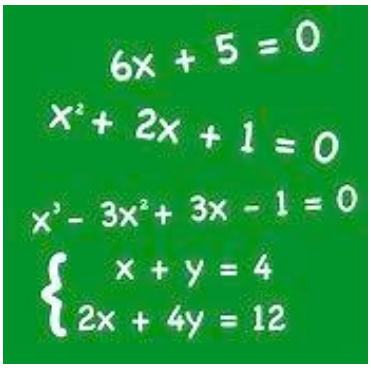
17.



"Courage and perseverance have a magical talisman, before which difficulties disappear and obstacles vanish into air."

– ***John Quincy Adams***

CUBIC AND QUARTIC EQUATIONS



This chapter is a collection of additional equations: Cubic (third degree) and Quartic (fourth degree). Although cubics and quartics can be incredibly difficult to solve, the equations in this chapter can all be solved using the Factoring method. (You can never be too good at factoring!)

❑ *CUBIC EQUATIONS USING THE GCF*

EXAMPLE 1: **Solve the cubic equation: $30x^3 + 99x^2 = 21x$**

Solution: Since factoring is how we solved quadratic equations earlier in the course, let's try factoring here. The factoring method requires one side of the equation to be **0**, so we start by subtracting $21x$ from each side of the equation:

$$30x^3 + 99x^2 - 21x = 0$$

Another theme in this chapter is that factoring always begins with an attempt to factor out the GCF, which in this case is $3x$:

$$3x(10x^2 + 33x - 7) = 0$$ [Check by distributing.]

Factoring the trinomial in the parentheses gives the final factorization of the left side of our equation:

$$3x(5x - 1)(2x + 7) = 0$$

Interpreting $3x$ is one of the factors, we have three factors whose product is 0; we therefore set each of the three factors to 0:

$$3x = 0 \quad \text{or} \quad 5x - 1 = 0 \quad \text{or} \quad 2x + 7 = 0$$

Solving each of these three linear equations gives us three solutions to our cubic equation:

$$\boxed{x = 0, \frac{1}{5}, -\frac{7}{2}}$$

Homework

1. Solve each cubic equation:

   a. $x^3 + 3x^2 + 2x = 0$

   b. $4n^3 - 18n^2 + 8n = 0$

   c. $x^3 = 16x$

   d. $3y^3 = -30y^2 - 75y$

   e. $a^3 + 9a = 0$

   f. $30x^3 + 25x^2 - 30x = 0$

2. Solve for x: $x^2(x + 1)(2x - 3)(x + 7)^3(x^2 - 4)(x^2 - 5x + 6) = 0$

❑ *CUBIC EQUATIONS USING GROUPING*

EXAMPLE 2: **Solve the cubic equation: $x^3 + 5x^2 = 9x + 45$**

Solution: The first step is to bring the terms on the right side of the equation to the left:

$$x^3 + 5x^2 - 9x - 45 = 0$$

By grouping the first two terms, we can factor the GCF in the first pair of terms and the last pair of terms:

$$x^2(x + 5) - 9(x + 5) = 0$$

Pull out the GCF, $x + 5$:

$$(x + 5)(x^2 - 9) = 0 \qquad \text{(partially factored)}$$

Continue by factoring the difference of squares:

$$(x + 5)(x + 3)(x - 3) = 0 \qquad \text{(fully factored)}$$

Set each factor to 0:

$$x + 5 = 0 \quad \text{or} \quad x + 3 = 0 \quad \text{or} \quad x - 3 = 0$$

Solving each linear equation gives:

$$x = -5 \quad \underline{\text{or}} \quad x = -3 \quad \underline{\text{or}} \quad x = 3$$

And now we have our three solutions:

$$\boxed{x = -5, -3, 3}$$

❑ *QUARTIC EQUATIONS*

EXAMPLE 3: **Solve the quartic equation: $x^4 - 26x^2 + 25 = 0$**

Solution: The factoring we learned in the chapter *Advanced Factoring* is exactly what we need here:

$$x^4 - 26x^2 + 25 = 0$$

$$\Rightarrow \quad (x^2 - 1)(x^2 - 25) = 0$$

Note: Each factor is a difference of squares.

$$\Rightarrow \quad (x + 1)(x - 1)(x + 5)(x - 5) = 0$$

$$\Rightarrow \quad x + 1 = 0 \quad \underline{\text{or}} \quad x - 1 = 0 \quad \underline{\text{or}} \quad x + 5 = 0 \quad \underline{\text{or}} \quad x - 5 = 0$$

$$\Rightarrow \quad x = -1 \text{ or } x = 1 \text{ or } x = -5 \text{ or } ____$$

$$\boxed{x = \pm 1, \pm 5}$$

By the way, a fourth-degree polynomial equation can have <u>at most</u> four solutions, but it may have fewer (see the next two examples).

EXAMPLE 4: **Solve for n: $2n^4 - 15n^2 = 27$**

Solution: The factoring technique requires one side of the equation to be 0, so our first step is to make that happen by subtracting 27 from each side of the equation:

$$2n^4 - 15n^2 - 27 = 0$$

$$\Rightarrow \quad (2n^2 + 3)(n^2 - 9) = 0$$

$$\Rightarrow \quad (2n^2 + 3)(n + 3)(n - 3) = 0$$

$$\Rightarrow \quad 2n^2 + 3 = 0 \text{ or } n + 3 = 0 \text{ or } n - 3 = 0$$

Let's solve the first equation:

$$2n^2 + 3 = 0 \Rightarrow 2n^2 = -3 \Rightarrow n^2 = -\frac{3}{2} \Rightarrow n = \pm\sqrt{-\frac{3}{2}}$$

which are numbers that are NOT in the set of real numbers, $\mathbb{R}$. So, assuming your algebra class involves only real numbers, we conclude that the equation $2n^2 + 3 = 0$ has No Solution. The other two equations should be easy for you by now. The final solution is

$$\boxed{x = \pm 3}$$

EXAMPLE 5: **Solve for a: $a^4 = -13a^2 - 36$**

Solution:

$$a^4 = -13a^2 - 36$$
$$\Rightarrow \quad a^4 + 13a^2 + 36 = 0$$
$$\Rightarrow \quad (a^2 + 9)(a^2 + 4) = 0$$
$$\Rightarrow \quad a^2 + 9 = 0 \text{ or } a^2 + 4 = 0$$

I hope it's clear that neither of these last two equations has a solution in the real numbers, $\mathbb{R}$. We're done:

No Solution

Homework

3. Solve each equation:

a. $x^3 + x^2 - 16x = 16$

b. $n^4 = 13n^2 - 36$

c. $2a^4 - 49a^2 = 25$

d. $x^4 + 17x^2 + 16 = 0$

Edith Hamilton:

"To be able to be caught up into the world of thought – that is educated."

Review Problems

Solve and check each equation:

4. $30x^3 = 2x^2 + 4x$

5. $n^3 + 5 = 5n^2 + n$

6. $x^4 + 900 = 109x^2$

7. $a^4 + 3a^2 - 4 = 0$

8. $t^4 + 49 + 50t^2 = 0$

9. $w^3 + 2w^2 - 12w - 9 = 0$ [Hard]
Hint: To factor, divide by $w - 3$.

Solutions

1. a. $x = 0, -1, -2$ b. $n = 0, \frac{1}{2}, 4$ c. $x = 0, 4, -4$
d. $y = 0, -5$ e. $a = 0$ f. $x = 0, \frac{2}{3}, -\frac{3}{2}$

2. There are quite a few solutions; you're on your own.

3. a. $x = \pm 4, -1$ b. $n = \pm 2, \pm 3$ c. $a = \pm 5$ d. No solution

4. $x = 0, -\frac{1}{3}, \frac{2}{5}$ **5**. $n = 5, \pm 1$ **6**. $x = \pm 3, \pm 10$

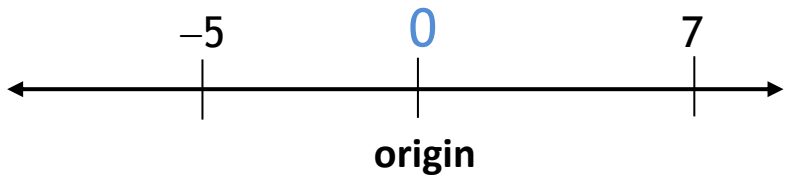
7. $a = \pm 1$ **8**. No solution

9. After you divide the cubic by $w - 3$, the other factor should be quadratic, but it is NOT factorable. So the only solution is $w = 3$.

EQUATIONS – DISTANCE AND THE ORIGIN

Sometimes the "size" of a number is more important than whether it's positive, zero, or negative. To measure the size of a number, we consider the ***distance*** between the number and zero (the origin) on the number line.

❑ *EXAMPLES*



What is the distance from the number 7 to 0? It's **7**.

How far from the origin is the number –5? It's **5** units away.

What is the distance from the origin to 2π? The distance is **2π**.

What is the distance between the number 0 and the origin? It's **0**.

❑ *NOTES*

1. We can calculate the distance from any number to 0 (the origin). In other words, the distance between the origin and any point on the line always exists, even if it's hard or impossible to calculate.
2. Distance must be greater than or equal to 0; that is, it's never negative. If $\boldsymbol{d}$ represents distance, then $\boldsymbol{d \geq 0}$.

Homework

1. For each number, calculate its distance to 0 (the origin):

   a. 18 b. –23 c. 0 d. $\frac{2}{3}$ e. $-2\frac{3}{5}$

   f. π g. $-\pi$ h. $\sqrt{5}$ i. $-\sqrt{17}$ f. $\sqrt{-9}$

2. For each expression, calculate its distance to 0 (the origin):

   a. $20 - 12$ b. $1 - 9$ c. $3.7 - 3.7$ d. $0 - 17$

   e. $\pi - \pi$ f. $\sqrt{2} + \sqrt{2}$ g. $\pi + \sqrt{2}$ h. $0.5 - 2$

3. If d represents the distance from any point on the line to the origin, then

   a. $d > 0$ b. $d \geq 0$ c. $d < 0$ d. $d \leq 0$

4. T/F: Given any number, we can always calculate its distance to the origin.

5. T/F: Given any number, its distance to the origin is always positive.

6. T/F: Given any number, its distance to the origin is never negative.

7. T/F: The distance between 0 and the origin does not exist.

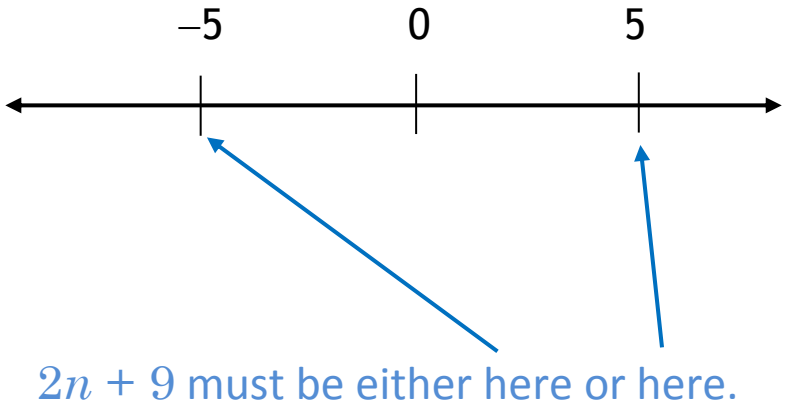
❑ *EXAMPLE*

I'm thinking of a number (or numbers). If the number is doubled and then 9 is added, the final result will be 5 units from the origin.

Solution: How about we let n represent the unknown number (or numbers). The problem states that

"*the number is doubled and then 9 is added*"

This sequence of events transforms the n into the expression $2n + 9$, and it is this quantity which must be 5 units from the origin. Well, what does it really mean for a quantity to be 5 units from the origin? It means that the quantity must be either +5 or –5 (since both +5 and –5 are 5 units from the origin).



Writing this idea with equations, we get two possibilities:

$2n + 9 = -5$	$2n + 9 = 5$
$2n = -14$	$2n = -4$
$n = -7$	$n = -2$

Therefore, the numbers I was thinking of were $\boxed{-7 \text{ and } -2}$

Homework

8. a. I'm 7 units from the origin on the number line. What number(s) am I?

   b. The distance between me and 0 on the number line is 100. What am I?

   c. I'm 0 units from the origin. What number(s) can I be?

   d. The distance between me and the number 0 is 0. Who am I?

   e. The distance between me and the origin is –10. What number(s) can I be?

   f. I'm –4 units from 0 on the number line. Who am I?

9. I'm thinking of a number or two...

   a. If 1 is added to the number, you get a number whose distance to the origin is 9.

   b. If 3 is subtracted from the number, the result is 10 units from 0 on the number line.

   c. If the number is doubled, the resulting number is 12 units from the origin.

   d. If the number is tripled, you get a number whose distance to the origin is 15.

   e. If 7 is subtracted from the number, the resulting number is 0 units away from 0.

   f. If 5 is added to the number, the distance between the final result and the origin is 0.

10. I'm thinking of a number or two...

a. If 3 is added to the number, you get a number whose distance to the origin is 14.

b. If 5 is subtracted from the number, the result is 12 units from 0 on the number line.

c. If the number is doubled, the resulting number is 8 units from the origin.

d. If the number is tripled, you get a number whose distance to the origin is 9.

e. If 4 is subtracted from the number, the resulting number is 0 units from the origin.

11. I'm thinking of a number (or numbers). If the number is quadrupled and then 8 is subtracted, the final result will be 24 units from the origin.

12. I'm thinking of a number (or numbers). If the number is tripled and then 6 is added, the final result will be 9 units from the origin.

13. I'm thinking of a number (or numbers). If the number is doubled and then 9 is subtracted, the final result will be 10 units from the origin.

14. I'm thinking of a number (or numbers). If the number is quintupled and then π is added, the final result will be -7 units from the origin.

Solutions

1. a. 18 b. 23 c. 0 d. $\frac{2}{3}$ e. $2\frac{3}{5}$

f. π g. π h. $\sqrt{5}$ i. $\sqrt{17}$ f. Not a number

2. a. 8 b. 8 c. 0 d. 17

e. 0 f. $2\sqrt{2}$ g. $\pi+\sqrt{2}$ h. 1.5

3. b. **4**. T **5**. F **6**. T **7**. F

8. a. 7 and –7 b. 100 and –100 c. 0 d. 0
e. Impossible; no distance can be –10.
f. Impossible; no distance can be –4.

9. a. –10 and 8 b. –7 and 13 c. 6 and –6
d. 5 and –5 e. 7 f. –5

10. a. 11 and –17 b. 17 and –7 c. 4 and –4
d. 3 and –3 e. 4

11. a. 8 and –4 **12**. 1 and –5

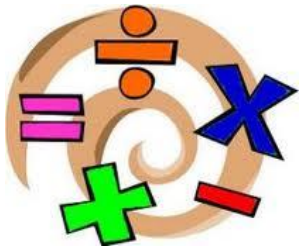
13. $\frac{19}{2}$ and $-\frac{1}{2}$ **14**. Impossible

"Next in importance to freedom and justice is education, without which neither freedom nor justice can be permanently maintained."

President James A. Garfield (1831 - 1881)

CH C – DISTRIBUTING TO SOLVE EQUATIONS

❑ *INTRODUCTION*



Now we have all the tools we need to solve linear equations containing parentheses. These tools are as follows:

- *The Distributive Property*: $3(x + 7) = 3x + 21$
- *Combining Like Terms*: $4w + 7 - 3w - 13 = w - 6$
- *Solving Equations*: $3n + 7 = 22 \Rightarrow 3n = 15 \Rightarrow n = 5$

Let's solve the equation in the Introduction to the previous chapter.

❑ *EXAMPLES*

EXAMPLE 1: **Solve for n: $8(n - 1) = -(3n + 7)$**

Solution: The distributive law comes to our rescue:

	$8(n - 1) = -(3n + 7)$	(the original equation)
$\Rightarrow$	$8n - 8 = -3n - 7$	(distribute)
$\Rightarrow$	$8n + \mathbf{3n} - 8 = -3n + \mathbf{3n} - 7$	(add $3n$ to each side)
$\Rightarrow$	$11n - 8 = -7$	(simplify)
$\Rightarrow$	$11n - 8 + \mathbf{8} = -7 + \mathbf{8}$	(add 8 to each side)
$\Rightarrow$	$11n = 1$	(simplify)
$\Rightarrow$	$\boxed{n = \frac{1}{11}}$	(divide each side by 11)

Homework

1. Solve each equation:

a. $2(2y - 3) = -8(2y - 9)$

b. $-6(-7b - 3) = -2(3b - 8)$

c. $-4(-2d + 9) = -5(2d - 9)$

d. $-4(-8j + 3) = 7(7j + 6)$

e. $3(3y - 5) = -(-y + 5)$

f. $-7(4t + 5) = -9(-5t + 3)$

g. $3(-2x + 8) = -3(-3x + 7)$

h. $7(2n + 3) = -6(3n + 7)$

EXAMPLE 2: **The Ultimate Challenge: Solve for x:**

$$\mathbf{2(3x - 7) - 5(1 - 3x) = -(-4x + 1) + (x + 7)}$$

Solution: The steps are

1) Distribute
2) Combine like terms
3) Solve the simplified equation

$2(3x - 7) - 5(1 - 3x) = -(-4x + 1) + (x + 7)$

$\Rightarrow \quad 6x - 14 - 5 + 15x = 4x - 1 + x + 7$ (distribute)

$\Rightarrow \quad 21x - 19 = 5x + 6$ (combine like terms)

$\Rightarrow \quad 21x - \mathbf{5x} - 19 = 5x - \mathbf{5x} + 6$ (subtract $5x$ from each side)

$\Rightarrow \quad 16x - 19 = 6$ (simplify)

$\Rightarrow \quad 16x - 19 \mathbf{+ 19} = 6 \mathbf{+ 19}$ (add 19 to each side)

$\Rightarrow \quad 16x = 25$ (simplify)

$\Rightarrow \quad \frac{16x}{\mathbf{16}} = \frac{25}{\mathbf{16}}$ (divide each side by 16)

$\Rightarrow \quad \boxed{x = \frac{25}{16}}$ (simplify)

Homework

2. Solve each equation:

a. $-6(7u - 7) + 8(1 - 7u) = 4(4u + 2) + 5(8 - 9u)$

b. $7(-b - 9) - 2(-8 + 7b) = 8(-5b + 5) + 3(-5b + 3)$

c. $-6(4n + 5) - 10(-10 + 9n) = 7(-n) - (-7n + 4)$

d. $-10(-3g) + 3(2g) = (-8g - 5) + 8(-g + 5)$

e. $9(4 - k) - 8(-9k + 1) = 3(6 + k) - 3(8 - 5k)$

f. $8(5w - 1) - 6(-w - 5) = 5(8 - 4w) + 6(9 + 8w)$

g. $-2(-4 - 2y) - 3(-9 + 8y) = 10(-3y - 8) - 2(-7 + 6y)$

h. $-4(a - 6) + (-5a - 3) = 6(2a + 1) - (5a + 4)$

Solutions

1. a. $y = \frac{39}{10}$ b. $b = -\frac{1}{24}$ c. $d = \frac{9}{2}$ d. $j = -\frac{54}{17}$

e. $y = \frac{5}{4}$ f. $t = -\frac{8}{73}$ g. $x = 3$ h. $n = -\frac{63}{32}$

2. a. $u = \frac{2}{69}$ b. $b = \frac{48}{17}$ c. $n = \frac{37}{57}$ d. $g = \frac{35}{52}$

e. $k = -\frac{34}{45}$ f. $w = 4$ g. $y = -\frac{101}{22}$ h. $a = \frac{19}{16}$

"*Opportunity* is missed by most people because it is dressed in overalls and looks like work."

– **Thomas Edison**

FRACTIONAL EQUATIONS

A ***fractional equation*** is an equation with one or more fractions in it. Since any fraction is really a division problem, it seems reasonable that we could solve such an equation by multiplying each side of the equation by something. What should that something be? The ***LCD*** (Least Common Denominator) of all the denominators in the equation works perfectly.

❑ *SOLVING FRACTIONAL EQUATIONS*

EXAMPLE 1: **Solve for z:** $\frac{z+6}{4} - \frac{z+4}{6} = \frac{2}{3}$

Solution:

First step: Determine the LCD; it's **12**. Reason: All three denominators, the 4, the 6, and the 3, divide into 12, AND 12 is the smallest number that does this.

Second step: Multiply each side of the equation by the LCD. [If this is done correctly, all denominators will be eradicated.]

Third step: Distribute, cross-cancel, and solve the equation.

Multiply each side of the equation by 12, the LCD:

$$12\left(\frac{z+6}{4} - \frac{z+4}{6}\right) = 12\left(\frac{2}{3}\right)$$

Distribute (and write 12 as $\frac{12}{1}$ if you'd like):

$$\frac{12}{1}\left(\frac{z+6}{4}\right) - \frac{12}{1}\left(\frac{z+4}{6}\right) = \frac{12}{1}\left(\frac{2}{3}\right)$$

Cross-cancel, thus removing all fractions from the equation:

$$\frac{{}^{3}\cancel{12}}{1}\left(\frac{z+6}{\cancel{4}_{1}}\right) - \frac{{}^{2}\cancel{12}}{1}\left(\frac{z+4}{\cancel{6}_{1}}\right) = \frac{{}^{4}\cancel{12}}{1}\left(\frac{2}{\cancel{3}_{1}}\right)$$

$$3(z+6) - 2(z+4) = 4(2)$$

Now solve for z in the usual way:

$$3z + 18 - 2z - 8 = 8$$

$$z + 10 = 8$$

$$\boxed{z = -2}$$

CHECK: Let $z = -2$ in the original equation. Do NOT do any algebra; just do simple arithmetic on each side:

$$\frac{z+6}{4} - \frac{z+4}{6} \qquad \frac{2}{3}$$

$$\frac{-\mathbf{2}+6}{4} - \frac{-\mathbf{2}+4}{6}$$

$$\frac{4}{4} - \frac{2}{6}$$

$$1 - \frac{1}{3}$$

$$\frac{2}{3} \qquad \checkmark$$

Homework

1. Solve and check each equation:

   a. $\frac{m}{5}+\frac{m}{4}=\frac{6}{5}$ b. $\frac{b}{2}-\frac{b}{6}=\frac{1}{3}$

   c. $\frac{n}{3}-\frac{n}{2}=1$ d. $\frac{r}{2}-\frac{r}{3}=\frac{1}{3}$

2. Solve and check each equation:

   a. $\frac{m-4}{6}+\frac{m+6}{5}=2$ b. $\frac{b-4}{5}-\frac{b+6}{4}=3$

   c. $\frac{c+3}{3}+\frac{c+5}{2}=\frac{1}{3}$ d. $\frac{k+4}{2}-\frac{k-4}{3}=1$

EXAMPLE 2: **Solve for x:** $\frac{2}{x}+\frac{y}{z}=w$

Solution: The LCD is xz (remember, the w can be written $\frac{w}{1}$). Therefore, we shall multiply each side of the equation by xz:

$\frac{2}{x}+\frac{y}{z}=w$ (the original equation)

$\Rightarrow \quad xz\left(\frac{2}{x}+\frac{y}{z}\right)=xz(w)$ (multiply each side by xz, the LCD)

$\Rightarrow \quad \cancel{x}z\left(\frac{2}{\cancel{x}}\right)+x\cancel{z}\left(\frac{y}{\cancel{z}}\right)=xz(w)$ (distribute and cross-cancel)

$\Rightarrow \quad 2z+xy=wxz$ (simplify)

$\Rightarrow \quad xy-wxz=-2z$ (x's to the left; the rest to the right)

$\Rightarrow \quad x(y-wz)=-2z$ (factor out the x, the GCF)

$\Rightarrow \quad \boxed{x=\frac{-2z}{y-wz}}$ (divide each side by $y-wz$)

Homework

3. Solve for x: $\frac{a}{b} - \frac{x}{a} = c$

4. Solve for y: $\frac{y}{t} + \frac{c}{d} = \frac{u}{w}$

5. Solve for n: $\frac{c}{h} + \frac{w}{n} = \frac{b}{x}$

6. Solve for z: $\frac{a}{z} - \frac{x}{n} = m$

7. Solve for c: $\frac{b}{m} + \frac{z}{c} = \frac{n}{w}$

8. Solve for n: $\frac{u}{n} + \frac{m}{t} = d$

EXAMPLE 3: **Solve for d:** $7 - \frac{6}{d-2} = \frac{1}{d-2}$

Solution: The LCD is $d - 2$, so this is what we'll multiply each side of the equation by:

$$\left[\frac{d-2}{1}\right]\left(7 - \frac{6}{d-2}\right) = \left[\frac{d-2}{1}\right]\left(\frac{1}{d-2}\right)$$

$$\Rightarrow (d-2)(7) - \cancel{(d-2)}\left(\frac{6}{\cancel{d-2}}\right) = \left(\frac{\cancel{d-2}}{1}\right)\left(\frac{1}{\cancel{d-2}}\right) \quad \text{(distribute)}$$

$\Rightarrow$ $7(d-2) - 6 = 1$ (cross-cancel)

$\Rightarrow$ $7d - 14 - 6 = 1$ (distribute)

$\Rightarrow$ $7d - 20 = 1$ (simplify)

$\Rightarrow$ $7d = 21$ (add 20)

$\Rightarrow$ $\boxed{d = 3}$ (divide by 7)

CHECK: Put 3 in for d in the original equation:

$7-\frac{6}{d-2}$	$\frac{1}{d-2}$
$7-\frac{6}{\mathbf{3}-2}$	$\frac{1}{\mathbf{3}-2}$
$7-\frac{6}{1}$	$\frac{1}{1}$
$7-6$	1
1	✓

The sides balance, so $d = 3$ is the solution.

EXAMPLE 4: **Solve for u:** $\mathbf{1-\frac{16}{u}+\frac{28}{u^2}=0}$

Solution: The LCD is u^2. Multiply each side of the equation by u^2:

$$u^2\left(1-\frac{16}{u}+\frac{28}{u^2}\right) = u^2(0)$$

$$\Rightarrow \quad u^2(1)-u^2\left(\frac{16}{u}\right)+u^2\left(\frac{28}{u^2}\right) = u^2(0) \quad \text{(distribute)}$$

$$\Rightarrow \quad u^2 - 16u + 28 = 0 \quad \text{(cross-cancel)}$$

$$\Rightarrow \quad (u-14)(u-2) = 0 \quad \text{(factor)}$$

$$\Rightarrow \quad u-14 = 0 \text{ or } u-2 = 0 \quad \text{(set each factor to 0)}$$

$$\Rightarrow \quad \boxed{u = 14 \text{ or } u = 2}$$

You should check these two potential solutions to ensure that both work in the original equation.

EXAMPLE 5: **Solve for m:** $\dfrac{m}{m+9} + 8 = \dfrac{-9}{m+9}$

Solution: We begin as usual by multiplying each side of the equation by the LCD; in this problem, it's $m + 9$:

$$[m+9]\left(\frac{m}{m+9}+8\right) = [m+9]\left(\frac{-9}{m+9}\right)$$

$$\Rightarrow \quad [m+9]\frac{m}{m+9}+[m+9]8 = [m+9]\frac{-9}{m+9} \qquad \text{(distribute)}$$

$$\Rightarrow \quad \cancel{[m+9]}\frac{m}{\cancel{m+9}}+[m+9]8 = \cancel{[m+9]}\frac{-9}{\cancel{m+9}}$$

$$\Rightarrow \quad m + 8m + 72 = -9 \qquad \text{(cross-cancel)}$$

$$\Rightarrow \quad 9m + 72 = -9 \qquad \text{(combine like terms)}$$

$$\Rightarrow \quad 9m = -81 \qquad \text{(subtract 72)}$$

$$\Rightarrow \quad \underline{m = -9} \qquad \text{(divide by 9)}$$

Now for the **check**; put –9 in for m in the original equation:

$\dfrac{m}{m+9}+8$	$\dfrac{-9}{m+9}$
$\dfrac{\mathbf{-9}}{\mathbf{-9}+9}+8$	$\dfrac{-9}{\mathbf{-9}+9}$
$\dfrac{-9}{0}+8$	$\dfrac{-9}{0}$

Hold it right there!! We have zero on the bottom of both fractions. Continuing with this calculation is meaningless, since we have two undefined fractions. We've reached the end of our check. Does this mean we made a mistake in solving the equation? Absolutely not! We did everything according to the rules, so the equation must be inherently flawed. Thus, our only candidate for a solution, –9, has failed to be a solution. We have

no choice but to admit that no number will make the original equation true. Our conclusion:

No solution

(Some teachers would write this as the null set: ∅)

EXAMPLE 6: **Solve for x:** $\dfrac{5}{x+1} - \dfrac{3}{x+3} = 4$

Solution: Multiply each side of the equation by the LCD: $(x + 1)(x + 3)$. Then distribute, cross-cancel, and solve the resulting quadratic equation by factoring.

Multiply each side of the equation by the LCD, $(x + 1)(x + 3)$:

$$(x+1)(x+3)\left(\frac{5}{x+1} - \frac{3}{x+3}\right) = (x+1)(x+3)(4)$$

Distribute the LCD, and then cross-cancel:

$$\cancel{(x+1)}(x+3)\left(\frac{5}{\cancel{x+1}}\right) - (x+1)\cancel{(x+3)}\left(\frac{3}{\cancel{x+3}}\right) = (x+1)(x+3)(4)$$

$5(x + 3) - 3(x + 1) = 4(x + 1)(x + 3)$

Distribute on the left and double distribute on the right:

$5x + 15 - 3x - 3 = 4(x^2 + 4x + 3)$

Combine like terms on the left, and distribute on the right:

$2x + 12 = 4x^2 + 16x + 12$

Since we have a quadratic equation, bring all terms to one side:

$0 = 4x^2 + 14x$

Turn the equation around and factor:

$2x(2x + 7) = 0$

Set each factor to 0:

$2x = 0$ or $2x + 7 = 0$

Solve each linear equation:

$$\boxed{x = 0 \text{ or } x = -\frac{7}{2}}$$

CHECK: I'm tired; you check both solutions.

EXAMPLE 7: **Solve for y:** $\dfrac{y-2}{y+1} = \dfrac{y+7}{y-8}$

Solution: Multiply each side of the equation by the LCD:

$$(y+1)(y-8)\left(\frac{y-2}{y+1}\right) = (y+1)(y-8)\left(\frac{y+7}{y-8}\right)$$

Cross-cancel the common factors on each side of the equation:

$$\cancel{(y+1)}(y-8)\left(\frac{y-2}{\cancel{y+1}}\right) = (y+1)\cancel{(y-8)}\left(\frac{y+7}{\cancel{y-8}}\right)$$

And simplify:

$$(y-8)(y-2) = (y+1)(y+7)$$

Notice that this equation can be found directly from the original equation by "cross-multiplying."

Double distribute on each side of the equation:

$$y^2 - 10y + 16 = y^2 + 8y + 7$$

Subtract y^2 from each side of the equation:

$$-10y + 16 = 8y + 7$$

Finally, solve for y in the standard way:

$$-18y + 16 = 7 \Rightarrow -18y = -9 \Rightarrow y = \frac{-9}{-18} = \frac{1}{2}$$

$$\boxed{y = \frac{1}{2}}$$

CHECK:

$\frac{y-2}{y+1}$	$\frac{y+7}{y-8}$
$\frac{\frac{1}{2}-2}{\frac{1}{2}+1}$	$\frac{\frac{1}{2}+7}{\frac{1}{2}-8}$
$\frac{-\frac{3}{2}}{\frac{3}{2}}$	$\frac{\frac{15}{2}}{-\frac{15}{2}}$
-1	-1 ✓

Homework

9. Solve and check each equation:

a. $\frac{4x}{x-8}+1 = \frac{9}{x-8}$

b. $\frac{1}{2n}+\frac{2}{n} = \frac{5}{6}$

c. $1-\frac{12}{a}-\frac{64}{a^2} = 0$

d. $g+\frac{24}{g} = -10$

e. $\frac{4}{t+1}-\frac{6}{t+3} = 2$

f. $\frac{9}{b-1}-\frac{9}{b+3} = 3$

g. $\frac{x+2}{x+4} = \frac{x+5}{x}$

h. $\frac{q-5}{-2} = \frac{-6}{q-1}$

i. $\frac{9}{x}-\frac{70}{x^2} = -1$

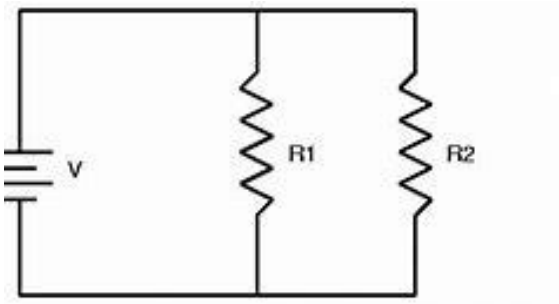
j. $\frac{-8}{g+1} = \frac{8g}{g+1}-9$

k. $\frac{1}{4a}+\frac{2}{5a} = \frac{8}{5}$

l. $\frac{2}{m+8}+\frac{1}{m+6} = 1$

❑ *ELECTRONICS*

The electrical circuit shows a voltage source along with two resistors wired in **parallel**. The TOTAL **resistance**, R_T, in the circuit is related to the resistors R_1 and R_2 by the formula



$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$$

Resistance can be measured in a unit called the **ohm**, (named after Georg Ohm) and abbreviated using the Greek capital letter omega, Ω.

EXAMPLE 8: **A parallel circuit contains two resistors, one with a resistance of $8\,\Omega$ and the other with a resistance of $5\,\Omega$. Find R_T, the total resistance in the circuit.**

Solution: Using the formula we can write

$$\frac{1}{R_T} = \frac{1}{8} + \frac{1}{5}$$

The LCD is $40R_T$, the product of the three denominators.

Multiplying both sides of the equation by the LCD gives

$$40R_T\left(\frac{1}{R_T}\right) = 40R_T\left(\frac{1}{8}\right) + 40R_T\left(\frac{1}{5}\right)$$

$$\Rightarrow \quad 40 = 5R_T + 8R_T$$

$$\Rightarrow \quad 40 = 13R_T$$

$$\Rightarrow \quad R_T = \frac{40}{13}$$

$$\Rightarrow \quad \boxed{R_T \approx 3.08\,\Omega}$$

Alternate Solution:

$$\frac{1}{R_T} = \frac{1}{8} + \frac{1}{5}$$

$$\Rightarrow \quad \frac{1}{R_T} = \frac{13}{40} \qquad \text{(add the fractions)}$$

$$\Rightarrow \quad R_T = \frac{40}{13} \qquad \text{(take the reciprocal of each side of the equation, giving the same answer)}$$

EXAMPLE 9: **Solve the total resistance formula for R_1.**

Solution: We start, of course, with the given formula:

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$$

The LCD is the product of all three denominators, which is $R_T R_1 R_2$, so we multiply each side of the formula by the LCD, which clears out the fractions:

$$\left[R_T R_1 R_2\right]\left(\frac{1}{R_T}\right) = \left[R_T R_1 R_2\right]\left(\frac{1}{R_1}\right) + \left[R_T R_1 R_2\right]\left(\frac{1}{R_2}\right)$$

Now simplify by cross-canceling:

$$R_1 R_2 = R_T R_2 + R_T R_1$$

$$\Rightarrow \quad R_1 R_2 - R_T R_1 = R_T R_2 \qquad \text{(bring all } R_1\text{'s to the left side)}$$

$$\Rightarrow \quad R_1(R_2 - R_T) = R_T R_2 \qquad \text{(factor out our unkown } R_1\text{)}$$

$$\Rightarrow \quad \boxed{R_1 = \frac{R_2 R_T}{R_2 - R_T}} \qquad \text{(divide each side by } R_2 - R_T\text{)}$$

Homework

10. A parallel circuit contains two resistors, one with a resistance of 5Ω and the other with a resistance of 0.2Ω. Find R_T, the total resistance in the circuit.

11. The total resistance in a parallel circuit containing two resistors is $\mathbf{8}\Omega$**.** If one of the resistors has a resistance of $\mathbf{10}\Omega$, find the resistance of the other resistor.

12. Solve the total resistance formula for R_T.

13. Solve the total resistance formula for R_2.

Review Problems

14. Solve for x: $\frac{x-2}{5} - \frac{2x-1}{6} = \frac{4}{15}$

15. Solve for x: $\frac{a}{y} - \frac{z}{x} = \frac{a}{b}$

16. Solve for a: $1 - \frac{25}{a} + \frac{150}{a^2} = 0$

17. Solve for x: $\frac{x}{x-3} + \frac{3}{2} = \frac{3}{x-3}$

18. Solve for x: $\frac{2}{x-3} + \frac{3}{x+1} = \frac{3}{2}$

Solutions

1. a. $m = \frac{8}{3}$ b. $b = 1$ c. $n = -6$ d. $r = 2$

2. a. $m = 4$ b. $b = -106$ c. $c = -\frac{19}{5}$ d. $k = -14$

3. $x = \frac{abc - a^2}{-b} = \frac{a^2 - abc}{b}$

4. $y = \frac{dtu - ctw}{dw}$

5. $n = \frac{-hxw}{cx - bh} = \frac{hxw}{bh - cx}$

6. $z = \frac{an}{mn + x}$

7. $c = \frac{-mwz}{bw - mn} = \frac{mwz}{mn - bw}$

8. $n = \frac{tu}{dt - m}$

9. a. $x = \frac{17}{5}$ b. $n = 3$ c. $a = 16, -4$

d. $g = -4, -6$ e. $t = 0, -5$ f. $b = 3, -5$

g. $x = -\frac{20}{7}$ h. $q = -1, 7$ i. $x = 5, -14$

j. No solution k. $a = \frac{13}{32}$ l. $m = -4, -7$

10. $R_T \approx 0.192\ \Omega$

11. $40\ \Omega$

12. $R_T = \frac{R_1 R_2}{R_1 + R_2}$

13. $R_2 = \frac{R_1 R_T}{R_1 - R_T}$

14. $x = -\frac{15}{4}$

15. $x = \frac{byz}{ab - ay}$

16. $a = 10, 15$

17. No solution

18. $x = 5, \frac{1}{3}$

"We cannot hold a torch

to light another's path

without brightening our own."

– Ben Sweetland

INTRO TO EQUATIONS

❑ INTRODUCTION

I'm thinking of a number. If I add 10 to the number, the result is 45. What number was I thinking of ?

Remember this question from the chapter *Constants*? Now we're ready to formalize the thought processes you went through to solve this problem. It may seem silly to make a big deal about this easy problem, but the skills you develop now will give you the tools needed to solve problems that could never be solved in your head.



An equation must always be in balance.

❑ ONE-STEP EQUATIONS

Considering the problem again, let's pretend we don't already know the answer. We'll choose some symbol to represent the unknown number; call it n. Here's what we need to do: Translate the English statement

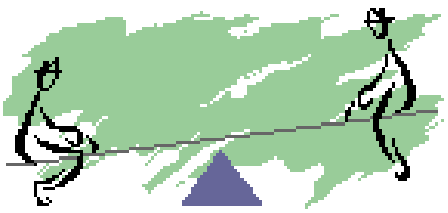
"*If I add* 10 *to the number, the result is* 45"

into an equation, a statement of equality. Since we let n represent the number, then the phrase "*If I add 10 to the number*" becomes "n + 10", the phrase "*the result is*" becomes " = ", and the 45 is, of course, just 45. The final translation from English to algebra results in the equation

$$\underbrace{n+10}_{\text{If I add 10 to the number}} \quad \underset{\text{the result is}}{=} \quad 45$$

An equation is like a seesaw in balance, perfectly level with a child at each end. If we were to drop a bowling ball onto the lap of one child, that child would plummet to the ground, throwing the second child into

the air — all in all, not a pleasant situation. But most importantly, the seesaw would not be in balance anymore; one end would be on the ground and the other up in the air. But what if we dropped a 10-lb bowling ball on the lap of each child at exactly the same moment? Even with kids screaming in pain, at least the seesaw would remain level, still in balance. By the way, math teachers don't care about kids in pain, as long as the equation remains balanced!



This is the essence of solving an equation:

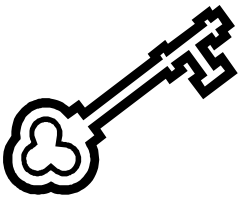
As long as we perform the same operation
on each side of an equation,
the resulting statement should
still be a balanced equation.

Thus, to solve the equation above, $n + 10 = 45$, we need to do something to each side of the equation that will tell us what n is. We have to "isolate" the n. To do that, we have to strip away the 10 from the expression $n + 10$.

Here's the thought process: The operation connecting the n with the 10 is addition. The operation which removes addition is subtraction. Therefore, we need to subtract 10 from the left side of the equation, which will leave the n all by itself (isolated). *But whatever we do to one side of an equation, we must do to the other side.*

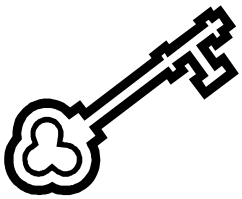
	$n + 10 = 45$	(the original equation)
$\Rightarrow$	$n + 10 \mathbf{- 10} = 45 \mathbf{- 10}$	(subtract 10 from each side of the equation)
$\Rightarrow$	$n = 35$	(simplify each side)

Therefore, the number I was thinking of was $\boxed{35}$



The key to solving any equation:

DO THE **SAME** THING TO EACH SIDE OF THE EQUATION.



EXAMPLE 1: **Solve each equation:**

A. $x + 17 = 50$

Since the x and the 17 are connected by addition, we will use subtraction to remove the 17. Remembering to always do the same thing to each side of the equation, we get

$x + 17 - \mathbf{17} = 50 - \mathbf{17}$ (subtract 17 from each side)

$\boldsymbol{x = 33}$ (simplify each side)

B. $y - 12 = 13$

In this equation, the 12 is being subtracted from the unknown y; to undo the subtracting, we add 12 to each side of the equation:

$y - 12 + \mathbf{12} = 13 + \mathbf{12}$ (add 12 to each side)

$\boldsymbol{y = 25}$ (simplify each side)

C. $7n = 35$

This time the operation to remove is multiplication, which is easily removed by division:

$\frac{7n}{\mathbf{7}} = \frac{35}{\mathbf{7}}$ (divide each side by 7)

$\boldsymbol{n = 5}$ (simplify each side)

D. $\frac{a}{8} = 2$

The operation connecting the unknown a with the 8 is division. Since the reverse of division is multiplication, we shall multiply each side of the equation by 8:

$\left(\frac{a}{8}\right)\mathbf{8} = (2)\mathbf{8}$ (multiply each side by 8)

$\left(\frac{a}{\cancel{8}}\right)\cancel{\mathbf{8}} = 16$ (cross-cancel the 8's)

$\boldsymbol{a = 16}$ (simplify)



This is what happens if you <u>don't</u> do the same thing to each side of the equation.

Homework

1. Solve each equation:

a. $w + 17 = 25$ b. $a - 9 = 4$ c. $x - 2 = 90$

d. $y + 99 = 105$ e. $z - 32 = 80$ f. $b + 50 = 160$

g. $8 + x = 12$ h. $12 + t = 100$ i. $30 + m = 30$

j. $n - 17 = 50$ k. $J + 20 = 44.9$ l. $3.3 + T = 100$

m. $w - 4.5 = 8.2$ n. $u + 4 = 1000$ o. $n + 0.88 = 0.88$

2. Solve each equation:

a. $3x = 18$ b. $2n = 144$ c. $5u = 95$

d. $4z = 30.4$ e. $1.3m = 128.7$ f. $0.25Q = 0.45$

g. $7x = 28$ h. $32a = 128$ i. $5R = 1000$

j. $1000k = 10{,}000$ k. $2x = 17$ l. $7h = 14$

m. $12c = 12$ n. $18a = 0$ o. $2x = 10$

3. Solve each equation:

a. $\frac{x}{3} = 6$ b. $\frac{y}{4} = 24$ c. $\frac{z}{2} = 33$

d. $\frac{a}{23} = 5$ e. $\frac{b}{17} = 34$ f. $\frac{c}{88} = 1$

g. $\frac{x}{7} = 1$ h. $\frac{a}{30} = 30$ i. $\frac{u}{17} = 3$

j. $\frac{y}{0.5} = 20$ k. $\frac{p}{1.7} = 0.4$ l. $\frac{q}{2.3} = 2.3$

4. Solve each equation:

a. $2x = 18$ b. $3y = 16$ c. $4n = 3$

d. $7 + x = 9$ e. $x - 3 = 42$ f. $y + 9.3 = 9.3$

g. $n(7) = 49$ h. $z - 1 = 1$ i. $a + 2.3 = 9.1$

j. $b - 0.7 = 2$ k. $c + 3 = 4.3$ l. $d - 2.13 = 18.77$

m. $\frac{x}{3} = 7$ n. $\frac{y}{9} = 1$ o. $\frac{z}{1.3} = 0.5$

❑ *TWO-STEP EQUATIONS*

Here's another word problem that can be done by guessing or working backwards:

> *"I'm thinking of a number. If I multiply it by 3 and then add 4, the result is 34. What is the number I was thinking of?"*

Translating this problem into algebra (assuming n is the unknown number) results in the equation

$$3n + 4 = 34$$

As we learned above, since the 3 is connected to the n by multiplying, we rid ourselves of the 3 by dividing. And because the 4 is being added to the $3n$, we remove the 4 by subtracting. And, of course, whenever we do something to the left side of the equation, we'll keep things in balance by doing the same thing to the right side.

This leaves one important question:

Which do we get rid of first, the 3 or the 4?



Just like unwrapping a gift, we start with the <u>final</u> operation in the expression, and work backwards until the variable n is alone. The final operation in the expression $3n + 4$ is addition, so we get rid of the 4 first:

	$3n + 4 = 34$	(the original equation)
$\Rightarrow$	$3n + 4 - \mathbf{4} = 34 - \mathbf{4}$	(subtract 4 from each side)
$\Rightarrow$	$3n = 30$	(simplify each side)
$\Rightarrow$	$\frac{3n}{\mathbf{3}} = \frac{30}{\mathbf{3}}$	(divide each side by 3)
$\Rightarrow$	$n = 10$	(simplify each side)

and we've found that the unknown number is $\boxed{10}$

Note: Since the expression $3n + 4$ was created using the Order of Operations – multiply, then add – we undo the order of operations by going in the reverse order: First we undo the last operation, addition, and second we undo the first operation, multiplication.

The next example shows us how we generally leave fractional answers in algebra.

EXAMPLE 2: **Solve for x: $-14 + 8x = 22$**

Solution:

Add 14 to each side of the equation: $-14 \mathbf{+ 14} + 8x = 22 \mathbf{+ 14}$

Simplify: $8x = 36$

Divide each side by 8: $\frac{8x}{\mathbf{8}} = \frac{36}{\mathbf{8}}$

Simplify: $\boxed{x = \frac{9}{2}}$

Notice that it's okay in algebra to leave an improper fraction as your final answer, as long as it's in reduced form. In this example, the answer $\frac{36}{8}$ must be reduced to $\frac{9}{2}$, but it need not be converted to a mixed number.

Some students find it easier by first using the commutative property of addition to flip the terms on the left side of the given equation:

$$8x - 14 = 22$$

EXAMPLE 3: **Solve for a: $\frac{a}{4} - 2 = 10$**

Solution: We notice that in this equation, the first operation on the variable a is the division, followed by the subtraction. Therefore, our plan is to reverse these steps.

The equation we're to solve: $\frac{a}{4} - 2 = 10$

Add 2 to each side of the equation: $\frac{a}{4} - 2 + \mathbf{2} = 10 + \mathbf{2}$

And simplify: $\frac{a}{4} = 12$

Last, multiply each side of the equation by 4: $\frac{a}{4}\left[\mathbf{4}\right] = 12\left[\mathbf{4}\right]$

And we're done: $\boxed{a = 48}$

EXAMPLE 4: **Solve for *n*:** $\frac{n+7}{3} = 9$

Solution: In this equation the final operation is division, so our first step in solving the equation is to multiply.

$\frac{n+7}{3}\left[\mathbf{3}\right] = 9\left[\mathbf{3}\right]$ (multiply each side by 3)

$\Rightarrow \quad n + 7 = 27$ (simplify each side)

$\Rightarrow \quad n + 7 - \mathbf{7} = 27 - \mathbf{7}$ (subtract 7 from each side)

$\Rightarrow \quad \boxed{n = 20}$ (simplify each side)

Homework

5. Solve each equation, leaving answers as fractions:

a. $3n + 7 = 22$ b. $2x - 1 = 49$ c. $4q + 17 = 93$

d. $7z - 7 = 7$ e. $13w + 26 = 65$ f. $23x + 18 = 38$

g. $18u - 7 = 11$ h. $12v + 80 = 80$ i. $7a + 3 = 10$

j. $n + 17 = 17$ k. $23T - 44 = 55$ l. $12 + 5k = 20$

m. $7x + 12 = 100$ n. $3y - 13 = 20$ o. $7t + 18 = 18$

p. $13c - 98 = 1$ q. $23 + 7h = 24$ r. $12 + 5w = 15$

6. Solve each equation:

a. $\frac{x+4}{5} = 2$ b. $\frac{a}{3} - 4 = 5$ c. $\frac{u-1}{3} = 3$

d. $\frac{c-3}{4} = 0$ e. $\frac{y}{7} + 1 = 5$ f. $\frac{w}{2} - 7 = 5$

g. $\frac{n+3}{5} = 3$ h. $\frac{b}{4} - 3 = 1$ i. $\frac{z-1}{4} = 4$

j. $\frac{d-7}{\pi} = 0$ k. $\frac{m}{6} + 2 = 5$ l. $\frac{g}{2} - 70 = 500$

Solutions

1. a. $w = 8$ b. $a = 13$ c. $x = 92$ d. $y = 6$
e. $z = 112$ f. $b = 110$ g. $x = 4$ h. $t = 88$
i. $m = 0$ j. $n = 67$ k. $J = 24.9$ l. $T = 96.7$
m. $w = 12.7$ n. $u = 996$ o. $n = 0$

2. a. $x = 6$ b. $n = 72$ c. $u = 19$ d. $z = 7.6$
e. $m = 99$ f. $Q = 1.8$ g. $x = 4$ h. $a = 4$
i. $R = 200$ j. $k = 10$ k. $x = 8.5$ l. $h = 2$
m. $c = 1$ n. $a = 0$ o. $x = 5$

3. a. $x = 18$ b. $y = 96$ c. $z = 66$ d. $a = 115$
e. $b = 578$ f. $c = 88$ g. $x = 7$ h. $a = 900$
i. $u = 51$ j. $y = 10$ k. $p = 0.68$ l. $q = 5.29$

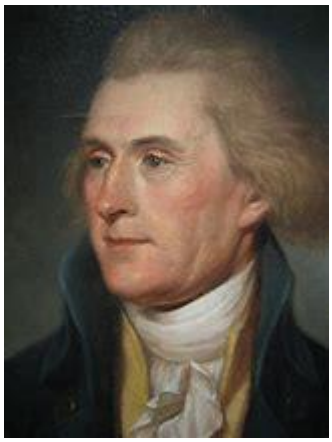
4. a. $x = 9$ b. $y = 5.33$ c. $n = 0.75$ d. $x = 2$
e. $x = 45$ f. $y = 0$ g. $n = 7$ h. $z = 2$
i. $a = 6.8$ j. $b = 2.7$ k. $c = 1.3$ l. $d = 20.9$
m. $x = 21$ n. $y = 9$ o. $z = 0.65$

5. For this problem, answers will be written in fraction form, just for variety. Notice that it's okay in algebra to leave an improper fraction as your final answer, as long as it's been reduced. For example, the answer $\frac{10}{8}$ must be reduced to $\frac{5}{4}$, but it need not be converted to a mixed number.

a. $n = 5$ b. $x = 25$ c. $q = 19$ d. $z = 2$
e. $w = 3$ f. $x = \frac{20}{23}$ g. $u = 1$ h. $v = 0$
i. $a = 1$ j. $n = 0$ k. $T = \frac{99}{23}$ l. $k = \frac{8}{5}$
m. $x = \frac{88}{7}$ n. $y = 11$ o. $t = 0$ p. $c = \frac{99}{13}$
q. $h = \frac{1}{7}$ r. $w = \frac{3}{5}$

6. a. $x = 6$ b. $a = 27$ c. $u = 10$
d. $c = 3$ e. $y = 28$ f. $w = 24$
g. $n = 12$ h. $b = 16$ i. $z = 17$
j. $d = 7$ k. $m = 18$ l. $g = 1{,}140$

"If a nation expects to be ignorant and free, it expects what never was, and will never be."



– Thomas Jefferson

SQUARE-ROOT EQUATIONS



An equation such as $\sqrt{x-4}-5 = 0$, with the variable inside the square-root sign, is called a ***square-root equation*** (or, in general, a *radical equation*). Just as subtraction in an equation is removed by addition, a square root is removed by squaring.

Here are the major steps to solve a general radical equation:

1. Isolate the radical (if necessary).
2. Raise each side of the equation to an appropriate power (squaring in this chapter).
3. Solve the resulting equation for potential solutions.
4. Check every potential solution in the original equation.

❑ *EXAMPLES*

EXAMPLE 1: **Solve for z:** $\sqrt{z} = 7$

The key to solving a square-root equation is the fact that, assuming $x \geq 0$,

$$\left(\sqrt{x}\right)^2 = x$$

Solution: The radical is already isolated, so on to step 2. Since the radical is a square root, we will square each side of the equation:

$$\sqrt{z} = 7 \qquad \text{(the original equation)}$$

$$\Rightarrow \quad \left(\sqrt{z}\right)^2 = 7^2 \qquad \text{(**square** each side of the equation)}$$

$$\Rightarrow \quad \boxed{z = 49} \qquad \text{(squaring undoes square rooting)}$$

Put 49 back into the original equation, and you'll see that we have a valid solution.

EXAMPLE 2: **Solve for x:** $\sqrt{x-4}-5 = 0$

Solution: Our plan is to isolate and square.

Start with the original equation: $\sqrt{x-4}-5 = 0$

First, isolate the radical: $\sqrt{x-4} = 5$

Second, to remove the square root sign, square each side of the equation: $\left(\sqrt{x-4}\right)^2 = 5^2$

Simplify each side of the equation: $x-4 = 25$

Now solve for x: $x = 29$

Now it's time to check our solution. Let's put our candidate, 29, into the original equation and perform the arithmetic separately on each side of the equation:

$\sqrt{\mathbf{29}-4}-5$	0
$\sqrt{25}-5$	
$5-5$	
0	✓

So 29 checks out; therefore, the solution is $\boxed{x = 29}$

EXAMPLE 3: **Solve for a:** $\sqrt{a^2+15}+10 = 2$

Solution: We isolate the radical, square each side, solve for a, and then CHECK the solutions.

	$\sqrt{a^2+15}+10 = 2$	(the original equation)
$\Rightarrow$	$\sqrt{a^2+15} = -8$	(isolate the radical)
$\Rightarrow$	$\left(\sqrt{a^2+15}\right)^2 = (-8)^2$	(square each side
$\Rightarrow$	$a^2 + 15 = 64$	(simplify each side)
$\Rightarrow$	$a^2 - 49 = 0$	(subtract 64 from each side)
$\Rightarrow$	$(a + 7)(a - 7) = 0$	(factor)
$\Rightarrow$	$a + 7 = 0$ or $a - 7 = 0$	(set each factor to 0)
$\Rightarrow$	$a = -7$ or $a = 7$	(solve for a)

These are two possible solutions, 7 and –7. Let's check out the **7**:

$\sqrt{7^2+15}+10$	2
$\sqrt{49+16}+10$	
$\sqrt{64}+10$	
$8 + 10$	
18	✘

The two sides of the equation do not balance; therefore, 7 is not a solution of the radical equation. Using –7 for a will result in the same calculations, indicating that –7 is also not a solution. So, what do we have here? We solved an equation and got two potential solutions, 7 and –7, but neither of them panned out (they are called *extraneous solutions*). That's it — it's over — nothing works.

No Solution

EXAMPLE 4: **Solve for x:** $\sqrt{11-x}+5 = x$

Solution: To isolate the radical, we need to subtract 5 from each side of the equation:

$$\sqrt{11-x} = x-5$$

Since we now have a square root to undo, we will square each side of the equation:

$$\left(\sqrt{11-x}\right)^2 = (x-5)^2$$ Notice the parentheses around the $x-5$.

Now we simplify each side of the equation:

$$11-x = x^2-10x+25$$ Careful when squaring $x-5$.

Recognizing this equation as quadratic, we'll put it into standard form by adding x to each side and subtracting 11 from each side:

$$0 = x^2-9x+14$$

You're welcome to switch the sides and write $x^2-9x+14 = 0$

Factoring produces

$$0 = (x-7)(x-2)$$

Setting each factor to 0 and solving gives us two candidates for a solution:

$$x = 7 \text{ or } x = 2$$

We now check each potential solution in the original equation:

$x = 7$

$\sqrt{11-7}+5$	**7**
$\sqrt{4}+5$	
$2+5$	
7	✓

$x = 2$

$\sqrt{11-2}+5$	**2**
$\sqrt{9}+5$	
$3+5$	
8	✘

What does all this mean? Easy — the **7** works while the **2** doesn't. So, even though we got two solutions when we solved the radical equation, only one actually worked out.

The 2 we obtained but failed to work in the original equation is called an *extraneous solution*. (By the way, just because one solution failed to work does not mean we made any mistakes when solving the problem.) Therefore, the final solution of the equation is

$$\boxed{x = 7}$$

Homework

Solve and check each equation:

1. $\sqrt{x} = 10$
2. $\sqrt{y} = 5$
3. $\sqrt{t} = 0$
4. $\sqrt{a} = -3$
5. $\sqrt{x+1} = 3$
6. $\sqrt{c-5} = 10$
7. $\sqrt{x+5} - 2 = 3$
8. $\sqrt{2y-1} + 3 = 6$
9. $\sqrt{8-n} - 4 = 0$
10. $\sqrt{7z+100} + 10 = 20$
11. $\sqrt{5x-1} = \sqrt{7x-25}$
12. $\sqrt{x} + 2 = x$
13. $\sqrt{n+1} + 3 = n - 8$
14. $\sqrt{2x-11} + x = 13$

❑ *To ∞ and Beyond*

A. Solve for x: $\sqrt[5]{7x-9} = 2$

B. Solve for x: $\sqrt{x+40} = 4 + \sqrt{x}$

C. Solve for y: $\sqrt[3]{(y-2)^2} = 4$

Solutions

1. $x = 100$ **2**. $y = 25$ **3**. $t = 0$ **4**. No solution

5. $x = 8$ **6**. $c = 105$ **7**. $x = 20$ **8**. $y = 5$

9. $n = -8$ **10**. $z = 0$

11. Square each side: $5x - 1 = 7x - 25 \Rightarrow x = 12$; check it in the original equation, doing just simple arithmetic on each side separately. You should find that $x = 12$ works.

12. Isolate the radical and square each side, leading to the equation $x^2 - 5x + 4 = 0$, whose solutions are 4 and 1. Only the 4 works; the value 1 is called an *extraneous* solution. The final answer is $x = 4$.

13. Isolate the radical and square both sides — when simplified, you'll have the quadratic equation $n^2 - 23n + 120 = 0$. Only $n = 15$ works.

14. The two candidates for a solution are 10 and 18. The 18 is extraneous and the final answer is $x = 10$.

"The only person who is educated is the one who has learned how to learn . . . and change." - Carl Rogers, Psychologist

USING THE GCF IN EQUATIONS AND FORMULAS

❑ INTRODUCTION

Consider the following equation

$$(x)(x - 7) = 0$$

There are two solutions to this equation — what are they? Before we present the formal process, here's what you should note: What would happen if we let $x = 0$ in the equation $(x)(x - 7) = 0$? We'd get

$$(\mathbf{0})(\mathbf{0} - 7) = (0)(-7) = 0 \quad ✓$$

And so we've stumbled upon a solution of the equation $(x)(x - 7) = 0$. Let's "stumble" one more time and choose $x = 7$ — check it out:

$$(\mathbf{7})(\mathbf{7} - 7) = (7)(0) = 0 \quad ✓$$

Can you see how we stumbled across these two solutions, 0 and 7? Each solution was chosen so that one of the two factors would turn into zero. That way, the product of that zero factor with the other factor (no matter what it may be) would have to be zero.

❑ SOLVING EQUATIONS THAT ARE ALREADY FACTORED

EXAMPLE 1: **Solve for n: $3n(2n - 13) = 0$**

Solution: We have something ($3n$) times something else ($2n - 13$) whose product is 0. We saw above that if something times something is 0, then either the first something is 0 or the second something is 0.

If $ab = 0$, then $a = 0$ or $b = 0$.

So we set each factor to 0 and get our two solutions for n:

Setting the first factor, $3n$, to 0:

$$3n = 0$$
$$\Rightarrow \quad n = 0$$

Setting the second factor, $2n - 13$, to 0:

$$2n - 13 = 0$$
$$\Rightarrow \quad 2n = 13$$
$$\Rightarrow \quad n = \frac{13}{2}$$

Let's check each solution:

$\underline{n = 0}$:

$$3(\mathbf{0})(2(\mathbf{0}) - 13) = 3(0)(-13) = 0 \quad \checkmark$$

$\underline{n = \frac{13}{2}}$:

$$3\left(\frac{\mathbf{13}}{\mathbf{2}}\right)\left(2\left(\frac{\mathbf{13}}{\mathbf{2}}\right) - 13\right) = 3\left(\frac{\mathbf{13}}{\mathbf{2}}\right)(13 - 13) = 3\left(\frac{\mathbf{13}}{\mathbf{2}}\right)(0) = 0 \quad \checkmark$$

Thus, there are two solutions for n: $\boxed{0, \frac{13}{2}}$

Homework

1. Solve each equation by setting each factor to 0:

a. $x(x - 3) = 0$

b. $a(a + 4) = 0$

c. $y(2y - 7) = 0$

d. $2c(c - 1) = 0$

e. $3d(d + 12) = 0$

f. $4w(2w + 1) = 0$

g. $5u(3u - 5) = 0$ h. $7z(z + 99) = 0$

i. $13n(13n + 12) = 0$ j. $b(7b + 1) = 0$

k. $-5w(9w - 2) = 0$ l. $-12a(13a - 13) = 0$

❑ *SOLVING EQUATIONS THAT ARE NOT ALREADY FACTORED*

In the Introduction and the homework we solved the equations using the following reasoning: Since the product of two factors was given to be 0, either of the factors could be 0. We then set each factor to 0 and solved the two resulting linear equations.

But what do we do if the expression on the left side of the equation is not in factored form? Simple — we factor it ourselves and then proceed as above. Also, if the right side of the equation is not 0, we can remove whatever's over there in order to force a 0 on the right side.

We now have a formal process for solving an equation using the **factoring method**:

1) Write the equation so that there's a 0 on one side.
2) Factor the expression.
3) Set each factor to zero.
4) Solve each linear equation.

EXAMPLE 2: **Solve for x: $6x^2 + 7x = 0$**

Solution: We first notice that one side of the equation is 0, which is necessary to use the factoring method.

$6x^2 + 7x = 0$ (the original equation)

$\Rightarrow \quad x(6x + 7) = 0$ (factor out the x)

$\Rightarrow \quad \underline{x = 0} \quad \text{or} \quad 6x + 7 = 0$ (set each factor to 0)

$6x = -7$ (solve each equation)

$\underline{x = -\frac{7}{6}}$

The two solutions are therefore $\boxed{0, -\frac{7}{6}}$

EXAMPLE 3: **Solve for y: $9y^2 = 6y$**

Solution: In this problem we don't have the 0 we need on one side of the equation. But we can make the right side into a 0 by subtracting $6y$ from each side of the equation and then proceeding to factor and set the factors to 0.

We start with the original equation: $9y^2 = 6y$

1) Make the right side 0 by subtracting $6y$ from each side of the equation: $9y^2 - 6y = 0$

2) Factor out the GCF ($3y$): $3y(3y - 2) = 0$

3) Set each factor to 0: $3y = 0$ or $3y - 2 = 0$

4) Solve each linear equation: $\underline{y = 0}$ or $3y = 2$

$\underline{y = \frac{2}{3}}$

We thus get two solutions to our equation: $\boxed{0, \frac{2}{3}}$

Homework

2. Solve each quadratic equation by factoring:

a. $3n^2 + 7n = 0$	b. $2x^2 - 8x = 0$	c. $3y^2 = y$
d. $7t^2 = 6t$	e. $5u^2 = 10u$	f. $7a^2 = -2a$
g. $7w^2 - 84w = 0$	h. $n^2 + 83n = 0$	i. $7x^2 = 7x$
j. $-5y^2 = 2y$	k. $13a^2 = -a$	l. $w^2 = 19w$
m. $13g^2 + 12g = 0$	n. $8n - 3n^2 = 0$	o. $23Q^2 = 19Q$

3. Solve each quadratric equation by factoring:

a. $3n^2 - 7n = 0$	b. $2x^2 + 10x = 0$	c. $7y^2 = y$
d. $7t^2 = -9t$	e. $10w^2 = 20w$	f. $17a^2 = -2a$
g. $7c^2 - 77c = 0$	h. $m^2 + 93m = 0$	i. $9x^2 = 9x$
j. $-7y^2 = 8y$	k. $14k^2 = -k$	l. $z^2 = 21z$
m. $13h^2 + 11h = 0$	n. $7n - 4n^2 = 0$	o. $29R^2 = 17R$

❑ *SOLVING MORE FORMULAS*

Do you remember how we solved for ***x*** in the formula (literal equation)

$$wx + y = A \ ?$$

We subtracted ***y*** from each side of the equation:

$$wx = A - y$$

and then we divided each side of the equation by ***w***:

$$x = \frac{A - y}{w}$$ and we've isolated the x — done.

This is all fine and dandy when the unknown occurs only once in a formula. But how do we isolate something that occurs more than once in a formula? For example, how do we solve for ***x*** in the formula

$$ax - c = bx\,?$$

There's an x-term on both sides of the equation. This x is going to be tough to isolate. What would you do if, instead of the symbols a, b, and c in the equation, they had been numbers?

For example, suppose the equation had been

$$7x - 10 = 4x$$

We subtract $4x$ from each side:	$7x - 4x - 10 = 0$
Then combine like terms:	$3x - 10 = 0$
Now add 10 to each side:	$3x = 10$
And last, divide each side by 3:	$x = \frac{10}{3}$

We follow a similar procedure for solving the formula $ax - c = bx$ for x.

The original formula:	$ax - c = bx$
Subtract bx from each side:	$ax - bx - c = 0$
Add c to each side:	$ax - bx = c$
How do we "combine the like terms" ax and $-bx$? Here's where factoring comes to the rescue – by factoring x out of $ax - bx$, we get $x(a - b)$, which contains only one x, not two:	**Factoring out the GCF** ↓ $x(a - b) = c$
Divide each side by $a - b$, and we're done:	$\boldsymbol{x = \frac{c}{a - b}}$

Note: There are no x's on the right side of the answer. Can you explain why this fact is so utterly important?

EXAMPLE 4: **Solve for n: $Qn - n + P = R$**

Solution:	$Qn - n + P = R$	(the original formula)
$\Rightarrow$	$Qn - n = R - P$	(subtract P from each side)
$\Rightarrow$	$n(Q - 1) = R - P$	**(factor out the n)**
$\Rightarrow$	$\frac{n(Q-1)}{Q-1} = \frac{R-P}{Q-1}$	(divide each side by $Q - 1$)
$\Rightarrow$	$\boxed{n = \frac{R-P}{Q-1}}$	(simplify)

EXAMPLE 5: **Solve for a: $c(a - d) + 3 = 5(e - a)$**

Solution:	$c(a - d) + 3 = 5(e - a)$	(the original formula)
$\Rightarrow$	$ac - cd + 3 = 5e - 5a$	(distribute)
$\Rightarrow$	$ac + 5a - cd + 3 = 5e$	(add $5a$ to each side)
$\Rightarrow$	$ac + 5a - cd = 5e - 3$	(subtract 3 from each side)
$\Rightarrow$	$ac + 5a = 5e - 3 + cd$	(add cd to each side)

Note: These steps were designed to get the variable a on one side of the equation, and the rest of the things on the other side.

$\Rightarrow$	$a(c + 5) = 5e - 3 + cd$	(**factor out the a**)
$\Rightarrow$	$\frac{a(c+5)}{\boldsymbol{c+5}} = \frac{5e-3+cd}{\boldsymbol{c+5}}$	(divide each side by $c + 5$)
$\Rightarrow$	$\boxed{a = \frac{5e-3+cd}{c+5}}$	(simplify)

Be sure you understand thoroughly why there must be no a's on the right side of the answer.

Homework

4. Solve each formula for n:

a. $cn + dn = 3$

b. $an - cn = d$

c. $Ln + n = M$

d. $tn = c - sn$

e. $rn = 3 + tn$

f. $m(n + 1) - Qn - R = 0$

g. $a(n + 3) + b(n + c) = R$

h. $an + bn + cn = d$

i. $an - n - a = 0$

j. $2(n + 1) + an = c$

5. Solve each formula for x:

a. $cx + 7x = 14$

b. $rx - ux = w + v$

c. $wx - x = w$

d. $ax - b = c - dx$

e. $u(x - a) + x = w$

f. $a(x + 1) + b(x - 1) = 0$

g. $mx - 3(x - w) = z + u$

h. $p(x - 3) = q(x + 2)$

i. $c(x + a) - a(x - 1) = a - b$

j. $a(b - x) + c(2 - x) = R - Q$

Review Problems

6. a. Solve for x: $-17x(34x + 14) = 0$

b. Solve for n: $-23n^2 + 46n = 0$

c. Solve for y: $6y^2 = 8y$

d. Solve for x: $ax = bx + c$

e. Solve for x: $a(x-1)+x = b$

f. Solve for x: $b(x-c)+d(n-x) = T$

g. Solve for x: $gx-a = hx-b$

h. Solve for x: $c(x-d)+b = e(x+n)-w$

7. a. Solve for n: $15n(12n-16) = 0$

b. Solve for n: $23n^2-69n = 0$

c. Solve for z: $6z^2 = 14z$

d. Solve for x: $cx = ax-b+c$

e. Solve for x: $b(x+1)+x = w$

f. Solve for x: $a(x+c)+d(n+x) = z$

g. Solve for u: $eu+a = hu+b$

h. Solve for x: $c(x+ad)-b = e(x-m)+u$

Solutions

1. a. $0, 3$ b. $0, -4$ c. $0, \frac{7}{2}$ d. $0, 1$

e. $0, -12$ f. $0, -\frac{1}{2}$ g. $0, \frac{5}{3}$ h. $0, -99$

i. $0, -\frac{12}{13}$ j. $0, -\frac{1}{7}$ k. $0, \frac{2}{9}$ l. $0, 1$

2. a. $0, -\frac{7}{3}$ b. $0, 4$ c. $0, \frac{1}{3}$ d. $0, \frac{6}{7}$

e. 0, 2 f. $0, -\frac{2}{7}$ g. 0, 12 h. 0, –83

i. 0, 1 j. $0, -\frac{2}{5}$ k. $0, -\frac{1}{13}$ l. 0, 19

m. $0, -\frac{12}{13}$ n. $0, \frac{8}{3}$ o. $0, \frac{19}{23}$

3. a. $0, \frac{7}{3}$ b. 0, –5 c. $0, \frac{1}{7}$ d. $0, -\frac{9}{7}$

e. 0, 2 f. $0, -\frac{2}{17}$ g. 0, 11 h. 0, –93

i. 0, 1 j. $0, -\frac{8}{7}$ k. $0, -\frac{1}{14}$ l. 0, 21

m. $0, -\frac{11}{13}$ n. $0, \frac{7}{4}$ o. $0, \frac{17}{29}$

4. a. $cn + dn = 3 \Rightarrow n(c+d) = 3 \Rightarrow \frac{n\cancel{(c+d)}}{\cancel{c+d}} = \frac{3}{c+d} \Rightarrow n = \frac{3}{c+d}$

b. $n = \frac{d}{a-c}$

c. $n = \frac{M}{L+1}$

d. $tn = c - sn \Rightarrow tn + sn = c \Rightarrow n(t+s) = c \Rightarrow n = \frac{c}{t+s}$

e. $n = \frac{3}{r-t}$

f. $m(n+1) - Qn - R = 0 \Rightarrow mn + m - Qn - R = 0$
$\Rightarrow mn - Qn + m - R = 0 \Rightarrow mn - Qn = R - m$
$\Rightarrow n(m-Q) = R - m \Rightarrow n = \frac{R-m}{m-Q}$

g. $a(n+3) + b(n+c) = R \Rightarrow an + 3a + bn + bc = R$
$\Rightarrow an + bn = R - 3a - bc \Rightarrow n(a+b) = R - 3a - bc$
$\Rightarrow n = \frac{R-3a-bc}{a+b}$

h. $n(a+b+c) = d \Rightarrow n = \frac{d}{a+b+c}$

i. $n(a-1) = a \Rightarrow n = \frac{a}{a-1}$

j. $2n+2+an = c \Rightarrow n(2+a) = c-2 \Rightarrow n = \frac{c-2}{a+2}$

5. a. $x = \frac{14}{c+7}$ b. $x = \frac{w+v}{r-u}$ c. $x = \frac{w}{w-1}$

d. $x = \frac{c+b}{a+d}$ e. $x = \frac{w+au}{u+1}$ f. $x = \frac{b-a}{a+b}$

g. $x = \frac{z+u-3w}{m-3}$ h. $x = \frac{2q+3p}{p-q}$

i. $x = \frac{-b-ac}{c-a}$, or, upon multiplying top and bottom by –1, $\frac{b+ac}{a-c}$

j. $x = \frac{R-Q-2c-ab}{-a-c}$, or, upon multiplying top and bottom by –1, $\frac{Q+2c+ab-R}{a+c}$

6. a. $0, -\frac{7}{17}$ b. 0, 2 c. $0, \frac{4}{3}$ d. $\frac{c}{a-b}$

e. $\frac{b+a}{a+1}$ f. $\frac{T-dn+bc}{b-d}$ g. $\frac{a-b}{g-h}$ h. $\frac{en-w-b+cd}{c-e}$

7. a. $0, \frac{4}{3}$ b. 0, 3 c. $0, \frac{7}{3}$ d. $\frac{-b+c}{c-a}$

e. $\frac{w-b}{b+1}$ f. $\frac{z-ac-dn}{a+d}$ g. $\frac{b-a}{e-h}$

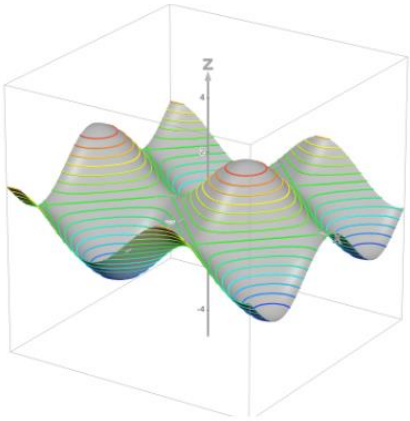
h. $\frac{-em+u-acd+b}{c-e}$

"Use what talents you possess. The woods would be very silent if no birds sang there except those that sang the best."



– Henry Van Dyke

EQUATIONS AND INEQUALITIES VIA DESMOS



Some equations, like $3x - 4 = 7x + 9$, are not too hard to solve. Others can be very difficult, if not impossible, to solve using algebra. For example, I have no idea how to solve the simple-looking equation

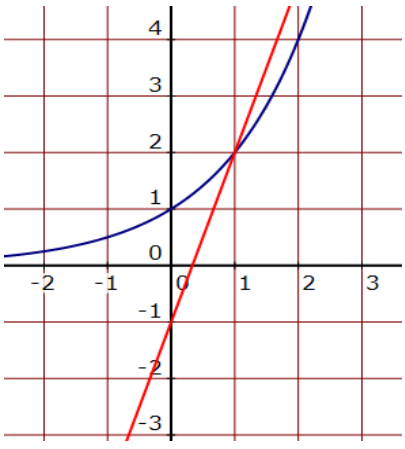
$$2^x = 3x - 1$$

using algebra. But we can find good (if not perfect) solutions using *GRAPHING*, as you'll see in the next example.

❑ *OBTAINING SOLUTIONS FROM GRAPHS*

EXAMPLE 1: **Solve the equation $2^x = 3x - 1$ graphically.**

Solution: I'm going to give you the graph of each side of the equation on the same grid:



From your knowledge of graphing lines, you should see that the red straight line is the graph of $y = 3x - 1$, the right side of the equation. And so the curvy blue graph is the graph of $y = 2^x$, the left side of the equation.

Now, the equation $2^x = 3x - 1$ is a statement of equality — we want to know what values of x will make each side of the equation result in the same number. To do that, we find any ***points of intersection*** of the two graphs. Looking at the graph, a point of intersection seems to be (1, 2). That means that when $x = 1$, both graphs have a y-value of 2, which also implies that when $x = 1$, both sides of the equation are equal (they're both equal to 2). In short, the solution of the equation $2^x = 3x - 1$ (as far as the graph is concerned) is

$$\boxed{x = 1}$$

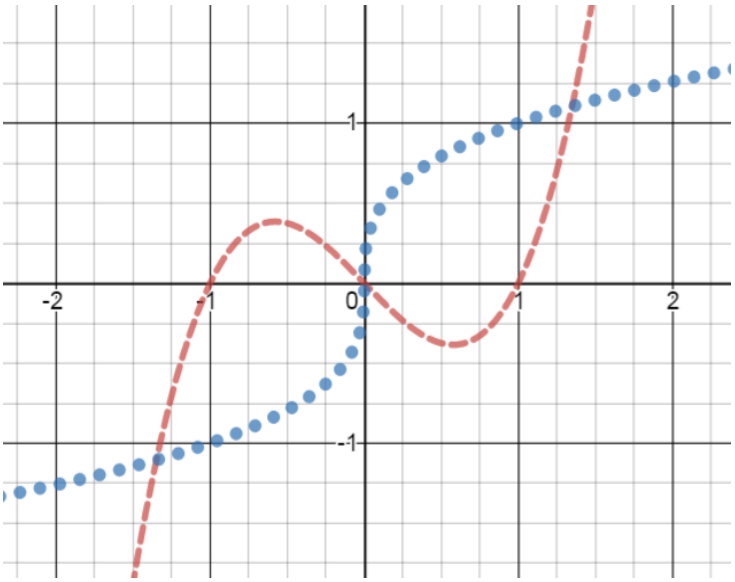
Two Caveats (things to watch out for):

1. How do we know that there's only one solution? Perhaps the two graphs cross somewhere higher up in Quadrant I, or maybe down in Quadrant III — using the graph we have, we can never know. [In fact, by simple substitution you can show that $x = 3$ is another solution of the equation. But are there others?]

2. It may appear that the point of intersection is (1, 2), but a graph is just a rough picture with limited resolution — no graph can be perfectly accurate. So maybe the true value of x at the point of intersection is 0.999997, or maybe 1.0000203. If it IS one of those, the graph will NEVER tell us that. So the graphing method is not an exact method, but if the equation can't be solved using algebra, a graph may be the best we can get — and may be quite useful for applications both inside and outside of mathematics.

EXAMPLE 2: **Solve the equation $x^3 - x = \sqrt[3]{x}$ graphically.**

Solution: For this problem, I'm not going to even bother telling you which graph goes with which side of the equation. Just know that each side of the equation has been graphed. Your job is to pick out any points of intersection (which are, remember, just approximations). The x-values of those points of intersection will be the solutions of the equation.



First, do you see three points of intersection? There's one in Quadrant I, there's one in Quadrant III, and it also appears that the origin, (0, 0), is a point of intersection. Second comes the guessing part: The x-values of the three points of intersection are approximately –1.3, 0, and 1.3. So these are our best shots at the solutions of the equation:

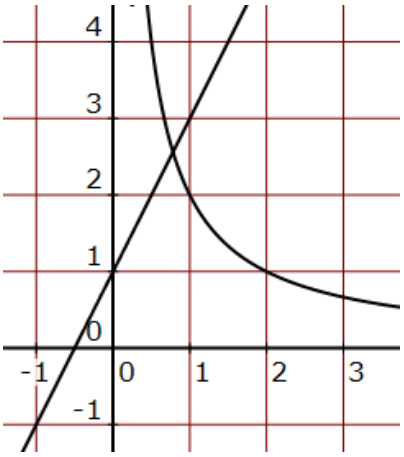
$x = -1.3$ $\quad x = 0$ $\quad x = 1.3$

For the rest of this chapter (after the following homework section), you graph the given equation yourself using DESMOS before you find the point(s) of intersection.

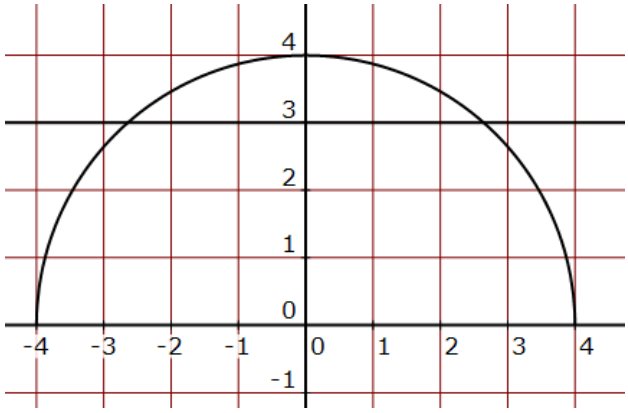
Homework

Solve each equation by using the associated graphs:

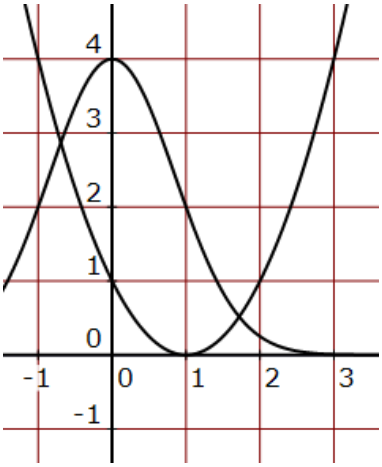
1. $\frac{1}{x} = 2x - 1$



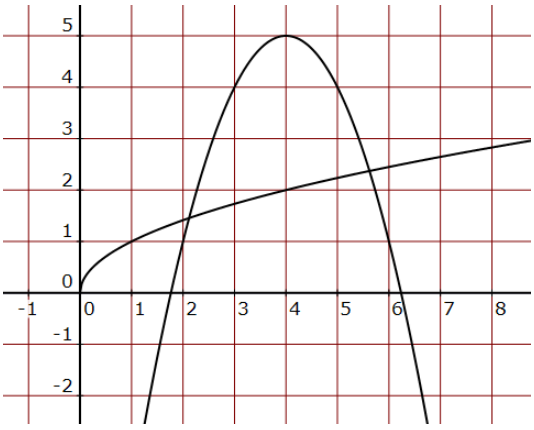
2. $\sqrt{16 - x^2} = 3$



3. $4^{-x^2} = (x-2)^2$



4. $-(x-4)^2 + 5 = \sqrt{x}$



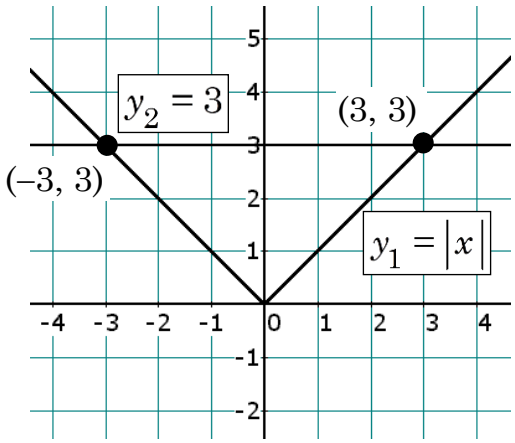
❑ *Absolute Value Equations*

EXAMPLE 3: **Solve the equation $|x| = 3$ graphically.**

Solution: We graph each side of the equation separately and look at any points of intersection we might find. To help us keep track, we'll call the left side of the equation y_1 and the right side y_2:

$y_1 = |x|$ $\quad y_2 = 3$

Now use Desmos to graph each equation on the same grid. It appears we have two points of intersection: (–3, 3) and (3, 3). Thus, if $x = -3$ or $x = 3$, the two sides of the equation balance, meaning we have two solutions to our equation:



$\boxed{x = 3 \text{ or } x = -3}$ [We could also write $x = \pm 3$]

Check:

$x = 3$:

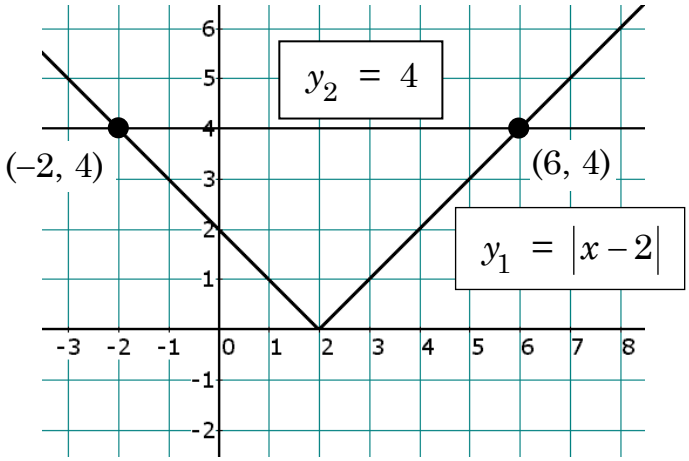
$\lvert x \rvert$	3
$\lvert 3 \rvert$	
3	✓

$x = -3$:

$\lvert x \rvert$	3
$\lvert -3 \rvert$	
3	✓

EXAMPLE 4: **Solve the equation $|x-2| = 4$ graphically.**

Solution: Let $y_1 = |x-2|$ and $y_2 = 4$. Graphing y_1 and y_2 on the same grid gives the following graphs:



It looks like y_1 and y_2 intersect at the points (–2, 4) and (6, 4). Therefore, the two solutions of the equation $|x-2| = 4$ are

$$x = -2 \text{ or } x = 6$$

Check:

$x = -2$:

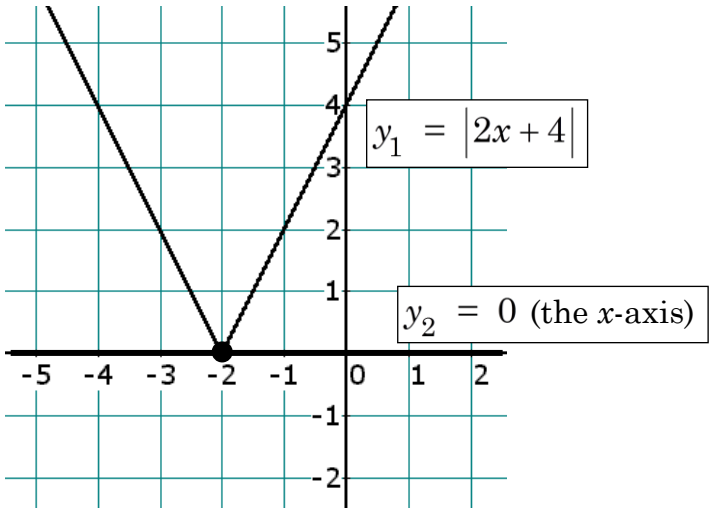
$$\begin{array}{l|l} |x-2| & 4 \\ |-2-2| & \\ |-4| & \\ 4 & \checkmark \end{array}$$

$x = 6$:

$$\begin{array}{l|l} |x-2| & 4 \\ |6-2| & \\ |4| & \\ 4 & \checkmark \end{array}$$

EXAMPLE 5: **Solve the equation $|2x+4| = 0$ graphically.**

Solution: Let $y_1 = |2x+4|$ and $y_2 = 0$. Noting that the graph of $y_2 = 0$ is just the x-axis, we graph y_1 and y_2 on the same grid:



The two graphs intersect at one point, (–2, 0). Thus, each formula has the same value (0) when $x = -2$. Thus, the only solution of the given equation is

$$\boxed{x = -2}$$

Check:

$\lvert 2x+4 \rvert$	0
$\lvert 2(-2)+4 \rvert$	
$\lvert -4+4 \rvert$	
$\lvert 0 \rvert$	
0	✓

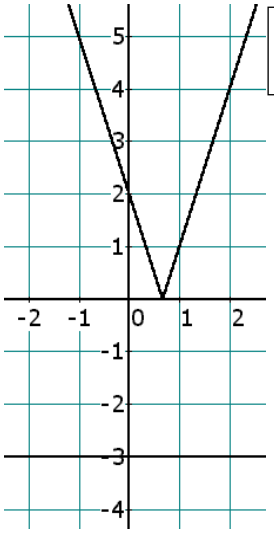
EXAMPLE 6: **Solve the equation $|3x-2| = -3$ graphically.**

Solution: Let's graph $y_1 = 3x-2$ and $y_2 = -3$:

Look at the two graphs. Notice that y_1 and y_2 have no point of intersection. This means that y_1 can never be equal to y_2.

What's our conclusion? The given equation has

No solution



$y_1 = |3x-2|$

$y_2 = -3$

Homework

5. Solve each absolute-value equation graphically:

a. $|x| = 2$

b. $|x| = -3$

c. $|x+1| = 3$

d. $|x-3| = 0$

e. $|2x-2| = 4$

f. $|2x+4| = -1$

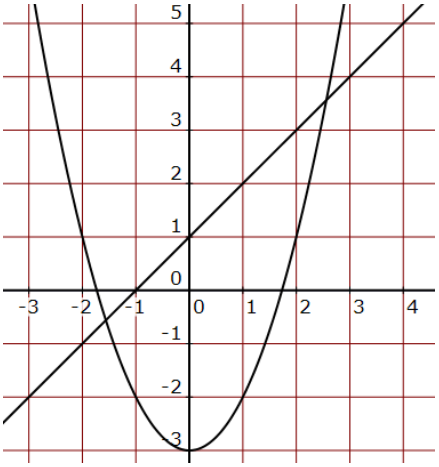
❑ *LINES AND CURVES*

EXAMPLE 7: **Solve the equation $x^2 - 3 = x + 1$ graphically.**

Solution: Let's graph $y_1 = x^2 - 3$ and $y_2 = x + 1$ on the same grid, and then check out if there are any points of intersection.

The formula $y_2 = x + 1$ is just a straight line, covered in the previous chapter. But the formula $y_1 = x^2 - 3$ might require a little more work. Let's make a table of x-y values:

x	y_1
−3	6
−2	1
−1	−2
0	−3
1	−2
2	1
3	6



Confirm these graphs with Desmos.

Now we try to determine where the line and the curve intersect. It appears (although there's no guarantee) that the points of intersection are **(–1.5, –0.5)** and **(2.5, 3.5)**. We can therefore estimate the two solutions of the original equation as

$$x = -1.5 \text{ or } x = 2.5$$

Note: The solutions we obtained in this problem are NOT the actual solutions — they are simply the best we can get from the picture. But note there are ways to solve the equation exactly.

Homework

Use DESMOS to approximate the solutions of each equation:

6. $x^2 - 2 = x + 4$

7. $x^2 - 1 = 2x - 2$

8. $x^2 + 2 = x - 1$

Use DESMOS to prove the following assertions:

9. The quadratic equation $2x^2 - 8x + 41 = 0$ has NO solution in $\mathbb{R}$.

10. The quadratic equation $25x^2 - 26x + 6.76 = 0$ has ONE solution in $\mathbb{R}$.

11. The quadratic equation $5x^2 + 97x - 168 = 0$ has TWO solutions in $\mathbb{R}$.

12. The equation $e^x = \ln x$ has NO solution in $\mathbb{R}$.

❑ *INEQUALITIES*

An inequality is quite different from an equation. Whereas an equation usually has one or a couple of solutions, inequalities tend to have infinitely many solutions. Here's an example:

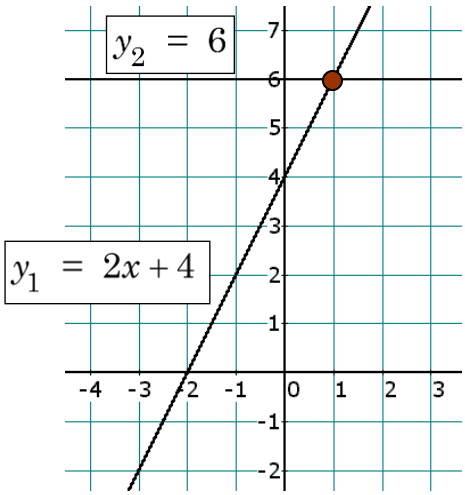
Solve for x: $5x - 3 > 12$

We're seeking any and all values of x which make the statement true. Suppose $x = 6$. Then $5(\mathbf{6}) - 3 = 30 - 3 = 27$, which is > 12. Thus $x = 6$ is a solution of the inequality; but it's "a" solution, not "the" solution. Why? Because there are others; $x = 5$ will work; $x = 3.7$ will work.

But will $x = 3$ work? Let's see: $5(\mathbf{3}) - 3 = 15 - 3 = 12$, which is not greater than 12. So $x = 3$ fails to satisfy the inequality. What about $x = 0$? $5(\mathbf{0}) - 3 = 0 - 3 = -3$, which is certainly not greater than 12. Let's summarize: 0 and 3 failed, but 3.7 and 5 and 6 worked. Our best guess at this point is that any number greater than 3 will work. Our solution is therefore $\boldsymbol{x > 3}$.

EXAMPLE 8: **Solve the inequality $2x + 4 < 6$ graphically.**

Solution: We employ the same technique we used for all the previous equations in this chapter, except we will take into account the "less than" sign after we construct our graphs. Let $y_1 = 2x + 4$ and $y_2 = 6$, both of which are lines.

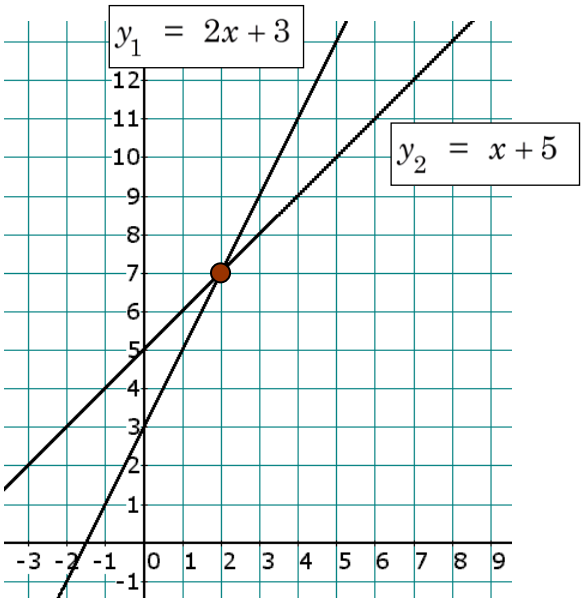


First we note that the two lines appear to intersect at the point (1, 6). But remember, we're trying to solve the inequality $2x + 4 < 6$, so here's the question: For what values of x is the quantity $2x + 4$ smaller than 6? Equivalently — and here's the key question — where on the graph is y_1 below y_2? By looking at the graphs of the lines, we see that y_1 is below y_2 to the left of the point of intersection, (1, 6). In other words, whenever x is smaller than 1, it will follow that $2x + 4$ is less than 6. That is, whenever $x < 1$, we can conclude that $2x + 4 < 6$. We have found the solution of the inequality $2x + 4 < 6$:

$\boxed{x < 1}$ which can be written $(-\infty, 1)$.

EXAMPLE 9: **Solve the inequality $2x + 3 \geq x + 5$ graphically.**

Solution: Graph $y_1 = 2x + 3$ and $y_2 = x + 5$ on the same grid.



We're looking for where y_1 is greater than or equal to y_2. Clearly, y_1 and y_2 are equal to each other at the point of intersection, (2, 7). Also, $y_1 > y_2$ wherever the graph of y_1 is above the graph of y_2. This occurs to the right of the point (2, 7); that is, $y_1 > y_2$ whenever $x > 2$. In short, $y_1 \geq y_2$ whenever $x \geq 2$. Thus, the solution of the inequality $2x + 3 \geq x + 5$ is

$\boxed{x \geq 2}$ and in interval notation: $[2, \infty)$

Homework

13. Solve each inequality graphically. When you get tired of graphing, use DESMOS:

a. $x + 3 < 6$

b. $x - 1 \geq 5$

c. $2x + 1 \leq 5$

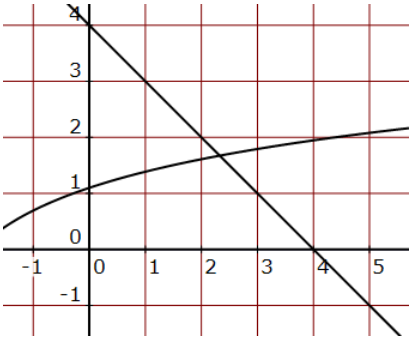
d. $x + 5 \geq 3x - 1$

e. $2x - 1 < x + 2$

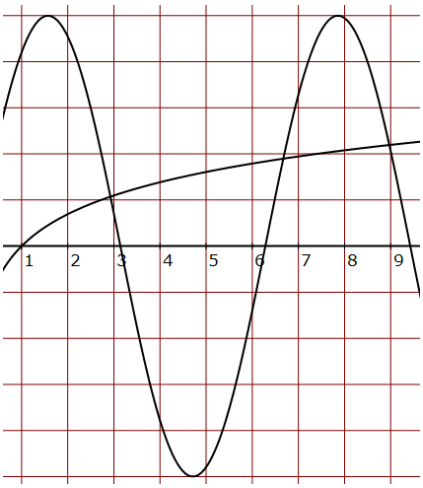
f. $3x + 1 \leq 2x - 3$

Review Problems

14. Solve the equation $\ln(x+3) = -x+4$ by using the associated graphs:



15. Solve the equation $5\sin x = \ln x$ by using the associated graphs:



16. Solve the equation $|2x-3| = 3$ graphically.

17. Solve the equation $|x+1|+3 = 3$ graphically.

18. Solve the equation $|x-1|+2 = 1$ graphically.

19. Solve the equation $x^2 - 4 = 2x - 3$ graphically.

20. Solve the inequality $2x - 2 \leq 4$ graphically.

21. Solve the inequality $2x + 3 > x - 2$ graphically.

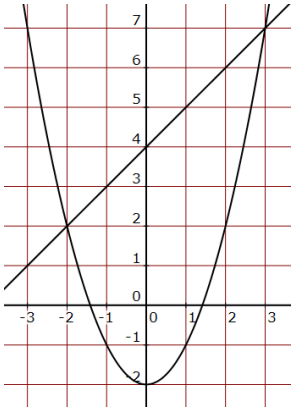
Solutions

Remember that these solutions are approximations; as long as your estimation is close to mine, you have the correct answer.

1. $x = 0.8$ **2**. $x = 2.6, -2.6$ **3**. $x = 1.7, -0.7$ **4**. $x = 2.1, 5.6$

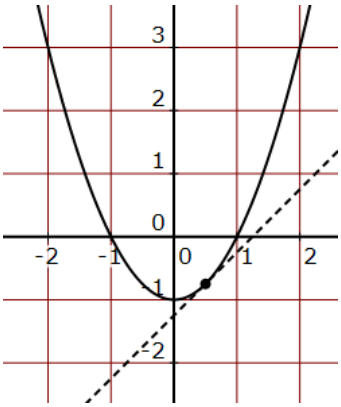
5. a. $x = 2$ or $x = -2$ b. No solution c. $x = 2$ or $x = -4$
d. $x = 3$ e. $x = 3$ or $x = -1$ f. No solution

6.



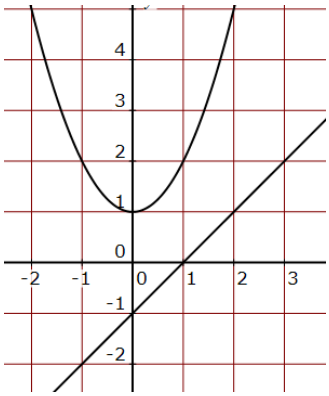
$x = -2, 3$

7.



$x = 0.5$

8.



No Solution

9. **10**. **11**. Graph each side of the equation, noting that $y_2 = 0$ is simply the x-axis. This means that the number of solutions is given by the number of points of intersection between the parabola on the left side of the equation and the x-axis on the right side.

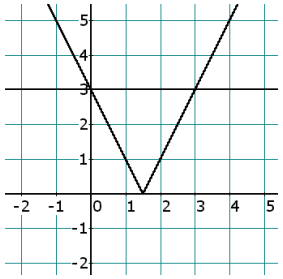
12. The graphs should prove to you that the equation has NO solution.

13. a. $x < 3$ b. $x \geq 6$ c. $x \leq 2$ d. $x \leq 3$
e. $x < 3$ f. $x \leq -4$

14. $x = 2.3$ (or anything reasonably close)

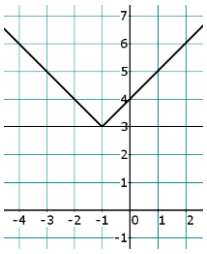
15. There are three points of intersection shown on the graph; the x-values are roughly 2.9, 6.7, and 9.

16.



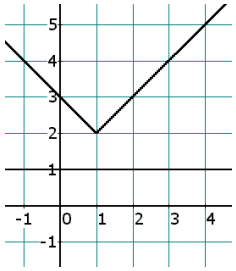
$x = 0, 3$

17.



$x = -1$

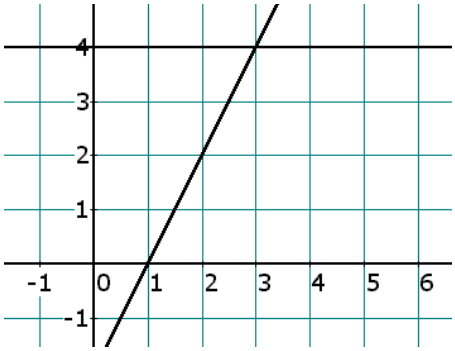
18.



No solution

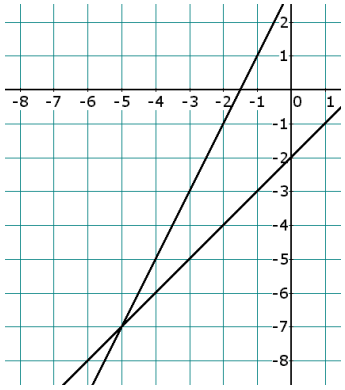
19. $x = -0.4$ or $x = 2.4$ (or anything reasonably close)

20.



$x \leq 3$

21.



$x > -5$

"The first step to getting the things you want out of life is this:

Decide what you want."

- Ben Stein

GRAPHING WITH BASE *e*

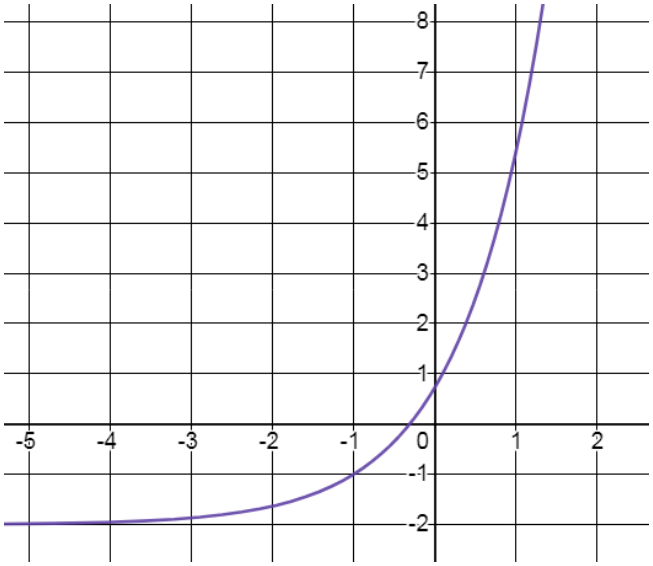
Since *e* is a positive real number not equal to 1 (it's greater than 1), it's allowed to be the base of an exponential function. In fact, the base-*e* exponential function, $y = e^x$, is one of the most important exponential functions in business, biology, chemistry, ecology, statistics, physics, and electronics. Let's graph one of these.



EXAMPLE 1: **Graph:** $y = e^{x+1} - 2$

Solution: Use your calculator to verify that each of the following ordered pairs is on the graph of the function:

(–4, –1.95) (–3, –1.86) (–2, –1.63) (–1, –1) (0, 0.72) (1, 5.39)



Exponential Growth

The **domain** of the function is $\mathbb{R}$. To determine the range, we need to verify what the graph appears to show, that the line $y = -2$ is a horizontal asymptote. Let x be a large negative number, for instance $x = -20$. Then the y-value is

$$y = e^{-20+1} - 2 = e^{-19} - 2 = .000000006 - 2 = \mathbf{-1.999999994}$$

which is extremely close to (and slightly above) –2. We can be pretty confident that the **horizontal asymptote** is $\boldsymbol{y = -2}$. From this fact, we can conclude that the **range** of the function is all real numbers greater than –2: **(–2, ∞)**.

For our last bit of analysis, we summarize the behavior of the function at extreme values of x by writing the limits

$$\lim_{x\to\infty}(e^{x+1} - 2) = \infty \qquad \text{and} \qquad \lim_{x\to-\infty}(e^{x+1} - 2) = \mathbf{-2}$$

The first of these two limits indicates that the graph is **increasing**; that is, as you move from left to right along the graph, the graph grows taller and taller (and never levels off). We can call this **exponential growth**. In our following example, we change one little (but critical) part of the equation (which part?) and we get a curve which represents **exponential decay**.

EXAMPLE 2: **Graph: $y = e^{-x} + 1$**

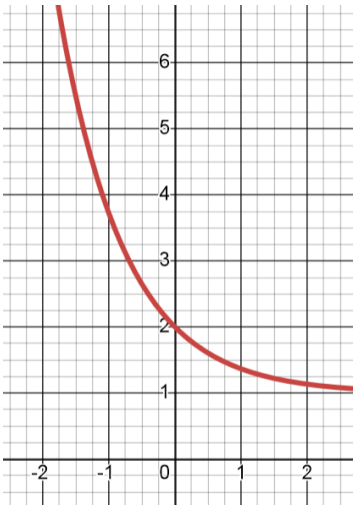
Solution: Let's go for the y-intercept first. Letting $x = 0$ produces $y = e^{-0} + 1 = e^{0} + 1 = 1 + 1 = 2$. The y-intercept is therefore **(0, 2)**. Use your calculator to verify some other points on the graph:

(–3, 21.09) (–2, 8.39) (–1, 3.72) (1, 1.37) (2, 1.14) (3, 1.05)

To emphasize the limits as x approaches ∞ or –∞, let's calculate two more points:

(–10, 22027.47) (10, 1.000045)

It appears that as $x \to -\infty$, $y \to \infty$. Also, as $x \to \infty$, $y \to 1$.



Exponential Decay

The domain of the function is $\mathbb{R}$, simply because any real number x can be used in the formula without a problem.

The range is all real numbers greater than 1: **(1, ∞)**

From the graph and the points we calculated, we see that there's a horizontal asymptote at $y = 1$.

In computer languages, the function $y = e^x$ cannot be written that way since computer code cannot contain an exponent. So instead we write

$$y = \exp(x).$$

Homework

Graph each of these exponential functions:

1. $y = e^x$
2. $f(x) = e^{-x}$
3. $g(x) = e^2 + 2$
4. $h(x) = e^{x-2}$
5. $f(x) = e^x - 1$
6. $y = e^{x-1} + 2$
7. $E(x) = e^{x+2} - 3$
8. $y = e^{-2x}$
9. $y = e^{-x} - 1$
10. $h(x) = e^{-x} + 2$
11. Describe the graph of $y = ex + e$.

❑ *THE BELL-SHAPED CURVE*

Some of you will take Statistics soon, and you will learn that the most important function in the course is a special bell-shaped curve called

the ***standard normal*** curve (or Gaussian Curve). It's an exponential function, and its formula is a bit complicated, but we have all the tools needed to find some of its points and then sketch it.

The formula for the standard normal curve is given by

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

Note that this function is a type of ***exponential function*** because the base is a constant (in this case, e) and the variable x occurs in the exponent. Let's decipher this formula before we try to plot any points: The coefficient of the exponential function is $\frac{1}{\sqrt{2\pi}}$, the base is e, and the exponent on the e is $-\frac{1}{2}x^2$.

Let's calculate some (x, y) pairs for this function and see what kind of graph we get. Let's start by letting $x = 0$. This, of course, will yield the y-intercept:

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(0)^2} = \frac{1}{\sqrt{2\pi}} e^{0} = \frac{1}{\sqrt{2\pi}}(1) \approx 0.3989$$

This gives us the approximate point **(0, 0.4)**. We're off to a good start.

Now we'll choose $x = 1$. Plugging this number into the function gives

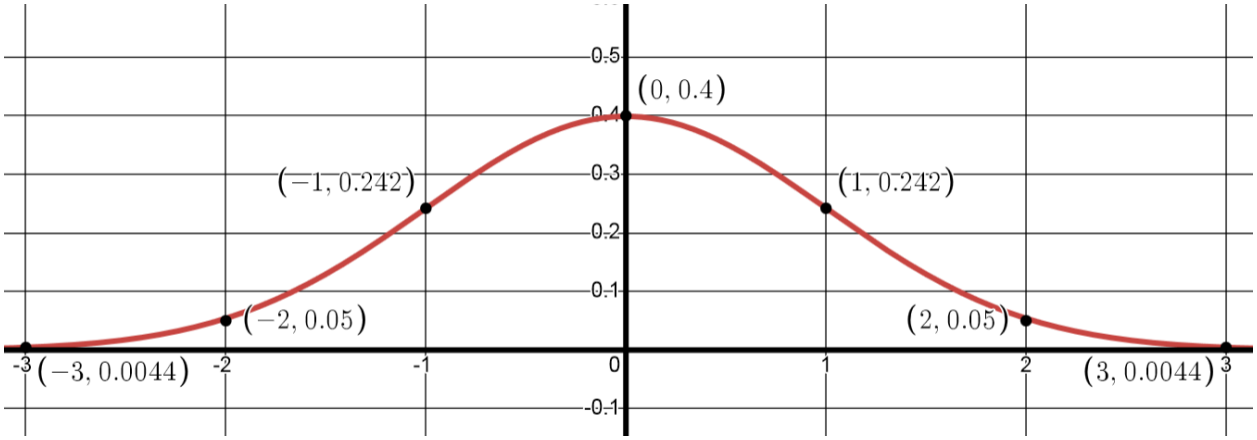
$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(1)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}} = \frac{1}{\sqrt{2\pi}}(0.6065) \approx 0.2420$$

Our second point is therefore **(1, 0.24)**. Now for $x = -1$. You should be able to do this one in your head. First, the –1 is squared, giving 1 — but this is exactly what we had in the previous calculation, and so the y-value must also be 0.2420. Our third point is therefore **(–1, 0.24)**.

Choosing $x = 2$ (or –2) gives the following calculation for y:

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\pm 2)^2} = \frac{1}{\sqrt{2\pi}} e^{-2} = \frac{1}{\sqrt{2\pi}} (0.1353) \approx 0.0540$$

We now have two more points for our graph: **(2, 0.05)** and **(–2, 0.05)**. Continuing in this manner, we can find the points **(3, 0.0044)** and **(–3, 0.0044)**. It appears that as we get farther from the origin, the *y*-values are getting smaller and smaller, but always remaining



positive. If we plot the seven points we've just computed and then connect them with a smooth curve, we get the following:

What can we infer from this graph? Lots of stuff:

A. The **domain** is $\mathbb{R}$, all real numbers. [The graph goes infinitely to the right and infinitely to the left.] This means that x can take on any value at all and the formula will work out just fine.

B. The **range** is all numbers that are bigger than 0 but less than or equal to *approximately* 0.4. In interval notation: **(0, 0.4]**.

C. If we fold the portion of the graph in Quadrant I over into Quadrant II, the curves will coincide. We therefore have ***y*-axis symmetry**. [You may have learned to change x to $-x$ and see if you get the same formula; you do.]

D. The calculations and the graph indicate these two limits:

$$\lim_{x \to \pm\infty} y = \mathbf{0}$$

E. And from the limits we deduce that the line $\boldsymbol{y = 0}$ (the x-axis) is a **horizontal asymptote** of the graph.

F. One reason for some of the numbers in the function is that we want the total **area** between the curve and the x-axis to be exactly **1**.

Review Problems

12. a. List the domain, range, intercepts, and asymptotes of $y = e^x$.

b. $\lim_{x \to \infty} e^x$ c. $\lim_{x \to -\infty} e^x$ d. $\lim_{x \to 0} e^x$

13. Let $f(x) = e^x$ and $g(x) = e^{x+\pi} + 100$. Describe the graph of g relative to the graph of f.

14. Graph: $y = e^{x-2} - 3$. Calculate the y-intercept and any asymptotes.

15. Graph: $y = e^{-x} - 3$. Calculate the y-intercept and any asymptotes.

16. Graph: $N(x) = e^{-x^2}$. Hint: If $x = 1$, then $e^{-1^2} = e^{-1} = \frac{1}{e} \approx 0.3679$. Prove that the graph has y-axis symmetry.

17. Explain why the graph of $y = x^2 + ex + \pi$ is a parabola, and then prove that it has no x-intercepts.

18. True/False:

a. $\lim_{x\to\infty} f(x) = L$ means the same thing as "As $x \to \infty$, $f(x) \to L$."

b. If $P(x) = \dfrac{6}{1+x^2}$, then $\lim_{x\to 0} P(x) = 6$.

c. The function $y = e^x$ is a polynomial.

d. The domain of the above function is $\mathbb{R}$.

e. The range of the above function is $(0, \infty)$.

f. Compared to the graph of $y = e^x$, the graph of $y = e^x + 2$ is two units higher.

g. The function $y = e^{-x}$ is decreasing.

h. The function $y = e^{-x}$ has a horizontal asymptote.

i. The function $y = e^{-x}$ has a vertical asymptote.

j. The slope of the line $y = \pi x + e$ is e.

k. The y-intercept of the graph of

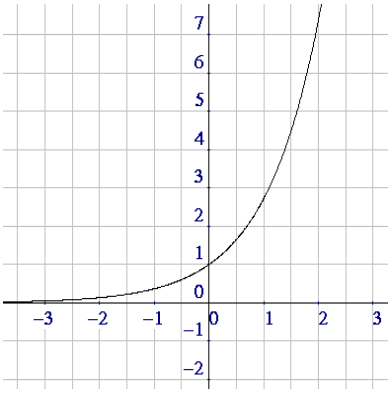
$$h(x) = e^{1-x} + 2$$

is $(0, e + 2)$.

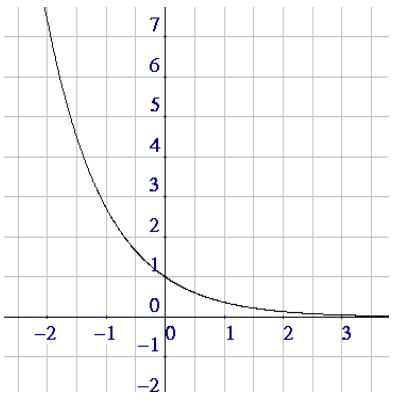
l. $e > \pi$

Solutions

1.



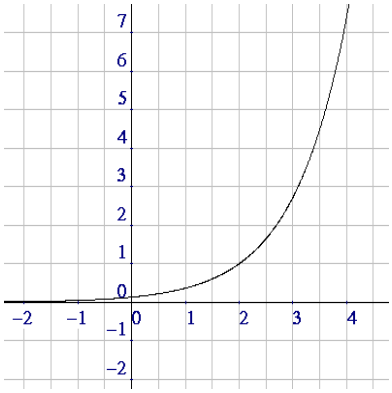
2.



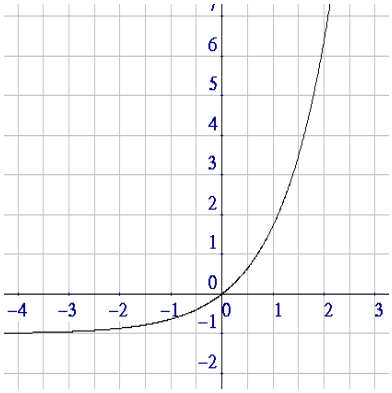
Graphing with Base e

3. I'd like to see what you think the graph is.

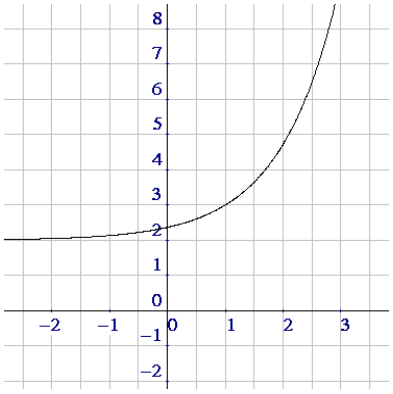
4.



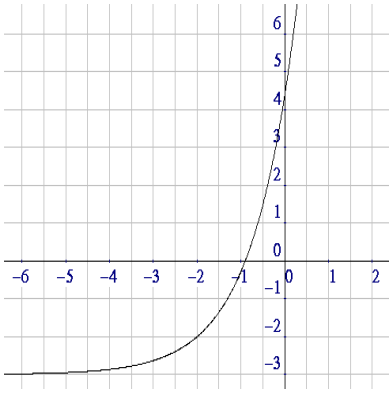
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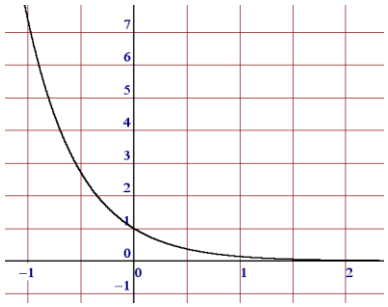
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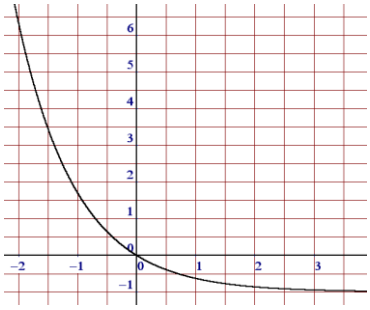
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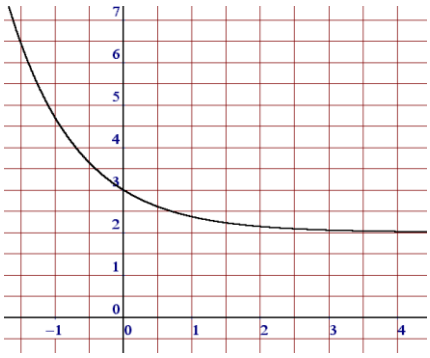
8.



9.



10.



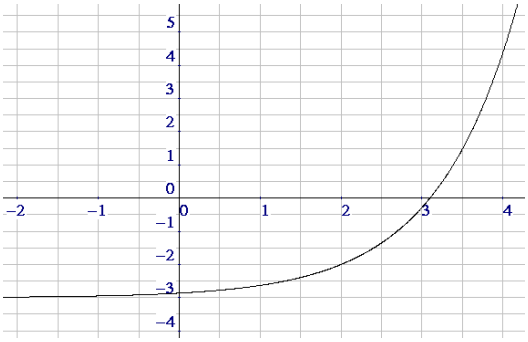
11. The graph of $y = ex + e$ is a straight line with slope e and having a y-intercept of $(0, e)$.

12. a. Domain = $\mathbb{R}$; Range = $(0, \infty)$;
x-int: none; y-int: $(0, 1)$;
vert asy: none; horiz asy: $y = 0$

b. ∞ c. 0 d. 1

13. The graph of g is the graph of f shifted π units to the left and 100 units up.

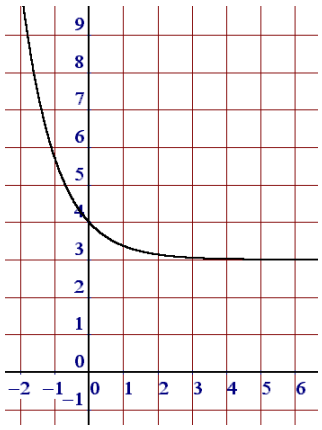
14.



y-int: $(0, e^{-2} - 3) \approx (0, -2.86)$

horiz asy: $y = -3$

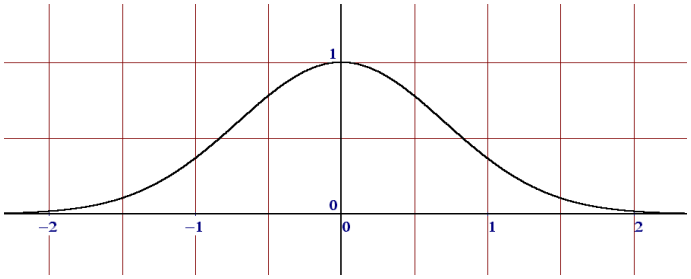
15.



y-int: (0, 4)

horiz asy: $y = 3$

16.



To prove *y*-axis symmetry, we change each *x* in the formula to –*x* and see what we get: $e^{-(-x)^2} = e^{-x^2}$, the same formula. We're done.

17. It's a parabola because it fits the form $y = ax^2 + bx + c$; in fact, $a = 1$, $b = e$, and $c = \pi$.
To find *x*-intercepts, set *y* to 0; solve the resulting quadratic equation for *x* using the Quadratic Formula. You should get a negative radicand, indicating no solutions in $\mathbb{R}$, and thus no *x*-intercepts.

18. a. T b. T c. F d. T e. T f. T g. T h. T
i. F j. F k. T l. F

"The direction in which education starts a man will determine his future life." Plato

THE NUMBER *e*



Recall the special number π. It is the ratio of the circumference of any circle to its diameter, and we learned that it's irrational (an infinite, non-repeating decimal). There's another extremely important irrational number and it's the topic of this chapter. It will be defined in terms of a limit, and will be the basis of applications in banking, biology, and radioactivity. In Calculus, it's absolutely the most important base for exponential functions.

❑ *NEW LIMIT NOTATION*

Let's review an example of limits. Consider the function

$$f(x) = \frac{1}{x}.$$

We know that as x grows larger and larger, $f(x)$ gets closer and closer to 0. We usually write:

$$\text{As } x \to \infty,\ f(x) \to 0\ .$$

In our new notation, we would write this fact as

$$\lim_{x \to \infty} f(x) = 0.$$

In general, instead of writing

$$\text{As } x \to a, f(x) \to L,$$

we write

$$\lim_{x \to a} f(x) = L.$$

These two statements mean exactly the same thing.

Homework

1. If $h(x) = x^2$, find $\lim_{x \to 3} h(x)$.

2. If $f(x) = \frac{2x+1}{x-3}$, find $\lim_{x \to \infty} f(x)$. Hint: Consider the graph of f.

3. If $E(x) = \frac{9}{1+x^2}$, find $\lim_{x \to 0} E(x)$.

❑ *EARNING INTEREST AT THE BANK*

In **simple interest** at the bank, the interest you earn on your investment is calculated just once at the end of the year. Suppose you place \$70 in the bank at 10% interest per year. At the end of the year you will have earned \$7 in interest, making your current balance \$77.

Now let's consider **compound interest**. As an example, we'll divide the year into four compounding periods (each one called a *quarter*), and we'll pretend we have \$1 invested at 100% annual interest. At the end of each quarter, you earn interest in the amount of 25% (100% divided into 4 equal parts) of your current balance. This means that from now on, your interest itself is earning interest.

Start: **\$1**

1st Quarter: previous balance + 25% of the previous balance

= \$1 + 25% of \$1

= \$1 + $\$\frac{1}{4}$ (**balance = \$1.25** after 1 quarter)

2nd Quarter: previous balance + 25% of the previous balance

= (\$1 + $\$\frac{1}{4}$) + 25% of (\$1 + $\$\frac{1}{4}$)

$= (\$1 + \$\frac{1}{4}) + \frac{1}{4}(\$1 + \$\frac{1}{4})$

$= (\$1 + \$\frac{1}{4})(\$1 + \$\frac{1}{4})$ (factor out $1 + \frac{1}{4}$)

$= (\$1 + \$\frac{1}{4})^2$ (**balance = $1.56** after 2 quarters)

3rd Quarter: previous balance + 25% of the previous balance

$= (\$1 + \$\frac{1}{4})^2$ + 25% of $(\$1 + \$\frac{1}{4})^2$

$= (\$1 + \$\frac{1}{4})^2 + \frac{1}{4}(\$1 + \$\frac{1}{4})^2$

$= (\$1 + \$\frac{1}{4})^2 (\$1 + \$\frac{1}{4})$ (factor out $(1 + \frac{1}{4})^2$)

$= (\$1 + \$\frac{1}{4})^3$ (**balance = $1.95** after 3 quarters)

4th Quarter: previous balance + 25% of the previous balance

$= (\$1 + \$\frac{1}{4})^3$ + 25% of $(\$1 + \$\frac{1}{4})^3$

$= (\$1 + \$\frac{1}{4})^3 + \frac{1}{4}(\$1 + \$\frac{1}{4})^3$

1 $= (\$1 + \$\frac{1}{4})^3 (\$1 + \$\frac{1}{4})$ (factor out $(1 + \frac{1}{4})^3$)

$= (\$1 + \$\frac{1}{4})^4$ (**balance = $2.44** after 1 full year)

Note: To compare simple interest with compound interest, $1.00 invested for one year at 100% simple interest would yield $1.00 in interest, for a final balance of **$2.00**. Using compound interest, we see that the last calculation in the 4th quarter above is

$(\$1 + \$\frac{1}{4})^4 = \$1.25^4 =$ **$2.44**

To summarize, investing $1 at 100% annual interest compounded four times a year yields a final balance, after one year in the bank, of

$(\$1 + \$\frac{1}{4})^4$

From this example we can generalize: Investing \$1 at 100% annual interest compounded $\boldsymbol{n}$ times a year yields a final balance of

$$\left(\$1+\$\frac{1}{\boldsymbol{n}}\right)^{\boldsymbol{n}}$$

at the end of one year. For example, investing \$1 at 100% annual interest compounded 360 times a year would give you a balance of

$$\left(1+\frac{1}{\mathbf{360}}\right)^{\mathbf{360}} = (1.002777778)^{360} = \mathbf{\$2.7145}$$

Summarizing,

Investing \$1 at 100% annual interest for one year, compounded *n* times a year, yields a final balance of

$$\left(1+\frac{1}{n}\right)^{n}$$

Homework

4. For each problem, use the formula $B = \left(1+\frac{1}{n}\right)^{n}$ to find the balance at the end of one year if the original investment was \$1, earning 100% annual interest, and assuming the given number of compounding periods:

   a. 1 b. 12 c. 365 d. 1,000 e. 5,000,000

❑ *THE MOST IMPORTANT EXPONENTIAL BASE OF ALL !*

Now suppose that we consider compounding so often that it occurs at every instant of time. This is called ***continuous compounding***. What will your account balance be after one year of continuous compounding? It would come from the formula

$$\left(1+\frac{1}{n}\right)^n$$

with n replaced by ∞: $\left(1+\frac{1}{\infty}\right)^{\infty}$. But this would be meaningless, so we use the new limit notation introduced a few pages back. We keep the n's in the formula, and then specify that n should approach infinity; that is, n becomes infinitely large:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Whatever every instant of time means, it drove the Greeks bonkers, and it wasn't until the 17th century that Newton and Leibniz made some sense of this when they invented Calculus.]

It's hard to see this now, but this limit is actually a specific real number. In fact, this number is one of the most important numbers in math, science, statistics, and business. It is given the name "***e***", perhaps due to the word "***e***xponential," or perhaps due to its creator Leonard ***E***uler.

Though ***e*** is defined as a limit, a decimal approximation of ***e*** will be discussed in the homework. In summary,

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Homework

5. Define e in two ways: as a limit, and as the result of a certain investment.

6. The number e is irrational, although I have no idea how to prove it. Nevertheless, explain exactly what it means for e to be irrational.

7. a. Use the definition of e with a value of n = 10,000,000 to approximate the value of e.

   b. Use the e^x button on your calculator to approximate e. Hint: $e = e^1$.

   c. Which approximation do you think is more accurate?

8. Use your calculator to approximate each of the following:

   a. $e^{3.7}$ b. $\frac{1}{e}$ c. $\sqrt[7]{e}$ d. $\frac{e^2 - \sqrt{e}}{\sqrt[3]{4e+1}}$

9. Some books define $\boldsymbol{e}$ as $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$. Prove that this is the same as our definition.

❑ THE LAWS OF EXPONENTS

All the laws for exponents with which we're familiar work for exponential expressions like 2^x and e^x just as well as they did for x^2. Here are the six main rules. Assume a and b are positive constants.

$$a^{-x} = \frac{1}{a^x} \qquad a^x a^y = a^{x+y} \qquad \frac{a^x}{a^y} = a^{x-y}$$

$$(a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x \qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

Homework

10. Simplify each expression:

a. $2^x 2^y$ b. e^0 c. e^{-5} d. $\frac{e^5}{e^2}$ e. $\frac{e^x}{e^y}$

f. $(e^x)^e$ g. $(10e)^x$ h. $\left(\frac{e}{2}\right)^n$ i. $(e^\pi)^{10x}$ j. $2^3 2^0$

Review Problems

11. By making a sketch of the function $y = \frac{2}{x}$, calculate:

a. $\lim_{x \to \infty} y$ b. $\lim_{x \to -\infty} y$

12. a. Define *e* as it applies to continuous compounding in the bank.
b. Define *e* as a limit.
c. Is *e* a real number?
d. Is *e* rational or irrational?
e. Explain exactly why *e* is a valid base for an exponential function.
f. Is $e > \pi$ or is $e < \pi$?

13. a. Simplify: $e^0 + \frac{e^7}{e^2} + (e^3)^4$ b. Approximate: $e^3 + \sqrt[3]{e}$

c. Approximate: $(\pi - e)^2$ d. Approximate: e^π.

Solutions

1. In this limit, x is approaching 3. This means that x is approaching 3 by taking values like 3.2, 3.1, 3.05, 3.001, 3.000001 – or perhaps x is taking values like 2.9, 2.95. 2.99, 2.999, etc. Now, as x does this, what does the functional value x^2 do? Well, it gets closer and closer to 9. Therefore, the limit is 9.

2. Since x is approaching infinity, this limit is equivalent to finding a horizontal asymptote. Take your calculator and evaluate the function for a really huge value of x. You should see that the limit is 2.

3. As x shrinks toward 0, the x^2 term in the denominator essentially disappears, leaving a functional value of 9, which is the answer to the limit question.

4. a. \$2.00 b. \$2.6130 c. \$2.7146 d. \$2.7169 e. \$2.7183

5. $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

If you invest \$1 at a 100% interest rate compounded continuously, your balance at the end of one year will be \$$e$.

6. e is an infinite, non-repeating decimal.

7. a. My TI-30X gives the answer 2.718288827, but your calculator might give 2.718281693, or something close.

b. 2.718281828 c. The second value

8. a. 40.4473 b. 0.3679 c. 1.1536 d. 2.5162

9. Since $1+\frac{1}{n} = \frac{n}{n}+\frac{1}{n} = \frac{n+1}{n}$, the definitions are equivalent.

10. a. 2^{x+y} b. 1 c. $\frac{1}{e^5}$ d. e^3 e. e^{x-y}

f. e^{ex} g. $10^x e^x$ h. $\frac{e^n}{2^n}$ i. $e^{10\pi x}$ j. 8

11. a. 0 b. 0

12. a. Invest \$1 for 1 year at an interest rate of 100%/yr compounded <u>continuously</u>. At the end of the year, the account balance is \$*e*.

b. $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ c. Yes d. Irrational

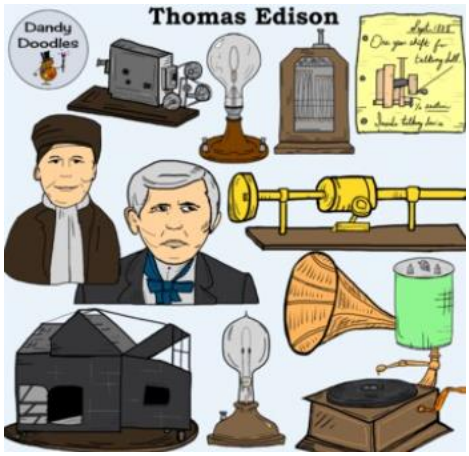
e. *e* is real number that is greater than 0 but not equal to 1.

f. $e < \pi$

13. a. $1 + e^5 + e^{12}$ b. 21.4811 c. 0.1792 d. 23.1407

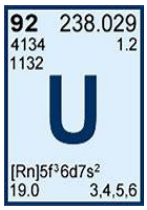
"Many of life's failures are people
who did not realize
how close they were to success
when they gave up."

– *Thomas Edison*



EXPONENTIAL EQUATIONS – NO LOGS

Watching your investments grow, tracking populations, the decaying of a radioactive substance — these are the kinds of problems that lead to exponential equations.



Uranium, atomic number 92, decays exponentially – the less that remains, the less that decays.

❑ *EXAMPLES*

An ***exponential equation*** is an equation with the variable in the exponent, something you might not be used to seeing. Let's start with two basic examples.

#1: Consider the exponential equation

$$5^x = 25$$

Five raised to what power equals 25? Well, 5 to the **2nd** power is 25, so $\boldsymbol{x = 2}$. Not so tough.

#2: How about the equation

$$3^{2n-1} = 81?$$

You might solve this equation by asking yourself, "3 to what power is 81?" Since 3 to the 4th power is 81, it follows that the exponent, $2n - 1$, must be equal to 4.

Writing this last phrase as an equation, we can find the value of n:

$$2n - 1 = 4 \Rightarrow 2n = 5 \Rightarrow \boldsymbol{n = \frac{5}{2}}. \text{ Done!}$$

Check: Placing $n = \frac{5}{2}$ into the equation $3^{2n-1} = 81$ gives

$$3^{2\left(\frac{5}{2}\right)-1} \stackrel{?}{=} 81$$

$$3^{\cancel{2}\left(\frac{5}{\cancel{2}}\right)-1} \stackrel{?}{=} 81$$

$$3^{5-1} \stackrel{?}{=} 81$$

$$3^4 \stackrel{?}{=} 81$$

$$81 = 81 \checkmark$$

EXAMPLE 1: **Solve for x: $3^{7x} = 3^{14}$**

Solution: Each side of the equation is an exponential expression. Notice that the bases are the same (the 3's), so the only way the two sides of the equation can be equal is *if the exponents are equal*. In other words, $7x$ must equal 14:

$$7x = 14$$

from which we determine that

$$\boxed{x = 2}$$

EXAMPLE 2: **Solve for y: $5^{4y} = 25$**

Solution: We're not as lucky here as we were in Example 1 — the bases are not the same. But maybe we can make them the same. Suppose we think of 25 as 5^2. Then, each side of the equation will have the same base, and we can set the exponents equal to each other to find the value of y. Let's try all of this:

$5^{4y} = 25$ (the original equation)

$\Rightarrow \quad 5^{4y} = 5^2$ (rewrite 25 with a base of 5)

$\Rightarrow \quad 4y = 2$ (set the exponents equal to each other, since the bases are the same)

$\Rightarrow \quad \boxed{y = \frac{1}{2}}$

EXAMPLE 3: **Solve for z: $27^{-4z} = \frac{1}{9}$**

Solution: This equation's terrible! The bases aren't the same: One is 27 and the other is 1/9. And it contains a fraction. But look at the 27 in the equation — it equals 3^3. And check out the 9 in the denominator — it can be written as 3^2. So maybe we can write everything in terms of the base **3**:

$27^{-4z} = \frac{1}{9}$ (the original equation)

$\Rightarrow \quad (\mathbf{3}^3)^{-4z} = \frac{1}{\mathbf{3}^2}$ (express 27 and 9 as powers of 3)

$\Rightarrow \quad 3^{-12z} = 3^{-2}$ (exponent rules)

$\Rightarrow \quad -12z = -2$ (bases are the same – set exponents equal)

$\Rightarrow \quad \boxed{z = \frac{1}{6}}$ (solve for z)

Homework

Solve each exponential equation:

1. $16^{6p} = 8$
2. $4^{10q} = 16$
3. $125^{3t} = \frac{1}{5}$
4. $16^{6w} = 16$
5. $4^{-2h} = 64$
6. $25^{6b} = \frac{1}{25}$
7. $25^{8w} = 125$
8. $625^{-4x} = \frac{1}{625}$
9. $125^{-10a} = \frac{1}{25}$
10. $16^{3v} = 2$
11. $4^{9r} = \frac{1}{2}$
12. $9^{-4p} = 729$
13. $625^{-3y} = \frac{1}{25}$
14. $81^{9x} = \frac{1}{729}$
15. $25^{-2k} = \frac{1}{25}$
16. $16^{-5x} = 512$
17. $27^{8b} = \frac{1}{729}$
18. $81^{6a} = 3$
19. $81^{-5r} = 81$
20. $625^{6b} = \frac{1}{25}$
21. $27^{-5v} = 3$
22. $81^{-4k} = \frac{1}{9}$
23. $125^{5p} = \frac{1}{25}$
24. $125^{-10d} = \frac{1}{5}$

EXAMPLE 4: **Solve each equation for x:**

A. $e^{5x} = e^{20}$

$\Rightarrow 5x = 20 \Rightarrow \boldsymbol{x = 4}$

B. $e^{2x+1} = (e^3)^{x-5}$

$\Rightarrow e^{2x+1} = e^{3x-15} \Rightarrow 2x+1 = 3x-15$

$\Rightarrow -x = -16 \Rightarrow \boldsymbol{x = 16}$

C. $\frac{1}{e} = e^{2x-1}$

$\Rightarrow\ e^{-1} = e^{2x-1}\ \Rightarrow\ -1 = 2x-1\ \Rightarrow\ \boldsymbol{x = 0}$

D. $\frac{1}{\sqrt[3]{e}} = \left(\frac{1}{e^2}\right)^x$

$\Rightarrow\ \frac{1}{e^{1/3}} = \frac{1^x}{(e^2)^x}\ \Rightarrow\ \frac{1}{e^{1/3}} = \frac{1}{e^{2x}}$

$\Rightarrow\ \frac{1}{3} = 2x\ \Rightarrow\ \boldsymbol{x = \frac{1}{6}}$

EXAMPLE 5: **Solve for x: $3^x = 5$**

Solution: Now we're really up a stump! How can we make the bases the same? We can't, so our current method for solving exponential equations dies at this point. In fact, the solution to this equation will not be a rational number as in the previous four examples, so the best we can do is approximate it. Here's one way to solve the problem, using a calculator (without using a logarithm key).

This is simply a guess-and-check method. Clearly, the solution x of the equation $3^x = 5$ is bigger than 1, since $3^1 = 3$; but x must be smaller than 2, since $3^2 = 9$. So the solution is between 1 and 2:

$$1 < x < 2$$

Now try 1.5. $3^{1.5} = 5.196$, a little too big. Let $x = 1.4$; then $3^{1.4} = 4.656$, a little too small. Thus, we've narrowed the solution to a number between 1.4 and 1.5:

$$1.4 < x < 1.5$$

Since 1.45 is halfway between 1.4 and 1.5, let's give it a try. $3^{1.45} = 4.918$, which is really close to 5. Consider 1.46: $3^{1.46} = 4.973$. Now nudge x up to 1.47: $3^{1.47} = 5.028$, a little too big. So the value of x is between 1.46 and 1.47:

$$1.46 < x < 1.47$$

This is getting boring, but I hope you're getting the point. Let's just take the average of 1.46 and 1.47, and figure we're close enough.

Our best guess is therefore

$$\boxed{x = 1.465}$$

In fact, if we check our "solution," we get $3^{1.465} = 5.0001$, so we have a really accurate value of x. The following chart is a summary of the calculations we made in this problem:

x	1	1.4	1.45	1.46	1.465	1.47	1.5	2
3^x	3	4.656	4.918	4.973	5.0001	5.028	5.196	9

In a future chapter we will study what are called "log" functions. Then we'll have a third (and much better) way of solving any kind of exponential equation, even if the base is e.

Homework

Solve each equation for exact solutions:

25. $2^{7x+1} = 2^{3x}$

26. $6^{8y-1} = 36$

27. $8^{-6z} = \frac{1}{2}$

28. $\frac{1}{25^x} = 125^{3x-1}$

29. $7^{3x} = 7^{5x}$

30. $81^{4-3x} = \sqrt{729}$

31. $e^{2x+1} = e$

32. $\frac{1}{e^2} = e^{1-7x}$

33. $\left(\frac{1}{\sqrt{e}}\right)^x = \sqrt[4]{e}$

Use your calculator to approximate the solution of each equation:

34. $2^x = 6$ 35. $10^y = 50$ 36. $e^z = 12$

37. Solve by inspection (this means stare at it and guess): $8^x = 5^x$

Solutions

1. $p = \frac{1}{8}$ **2**. $q = \frac{1}{5}$ **3**. $t = -\frac{1}{9}$ **4**. $w = \frac{1}{6}$

5. $h = -\frac{3}{2}$ **6**. $b = -\frac{1}{6}$ **7**. $w = \frac{3}{16}$ **8**. $x = \frac{1}{4}$

9. $a = \frac{1}{15}$ **10**. $v = \frac{1}{12}$ **11**. $r = -\frac{1}{18}$ **12**. $p = -\frac{3}{4}$

13. $y = \frac{1}{6}$ **14**. $x = -\frac{1}{6}$ **15**. $k = \frac{1}{2}$ **16**. $x = -\frac{9}{20}$

17. $b = -\frac{1}{4}$ **18**. $a = \frac{1}{24}$ **19**. $r = -\frac{1}{5}$ **20**. $b = -\frac{1}{12}$

21. $v = -\frac{1}{15}$ **22**. $k = \frac{1}{8}$ **23**. $p = -\frac{2}{15}$ **24**. $d = \frac{1}{30}$

25. $-\frac{1}{4}$ **26**. $6^{8y-1} = 6^2 \Rightarrow 8y - 1 = 2 \Rightarrow y = \frac{3}{8}$

27. $8^{-6z} = \frac{1}{2} \Rightarrow (2^3)^{-6z} = 2^{-1} \Rightarrow 2^{-18z} = 2^{-1}$

$\Rightarrow -18z = -1 \Rightarrow z = \frac{1}{18}$

28. $\frac{3}{11}$ **29**. 0 **30**. $\frac{13}{12}$ **31**. 0 **32**. $\frac{3}{7}$

33. $\left(\frac{1}{\sqrt{e}}\right)^x = \sqrt[4]{e} \Rightarrow \left(e^{-1/2}\right)^x = e^{1/4} \Rightarrow e^{-\frac{1}{2}x} = e^{\frac{1}{4}}$

$\Rightarrow -\frac{1}{2}x = \frac{1}{4} \Rightarrow x = -\frac{1}{2}$

34. 2.58 **35**. 1.7 **36**. 2.48

37. Hint: The solution is the only number with no reciprocal.

"*Education* is not merely a means for earning a living or an instrument for the acquisition of wealth. It is an initiation into life of spirit, a training of the human soul in the pursuit of truth, and the practice of virtue."

– Vijaya Lakshmi Pandit

EXPONENTIAL EQUATIONS – WITH LOGS

When we studied the growth and decay formula a while back, we found that we had no way to determine the growth rate or the amount of time. The chapter *The Laws of Logs* will give us the necessary tools.

$$A = A_0 e^{kt}$$



The exponential decay of radioactive elements allows us to determine the age of dinosaur bones.

❑ EXPONENTIAL EQUATIONS

Remember the hassles in the chapter we encountered in trying to solve the exponential equation $3^x = 5$? We had to use a calculator to guess an answer, and then adjust and guess again, and so on, until we had a couple of digits of accuracy. We're now ready to solve this problem more efficiently.

EXAMPLE 1: **Solve for x: $3^x = 5$**

Solution: The variable is in the exponent. This is a dilemma — how do we get the unknown out of the exponent so that we can solve for it? Do you recall the third law of logs? It states that

$$\log_b a^x = x \log_b a$$

It allows us to move the exponent to the front (making it a coefficient), but only if we're taking the log of an expression. So the procedure here will be to take a log (we'll choose *ln*, since it's on your calculator and ln is used in calculus), bring down the exponent, and then solve for it.

	$3^x = 5$	(the original equation)
$\Rightarrow$	$\mathbf{ln}\ 3^x = \mathbf{ln}\ 5$	(take the *ln* of both sides)
$\Rightarrow$	$x \ln 3 = \ln 5$	(the third law of logs)
$\Rightarrow$	$x = \frac{\ln 5}{\ln 3}$	(simple algebra – solve for x)
$\Rightarrow$	$x = \frac{1.609437912}{1.098612289}$	(use your calculator)
$\Rightarrow$	$\boxed{x = 1.464973521}$	

When we did this problem by guessing with a calculator, a lot of work produced a guess of 1.465. Clearly, our new method is superior. Note that the true answer, $x = \frac{\ln 5}{\ln 3}$, is an irrational number, while the answer in the box is a rational approximation of the true answer.

EXAMPLE 2: **Solve for *n*: $2^{3n+2} = 7$**

Solution:

	$2^{3n+2} = 7$	(the given equation)
$\Rightarrow$	$\mathbf{ln}\, 2^{3n+2} = \mathbf{ln} 7$	(take the *ln* of each side)
$\Rightarrow$	$(3n+2) \ln 2 = \ln 7$	(third law of logs)

[Notice the parentheses around the $3n + 2$]

$\Rightarrow \quad (3\ln 2)\,n + 2\ln 2 = \ln 7$ (distribute)

$\Rightarrow \quad (3\ln 2)n = \ln 7 - 2\ln 2$ (subtract 2 ln2)

$\Rightarrow \quad n = \dfrac{\ln 7 - 2\ln 2}{3\ln 2}$ (divide each side by 3 ln 2 to get the **exact** answer)

$\Rightarrow \quad \boxed{n = 0.269118307}$ (use your calculator to get a rational **approximation**)

EXAMPLE 3: **Solve for a: $5^{2a-3} = 6^{a+1}$**

Solution:

$5^{2a-3} = 6^{a+1}$ (the original equation)

$\Rightarrow \quad \mathbf{ln}\,5^{2a-3} = \mathbf{ln}\,6^{a+1}$ (take the *ln* of each side)

$\Rightarrow \quad (2a-3)\ln 5 = (a+1)\ln 6$ (third law of logs)

$\Rightarrow \quad 2a\ln 5 - 3\ln 5 = a\ln 6 + \ln 6$ (distribute)

$\Rightarrow \quad 2a\ln 5 - a\ln 6 = \ln 6 + 3\ln 5$ (variables to the left and constants to the right)

$\Rightarrow \quad a(2\ln 5 - \ln 6) = \ln 6 + 3\ln 5$ (factor out the variable)

$\Rightarrow \quad \boxed{a = \dfrac{\ln 6 + 3\ln 5}{2\ln 5 - \ln 6}}$ (divide to isolate the a)

This is the exact answer. A rational approximation would be $a = 4.638776075$.

Homework

Solve each equation and round your answers to the nearest ten thousandths place:

1. $2^x = 72$
2. $5^{-y} = 3$
3. $3^{4n-1} = 5$
4. $3^{z+1} = 8^{3-z}$
5. $3^{5c} = 7^{10c}$
6. $e^{3x+4} = 25$
7. $3^{-n} = 43$
8. $7^{4-3x} = 2$
9. $e^x = 2^{x-6}$

❑ THE GROWTH AND DECAY FORMULA REVISITED

The growth and decay formula

$$A = A_0 e^{kt}$$

worked just fine before, when we were searching for either the starting amount, A_0, or the final amount, A. But when the unknown was in the exponent (the k or the t), we were stuck. Now we're not stuck.

EXAMPLE 4: **Assuming an initial population of 7500, a final population of 12,000, and a time period of 7 years, find the annual growth rate.**

Solution: We will write the growth formula, substitute the given values, and then solve for the unknown k:

$$A = A_0 e^{kt} \qquad \text{(the growth formula)}$$

$$\Rightarrow \quad 12{,}000 = 7500 e^{k \cdot 7} \qquad \text{(substitute the given values)}$$

$$\Rightarrow \quad e^{7k} = \frac{12{,}000}{7500} \qquad \text{(isolate the } e^{7k}\text{)}$$

$$\Rightarrow \quad e^{7k} = 1.6 \qquad \text{(calculator)}$$

$\Rightarrow \quad \ln e^{7k} = \ln 1.6$ (take the *ln* of each side)

$\Rightarrow \quad 7k \ln e = \ln 1.6$ (the third law of logs)

$\Rightarrow \quad 7k = \ln 1.6$ ($\ln e = 1$)

$\Rightarrow \quad k = \frac{\ln 1.6}{7}$ (solve for k)

$\Rightarrow \quad k = 0.067$ (calculator gives a decimal)

And therefore the annual growth rate is

6.7%

EXAMPLE 5: **How long will it take for an investment of $10,000 to reach a total of $32,000 if the interest rate is 7.3% per year compounded continuously?**

Solution: The phrase "compounded continuously" justifies the use of our growth formula.

$A = A_0 e^{kt}$ (the growth formula)

$\Rightarrow \quad 32{,}000 = 10{,}000 e^{0.073t}$ (remember: 7.3% = .073)

$\Rightarrow \quad \frac{32{,}000}{10{,}000} = \frac{10{,}000 e^{0.073t}}{10{,}000}$ (divide each side by 10,000)

$\Rightarrow \quad e^{0.073t} = 3.2$ (simplify and reverse)

$\Rightarrow \quad \ln e^{0.073t} = \ln 3.2$ (take the *ln* of each side)

$\Rightarrow \quad 0.073t = \ln 3.2$ (third law of logs & $\ln e = 1$)

$\Rightarrow \quad t = \frac{\ln 3.2}{0.073} = 15.933$ (solve for t)

Thus, the amount of time it will take to reach the goal is

15.933 years

EXAMPLE 6: **A population is growing at 13% per year, compounded continuously. How long will it take for the population to triple?**

Solution: It would appear that the answer should depend on the starting or ending population. But an interesting aspect of the exponential growth/decay formula is that it doesn't matter. If it takes 10 years for the population to triple from 12 to 36, then it takes the same 10 years for the population to triple from 5,000 to 15,000. The following calculations should prove this.

$$A = A_0 e^{kt}$$

$$36 = 12e^{0.13t} \qquad\qquad 15{,}000 = 5000e^{0.13t}$$

In each case we divide by the initial population:

$$\frac{36}{12} = e^{0.13t} \qquad\qquad \frac{15{,}000}{5000} = e^{0.13t}$$

$$3 = e^{0.13t} \qquad\qquad 3 = e^{0.13t}$$

Note that under either scenario we arrived at exactly the same equation:

$$\begin{aligned} & e^{0.13t} = 3 \\ \Rightarrow \quad & \ln e^{0.13t} = \ln 3 \\ \Rightarrow \quad & 0.13t = \ln 3 \\ \Rightarrow \quad & t = \frac{\ln 3}{0.13} = 8.45 \end{aligned}$$

Therefore, the amount of time needed for the population to triple, regardless of the actual populations, is approximately

$$\boxed{8.45 \text{ years}}$$

Half-life The ***half-life*** of a radioactive substance is the time required for **half** of the material to disintegrate into energy and other particles, which therefore means it's the time it takes for the final amount of the substance to be half of the starting amount. For instance, start with 100 grams of plutonium, and assume the half-life is 7 years. After 7 years, 50 g remain. After another 7 years, 25 g remain. Another 7 years and $12\frac{1}{2}$ g remain, etc., etc. The concept of half-life also pertains to the time it takes for the amount of medication remaining in the body to be half of its starting amount.

EXAMPLE 7: **Find the half-life of a radioactive substance which decays at an annual rate of 9%.**

Solution: Recall the preceding example on tripling the population. It didn't matter whether we went from 12 to 36 or from 5,000 to 15,000 — the tripling time was the same either way. In the same manner, the half-life of a substance does not depend on the starting and ending amounts, as long as the ending amount is half of the starting amount. We can prove this general statement by using variables instead of numbers for the starting and ending amounts.

Let's assume a starting amount of A_0 grams. When half of it decays, the remaining amount is half of A_0, or $\frac{1}{2}A_0$. In other words, $A = \frac{1}{2}A_0$.

$$A = A_0 e^{kt} \qquad \text{(the growth/decay formula)}$$

$$\Rightarrow \quad \tfrac{1}{2}A_0 = A_0 e^{-0.09t} \qquad \text{(note: } k \text{ is negative)}$$

$$\Rightarrow \quad \tfrac{1}{2} = e^{-0.09t} \qquad \text{(divide both sides by } A_0\text{)}$$

$$\Rightarrow \quad \mathbf{ln}\,\tfrac{1}{2} = \mathbf{ln}\,e^{-0.09t} \qquad \text{(take the } ln \text{ of each side)}$$

$$\Rightarrow \quad -0.09t = \ln\frac{1}{2} \qquad \text{(simplify and flip the equation)}$$

$$\Rightarrow \quad t = \frac{\ln\frac{1}{2}}{-0.09} = 7.7$$

And thus the half-life of the substance is about

7.7 years

Homework

10. In the growth formula $A = A_0 e^{kt}$, solve for
    a. A_0 b. k c. t

11. Find the interest rate if an investment of \$7200 reached a total of \$18,000 in 5 years.

12. If the population is growing 9% per year, how long will it take for a population of 25,600 to reach a population of 100,000?

13. How long will it take for a population to double in size if the growth rate is 18% per year?

14. Find the half-life of a radioactive substance whose annual decay rate is 23%.

15. At an annual rate of 8%, how many years will it take for you to quadruple your money?

16. The decay rate of a radioactive substance is k (where $k < 0$). Prove that the half-life of the substance is $t = \frac{-\ln 2}{k}$.

Review Problems

17. Solve for a: $5^a = 3$

18. Solve for x: $T = T_0 e^{ax}$

19. Solve for x: $7^{3x-1} = 2^{6-5x}$

20. Solve for n: $2^{2n+3} = 3^{4-5n}$

21. Solve for b: $e^{3b} = 2^{b+1}$

22. Solve for x: $10^{7x-14} = 1$

23. Find the annual growth rate if a population increased from 2000 to 7,500 in a period of 9 years.

24. $25,000 is invested in a money market account paying 9.5% per year compounded continuously. How many years will it take for that investment to reach a total of $75,000?

25. How long would it take a population to quintuple in size if the growth rate is 12.7% per year?

26. Find the half-life of a substance whose annual decay rate is 6.3%.

27. True/False:

    a. The solution of the exponential equation $6^x = 4$ is about 0.774.

    b. The exact solution of the equation $2^{3x+2} - 7 = 0$ is $\frac{\ln 1.75}{\ln 8}$.

    c. If it took 12 years for the population to increase from 500 to 2,250, then the annual growth rate was 20%.

    d. If the amount in the 7%/year continuous compounding savings account went from $7,500 to $14,000, then the time span was about 8.916 years.

e. A population is growing at 23% per year compounded continuously. It should take about 1.76 years for the population to grow 50%.

f. The half-life of a substance which decays at an annual rate of 1.3% is about 35 years.

g. If $A = A_0 e^{kt}$, then $t = \frac{1}{k}\ln\frac{A}{A_0}$.

Solutions

1. 6.1699 **2**. –0.6826

3. $3^{4n-1} = 5 \Rightarrow \ln(3^{4n-1}) = \ln 5 \Rightarrow (4n-1)\ln 3 = \ln 5$

$\Rightarrow n = \frac{\frac{\ln 5}{\ln 3}+1}{4} \approx .6162$

4. 1.6173

5. $3^{5c} = 7^{10c} \Rightarrow \ln(3^{5c}) = \ln(7^{10c}) \Rightarrow 5c\ln 3 = 10c\ln 7$

$\Rightarrow 5c\ln 3 - 10c\ln 7 = 0 \Rightarrow c(5\ln 3 - 10\ln 7) = 0$

$\Rightarrow c = \frac{0}{5\ln 3 - 10\ln 7} \Rightarrow c = 0$

6. –0.2604 **7**. –3.4236 **8**. 1.2146 **9**. –13.5533

10. a. $A_0 = \frac{A}{e^{kt}}$

b. $A = A_0 e^{kt} \Rightarrow \frac{A}{A_0} = e^{kt} \Rightarrow \ln\frac{A}{A_0} = \ln e^{kt}$

$\Rightarrow \ln A - \ln A_0 = kt \Rightarrow k = \frac{\ln A - \ln A_0}{t}$

c. $t = \dfrac{\ln A - \ln A_0}{k}$

11. 18% **12**. 15 yrs **13**. 3.9 yrs **14**. 3 yrs **15**. 17.3 yrs

16. $A = A_0 e^{kt}$. We assume that the final amount is half of the starting amount: $A = \frac{1}{2}A_0$. Also assume that $k < 0$, since it's a decay problem and not a growth problem.

$$\tfrac{1}{2}A_0 = A_0 e^{kt} \Rightarrow \ln\tfrac{1}{2} = \ln e^{kt} \Rightarrow kt = \ln 1 - \ln 2$$

$$\Rightarrow t = \frac{0 - \ln 2}{k} \Rightarrow t = \frac{-\ln 2}{k} \quad \text{Q.E.D.}$$

17. $a = \dfrac{\ln 3}{\ln 5} \approx 0.6826$ **18**. $x = \dfrac{\ln T - \ln T_0}{a}$

19. $x = \dfrac{6\ln 2 + \ln 7}{3\ln 7 + 5\ln 2} \approx 0.6562$ **20**. $n = \dfrac{4\ln 3 - 3\ln 2}{2\ln 2 + 5\ln 3} \approx 0.3365$

21. $b = \dfrac{\ln 2}{3 - \ln 2} \approx 0.3005$ **22**. $x = 2$

23. $k = 14.7\%$ **24**. 11.56 yrs **25**. 12.67 yrs **26**. 11 yrs

27. a. T b. T c. F d. T e. T f. F g. T

It's better to light a candle

than to curse the darkness.

Chinese proverb

EXPONENTIAL FUNCTIONS – AN INTRO

❑ *INTRODUCTION*

This chapter assumes that you have studied linear functions (straight lines) and quadratic functions (parabolas).

❑ *THE RACE*

Three functions are having an automobile race. Each of the three functions has a different type of engine, given by the following formulas:

$2t$ t^2 2^t [where ***t*** is time]

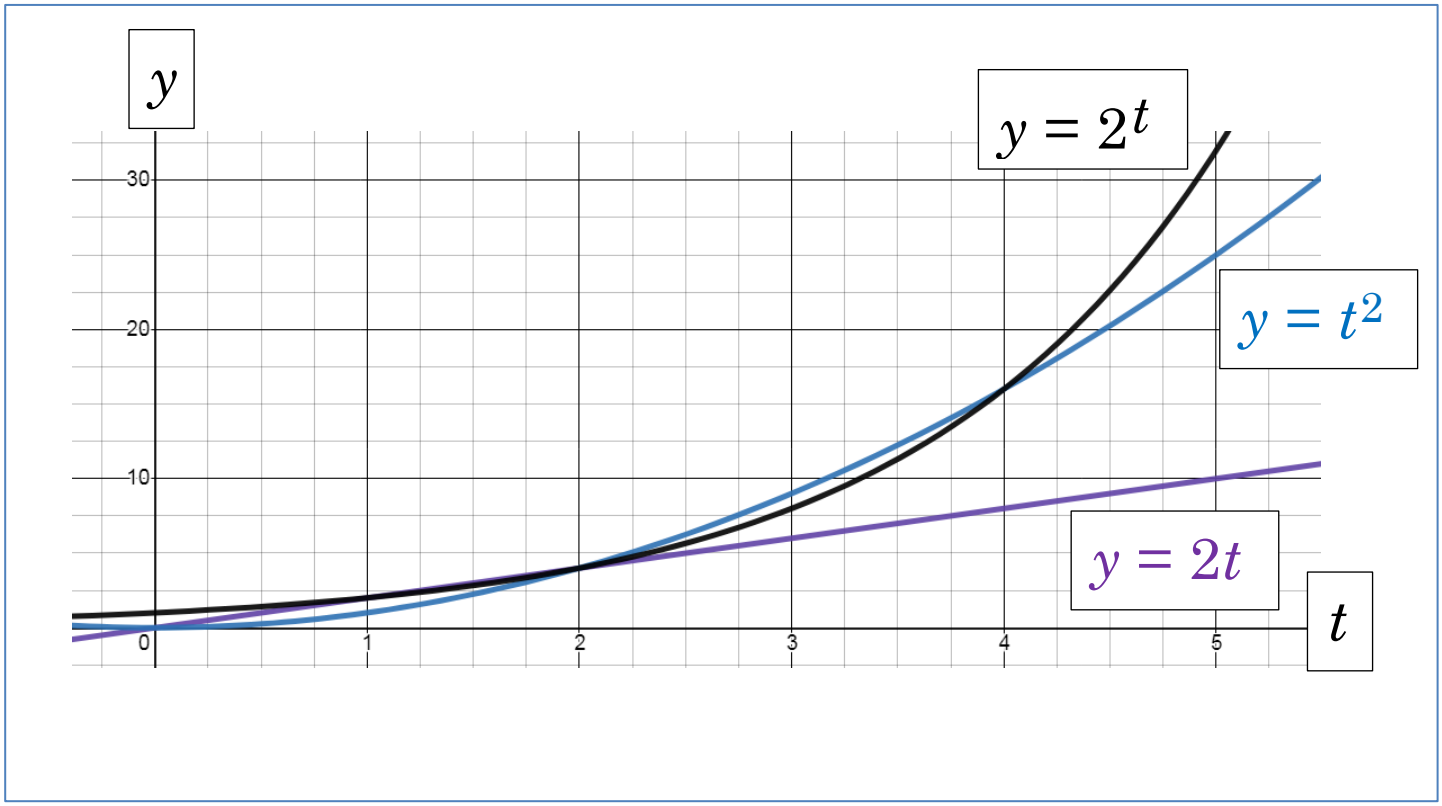
We're going to make a table of all three functions of time, and use it to figure out which one is winning the race at various moments in time, and which one will be the ultimate winner if we let *time* grow larger and larger (furthur into the future).

t	$2t$	t^2	2^t	
0	0	0	1	2^t is in the lead.
1	2	1	2	Now $2t$ and 2^t are tied.
2	4	4	4	At t = 2, there's a 3-way tie.
3	6	9	8	t^2 pulls ahead.
4	8	16	16	$2t$ is fading, while t^2 and 2^t are tied.
5	10	25	32	2^t now has the edge.

6	12	36	64	$2t$ and t^2 are fading away.
10	20	100	1,024	Now it's not even a race anymore.
20	40	400	1,048,576	This is getting ridiculous!

For all the t-values from 0 to 4, the function 2^t is sometimes ahead, sometimes tied, and sometimes behind. But, starting at $t = 5$, the function 2^t grows leaps and bounds beyond the other two functions. We say that the function 2^t is growing ***EXPONENTIALLY.***

You may recall that $2t$ is a linear function, while t^2 is a quadratic function. We're now going to graph all three functions on the same grid to get a picture of what's going on.



Moreover, here's how powerful an exponential function can be even when compared to a "massive" quadratic function:

Let's compare the two functions

$y_1 = 100x^2$ and $y_2 = 2^x$ [The 100 makes y_1 quite large.]

For $x = 5$: $y_1 = 2{,}500$ and $y_2 = 32$ [y_2 seems quite weak.]

For $x = 10$: $y_1 = 10{,}000$ and $y_2 = 1{,}024$ [y_2 is still way behind.]

For $x = 12$: $y_1 = 14{,}400$ and $y_2 = 4{,}096$ [Does y_2 even have a chance?]

For $x = 25$: $y_1 = 62{,}500$ and $y_2 = 33{,}554{,}432$ [Now y_2 can't be stopped!]

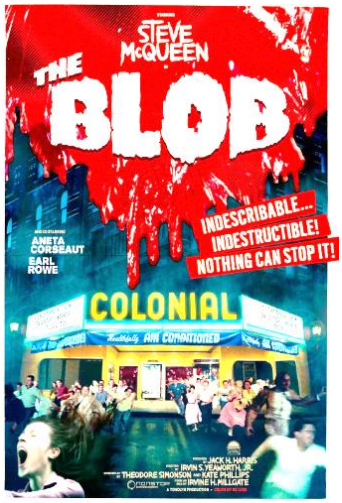
And for my next statement, you'll have to trust me:

No matter what large number
you put in front of the $\boldsymbol{x^2}$*,*
the $\boldsymbol{2^x}$ *will eventually overtake it*
and remain in the lead forever,
with the gap between them
growing larger and larger.

In fact, you can even change the exponent on the x^2 to a really large number, and 2^x will still eventually leap ahead. So, for example, in a race between x^{78} and 2^x, the 2^x will eventually win the race (although an unbelievably gigantic value of x would be needed for this to happen, perhaps even beyond the capability of your calculator.)

EXPONENTIAL FUNCTIONS

What worried Steve McQueen was not that The Blob was growing by a constant amount every hour, but rather that it was **doubling** in size every hour. If our hero had known his math, he could have warned the town that The Blob was not growing linearly, but *exponentially*.



❑ *EXPONENTIAL FUNCTIONS*

EXAMPLE 1: **Analyze The Blob function.**

Solution: To give you an idea of the concept of an exponential function, let's look at two scenarios regarding the rate of growth of The Blob. An example of a linear rate of growth might be the formula $B = 4t$, where t is the time in hours and B is the size of The Blob. An exponential formula could be $B = 2^t$. If we construct a table, showing the time and the amount of The Blob for each formula, we can see the true effect of exponential growth.

t	1	2	3	4	5	6	7	8	9	10	11
$4t$	4	8	12	16	20	24	28	32	36	40	44
2^t	2	4	8	16	32	64	128	256	512	1,024	2,048

For the first three hours, there's more blob in the linear formula than in the exponential formula. At $t = 4$, the blob amounts are equal. But after that, it's not even a contest — the exponential

formula shows that The Blob will probably eat the town, the state, and eventually the entire Earth!

EXAMPLE 2: **Find some ordered pairs for the exponential function $f(x) = 4^x$.**

Solution:

$$f(1) = 4^1 = 4 \quad \Rightarrow \quad (1, 4)$$

$$f(-2) = 4^{-2} = \frac{1}{4^2} = \frac{1}{16} \quad \Rightarrow \quad (-2, \tfrac{1}{16})$$

$$f(-1) = 4^{-1} = \frac{1}{4} \quad \Rightarrow \quad (-1, \tfrac{1}{4})$$

$$f(0) = 4^0 = 1 \quad \Rightarrow \quad (0, 1)$$

$$f(\tfrac{1}{2}) = 4^{1/2} = \sqrt{4} = 2 \quad \Rightarrow \quad (\tfrac{1}{2}, 2)$$

$$f(2) = 4^2 = 16 \quad \Rightarrow \quad (2, 16)$$

We now try to determine exactly what the formula for an exponential function looks like, and how it differs from that of a polynomial. Look at the exponential functions given in the previous examples:

$$B = 2^t \qquad f(x) = 4^x$$

Notice that in each function, the base is a constant (the 2 and the 4) and the exponent is a variable (the t and the x). Thus, an ***exponential function*** is a function of the form

$$f(x) = b^x$$

variable

constant

where b is some appropriate real number (a constant)

This is in sharp contrast to the notion of a *polynomial*, which is the other way around. Thus,

$y = 10^x$ is an *exponential* function,

$y = x^{10}$ is a *polynomial* function, and

$y = x^x$ is neither an exponential nor a polynomial function.

The question of which real numbers ***b*** serve nicely as the base of an exponential function will be discussed later.

Homework

1. a. Fill in the following chart, similar to Example 1:

t	1	2	3	4	5	6	7	8	9	10
$9t$										
3^t										

b. Is the $9t$ row of the chart a linear or exponential function?

c. Is the 3^t row of the chart a linear or exponential function?

d. For how many hours does the linear growth produce more blob than the exponential growth?

e. At what hour do both growths give the same amount of blob?

f. At 10 hours, what is the <u>ratio</u> of the exponential amount of blob to the linear amount of blob?

2. Let's find some more ordered pairs in the function $f(x) = 4^x$ from Example 3.

a. (3, ___) b. (–3, ___) c. $(-\frac{1}{2}, \text{___})$ d. $(\frac{3}{2}, \text{___})$

3. Take a guess what the domain of $f(x) = 4^x$ is.

❑ GRAPHING EXPONENTIAL FUNCTIONS

Let's use the function $y = 4^x$ discussed in the previous section to make our first exponential graph.

EXAMPLE 3: **Graph: $y = 4^x$**

Solution: Here are some ordered pairs for this function that we found in Example 2 and Homework #2:

$(-1, \frac{1}{4})$ $(-\frac{1}{2}, \frac{1}{2})$ $(0, 1)$ $(\frac{1}{2}, 2)$ $(1, 4)$ $(2, 16)$

Notice that the point (0, 1) is the ***y*-intercept**.

Next we analyze a pair of **limits**. First we'll let $x \to \infty$. As it does, the functional value 4^x approaches ∞ much faster than x does. For example, as x takes the values 6, 8, 10, the y-values go 4096, 65536, 1048576. Wow! The functional values are growing like crazy. The following limit should now be clear:

As $x \to \infty$, $y \to \infty$ [an amazingly fast increase]

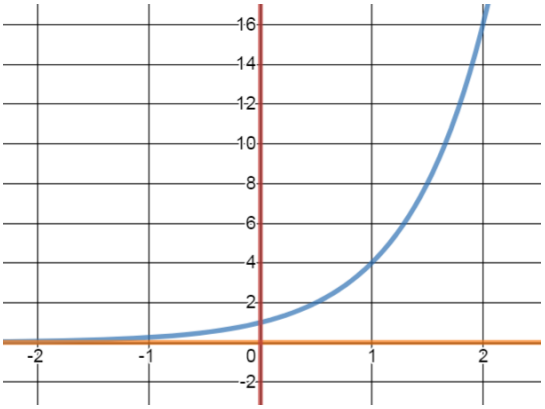
Second, we analyze what the y-values do when $x \to -\infty$. Consider the three ordered pairs (confirm with your calculator):

$(-5, 0.000977)$ $(-8, 0.000015)$ $(-10, 0.000000954)$

It appears that as x grows smaller (towards $-\infty$), the y-values are positive numbers shrinking toward zero. That is,

As $x \to -\infty$, $y \to 0$

The ordered pairs we listed and the limits we calculated lead us to the following graph:



The **domain** is $\mathbb{R}$. Also, there is no vertical **asymptote**, but the line $y = 0$ (the x-axis) is a **horizontal asymptote**.

Last, it appears that there is no ***x*-intercept** on our graph. Even more importantly, this confirms that

The equation $4^x = 0$ has <u>no</u> solution.

EXAMPLE 4: **Graph:** $y = \left(\frac{1}{2}\right)^x$

Solution: Let's get right to some ordered pairs.

If $x = -3$, then $y = \left(\frac{1}{2}\right)^{-3} = \frac{1}{\left(\frac{1}{2}\right)^3} = \frac{1}{\frac{1}{8}} = 8$, which gives us the ordered pair $(-3, 8)$. It's now your job to verify each of the following ordered pairs in our function:

$(0, 1)$ $(1, \frac{1}{2})$ $(2, \frac{1}{4})$ $(3, \frac{1}{8})$ $(4, \frac{1}{16})$

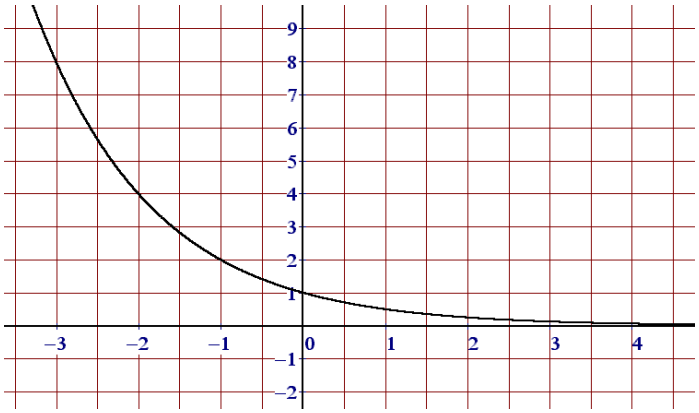
$(-1, 2)$ $(-2, 4)$ $(-3, 8)$ $(-4, 16)$

The y-intercept is $(0, 1)$. There is no x-intercept, since the equation $\left(\frac{1}{2}\right)^x = 0$ has no solution. The ordered pairs listed above give credence to the following limits:

As $x \to \infty$, $y \to 0$ and As $x \to -\infty$, $y \to \infty$

The ordered pairs and the limits lead us to the following graph:

We can see that the domain of this function is $\mathbb{R}$. Notice also that we have a horizontal asymptote at $y = 0$, but there is no vertical asymptote.



EXAMPLE 5: **Graph: $f(x) = 2^{-x}$**

Solution: Two observations and we'll be done in a jiffy. First, we know that $f(x)$ can be written simply as y. Second, check out the following calculation:

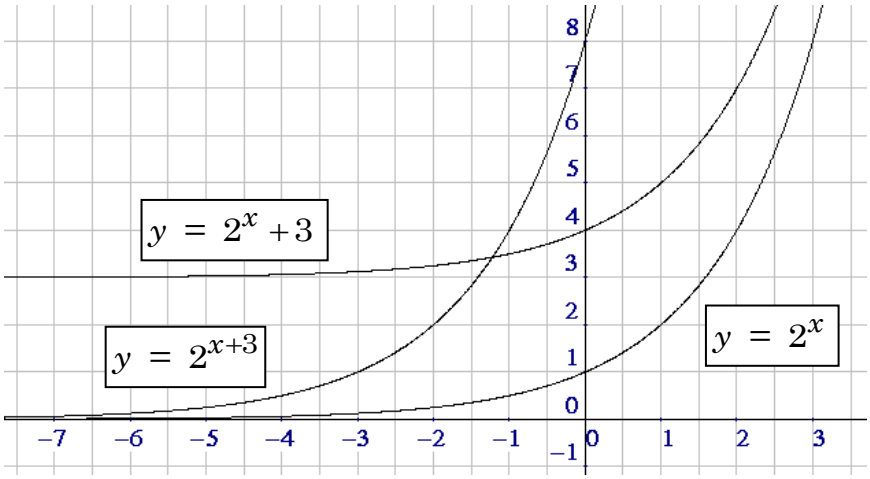
$$2^{-x} = \frac{1}{2^x} = \frac{1^x}{2^x} = \left(\frac{1}{2}\right)^x$$

In other words, the original formula can be written $y = \left(\frac{1}{2}\right)^x$, which we just finished graphing. So the solution to this problem is identical to that of the previous example.

EXAMPLE 6: **Graph: $y = 2^x$ and $y = 2^x + 3$ and $y = 2^{x+3}$.**

Solution: Let's make one table showing x and all the y-values at once and a single grid containing all the graphs at once.

x	2^x	$2^x + 3$	2^{x+3}
–6	1/64	3 1/64	1/8
–5	1/32	3 1/32	1/4
–4	1/16	3 1/16	1/2
–3	1/8	3 1/8	1
–2	1/4	3 1/4	2
–1	1/2	3 1/2	4
0	1	4	8
1	2	5	16
2	4	7	32
3	8	11	64



You should note that the graph of $2^x + 3$ is just the graph of 2^x but shifted 3 units up. Also, we see that the graph of 2^{x+3} is the result of taking the graph of 2^x and shifting it 3 units to the left.

Homework

4. Graph: $f(x) = 3^x$
5. Graph: $y = 3^x - 2$
6. Graph: $y = 3^{x+2}$
7. Graph: $g(x) = \left(\frac{1}{3}\right)^x$
8. Graph: $h(x) = 3^{-x}$

❑ THE LEGAL BASES OF AN EXPONENTIAL FUNCTION

In the previous section we graphed exponential functions with bases 4, $\frac{1}{2}$, 2, 3, and $\frac{1}{3}$. Now it's time to figure out exactly which bases we'll allow in the exponential function

$$f(x) = b^x$$

Whatever values of ***b*** we allow to be the base of an exponential function, we'd like the domain of the function (the legal x-values) to be

$\mathbb{R}$, the set of real numbers. And we don't want the exponential function to degenerate into some simple function that doesn't possess the "exponential" properties we've seen up until now.

$b < 0$ What about negative bases? Consider $f(x) = (-4)^x$. If we choose $x = \frac{1}{2}$, the functional value is $(-4)^{1/2} = \sqrt{-4}$, not a real number. We thus disallow any base b that is negative.

$b = 0$ Now consider $f(x) = 0^x$. But 0^x is fraught with problems. For example, if $x = 0$, we get 0^0. Can we assign a value to 0^0? On the one hand, 0 to any power should be 0. On the other hand, anything to the 0 power is supposed to be 1. So 0^0 is meaningless (but can be solved in Calculus). Even worse, consider 0^{-2}. Since a negative exponent indicates reciprocal, we get $\frac{1}{0^2} = \frac{1}{0}$, which is undefined. All in all, a base of 0 really stinks.

$0 < b < 1$ These bases are just fine. We used bases of $\frac{1}{2}$ and $\frac{1}{3}$ in the previous section. Even the number $\frac{1}{\pi}$ would be a legal base, although I've never seen it used.

$b = 1$ This gives us the function $f(x) = 1^x$, which is the function $f(x) = 1$, a constant function (the horizontal line $y = 1$). Exponential functions aren't supposed to be flat, so b shouldn't be allowed to be 1.

$b > 1$ Any base bigger than 1 is appropriate. In fact, in computer science a base of **2** is very popular. In basic science, the best base is **10** (for things like acids, earthquakes, and the volume of sound). And in calculus and the more advanced sciences, we use a strange number you may have seen: "***e***".

Homework

9. Describe precisely the legal bases for an exponential function.

10. Explain why –9 is not a good base for an exponential function.

11. Which of the following real numbers are legal bases for an exponential function?

-1 $\quad -.01$ $\quad 0$ $\quad \frac{2}{3}$ $\quad 0.987$ $\quad 1$ $\quad \pi$ $\quad 200$

Review Problems

12. T/F: $y = x^3$ is an exponential function.
13. Describe the real numbers which can be used as the base of an exponential function.
14. Explain why 1 is not a good base for an exponential function.
15. Give a function which is both an exponential function and a polynomial function.
16. Graph $y = 5^x$.
17. Graph $y = 3^{-x} - 2$ and state its horizontal asymptote.
18. Let $f(x) = 2^x$. Now let g be the graph which results from taking the graph of f and shifting it 7 units to the left and 4 units up. Find a formula for g.
19. T/F: $y = 3^x$ is an exponential function.
20. T/F: In the exponential function $f(x) = b^x$, b can be any positive real number.

21. What is the domain of the function $y = 10^{7x+1} - 10$?

22. Find all the asymptotes of the function $f(x) = 99^x$.

23. Explain why the graphs of $g(x) = \left(\frac{1}{3}\right)^x$ and $h(x) = 3^{-x}$ are the same.

24. How does the graph of $f(x) = 5^{x-2} + 4$ compare with that of $y = 5^x$?

25. T/F: All exponential functions are increasing functions.

26. Explain why 0 is not a good base for an exponential function.

27. True/False:

    a. $y = x^3$ is an exponential function.

    b. $y = \pi^x$ is an exponential function.

    c. The domain of the function $y = 4^x$ is $[0, \infty)$.

    d. The function $y = x^x$ is neither polynomial nor exponential.

    e. The function $y = 2^x - 1$ has an x-intercept.

    f. For the function above, as $x \to -\infty$, $y \to 0$.

    g. Consider the function $g(x) = \left(\frac{1}{3}\right)^x$. As $x \to \infty$, $y \to \infty$.

    h. Compared to the graph of $y = 4^x$, the graph of $y = 4^{x-5}$ is five units lower.

    i. Any real number $b > 0$ is a legal base for an exponential function.

    j. Any real number $b \geq 0$, but not equal to 1, is a legal base for an exponential function.

    k. The number $\pi + \sqrt{2}$ is a legal base for an exponential function.

    l. All exponential functions are decreasing functions.

Solutions

1. a.

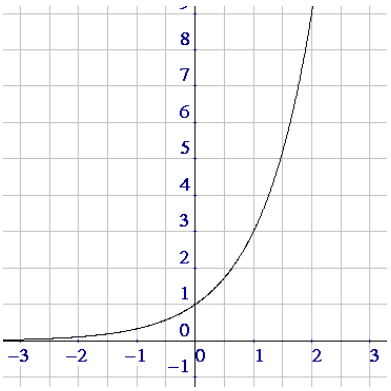
t	1	2	3	4	5	6	7	8	9	10
$9t$	9	18	27	36	45	54	63	72	81	90
3^t	3	9	27	81	243	729	2187	6561	19683	59049

b. linear c. exponential d. first two hours
e. $t = 3$ f. $59{,}049 / 90 \approx 656$

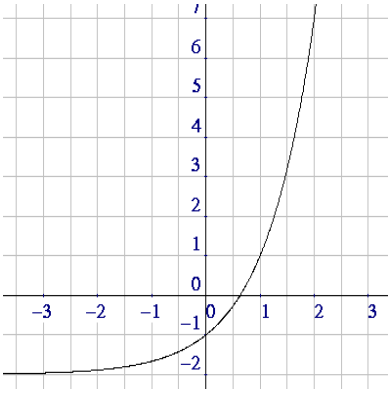
2. a. 64 b. $\frac{1}{64}$ c. $\frac{1}{2}$ d. 8

3. x can be all kinds of numbers, so the domain is probably $\mathbb{R}$.

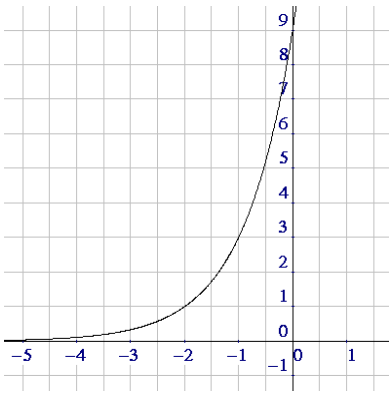
4.



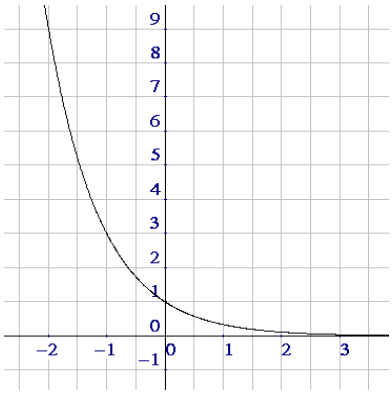
5.



6.



7.



8. Same graph as #14, since $\left(\frac{1}{3}\right)^x = \frac{1^x}{3^x} = \frac{1}{3^x} = 3^{-x}$.

9. The base must be positive but not equal to 1. That is, the function $f(x) = b^x$ is an exponential function if $b > 0$, but $b \neq 1$. In other words, the base b must be in the set: $(0, \infty) - \{1\}$.

10. If we consider the exponential function $y = (-9)^x$, then we could not use $x = \frac{1}{2}$, since $y = (-9)^{\frac{1}{2}} = \sqrt{-9} \notin \mathbb{R}$.

11. 2/3, 0.987, π, 200

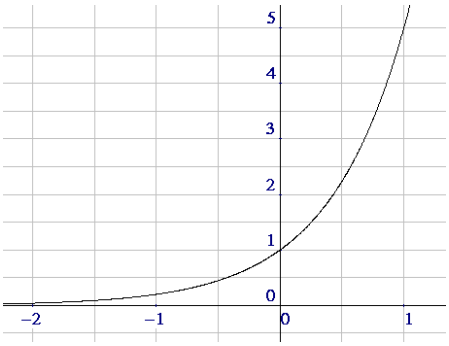
12. False

13. $\{x \in \mathbb{R} \mid x > 0, x \neq 1\}$ –OR– Any positive real number ≠ 1
–OR– $(0, 1) \cup (1, \infty)$ –OR– $(0, \infty) - \{1\}$

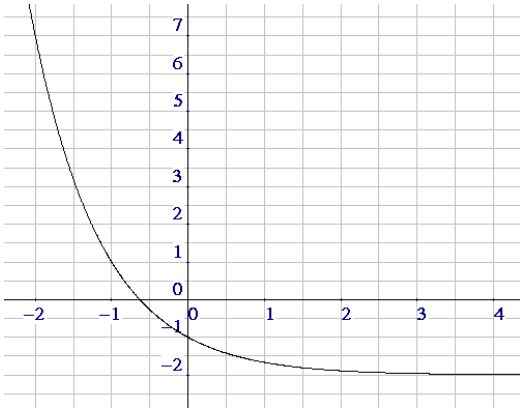
14. If b were 1, then $f(x) = b^x$ would become $f(x) = 1^x = 1$, which is a simple constant function whose graph is a horizontal line, rather useless to describe exponential growth and decay.

15. Ain't no such animal

16.



17.



Horizontal asymptote: $y = -2$

18. $g(x) = 2^{x+7} + 4$

19. T **20**. F (any positive real number $\neq 1$)

21. $\mathbb{R}$ **22**. horiz: $y = 0$; vert: none

23. Because $\left(\frac{1}{3}\right)^x = \frac{1^x}{3^x} = \frac{1}{3^x} = 3^{-x}$.

24. The graph of f is the graph of y shifted 2 units to the right and 4 units up.

25. F

26. Because if $f(x) = 0^x$, then $f(x) = 0$, which is just a horizontal line. Even a better reason: If $f(x) = 0^x$ and we choose $x = -4$, we get an output of 0^{-4}, which is $1/0^4$, which is 1/0, which is verboten!

27. a. F b. T c. F d. T e. T f. F
g. F h. F i. F j. F k. T l. F

"The most beautiful experience we can have is the mysterious. It is the fundamental emotion which stands at the cradle of true art and true science."

Albert Einstein

INTRO TO EXPONENTS

❑ *INTRODUCTION*



You may have seen an exponent of 2, ***squaring***, in your previous math classes: The area of a circle is given by the formula $A = \pi r^2$, and the Pythagorean Theorem, $a^2 + b^2 = c^2$, has squaring in three different places. Even my favorite formula in the world, Albert Einstein's $E = mc^2$, contains squaring.



Some formulas involve ***cubing***. For example, the volume of a cube, where s is the length of a side, is given by $V = s^3$ (which is why raising something to the third power is called *cubing*). Even the unit "cubic meter" (the volume of a box one meter on all sides) is often written m^3.

Larger exponents are possible; consider the following examples:

$$3^4 = \underbrace{3 \cdot 3 \cdot 3 \cdot 3}_{\text{4 factors of 3}} = \mathbf{81}$$

$$2^5 = \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{\text{5 factors of 2}} = \mathbf{32}$$

$$1^{10} = \underbrace{1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1}_{\text{10 factors of 1}} = \mathbf{1}$$

$$(-4)^6 = \underbrace{(-4)(-4)(-4)(-4)(-4)(-4)}_{\text{6 factors of } -4} = \mathbf{4096}$$

$$(-2)^7 = \underbrace{(-2)(-2)(-2)(-2)(-2)(-2)(-2)}_{\text{7 factors of } -2} = \mathbf{-128}$$

Homework

1. Evaluate each expression:

a. the square of 7	b. the cube of 5	c. the cube of 3
d. the square of 20	e. the square of –3	f. the cube of –4
g. the square of 1	h. the cube of 0	i. the cube of 1
j. the square of 0	k. the cube of 10	l. the square of –10
m. 3^5	n. $(-7)^2$	o. 1000^2
p. $(-2)^4$	q. $(-1)^7$	r. $(-2)^5$
s. $(-1)^{20}$	t. 10^6	u. 5^4
v. 1^{12}	w. 2^8	x. 0^{34}
y. $(-3)^4$	z. $(-3)^5$	

❑ *STRANGE EXPONENTS*

Let's use the base of 2 to discuss some more exponents. We know what an exponent of 2 or higher means. These examples should be familiar:

$2^2 = 4$ $\qquad$ $2^4 = 16$ $\qquad$ $2^8 = 256$

But numbers like 2^1, 2^0, and 2^{-3} might seem a bit strange (or extremely strange!). Let's attack each kind of exponent one at a time.

An Exponent of One

What do we mean by 2^1? Since the exponent tells us how many times to use the base as a factor (for example, 2^{10} represents 10 factors of 2), it follows that $2^1 = 2$, and that's it.

An Exponent of Zero

Now for the interesting exponent of zero — what in the world could 2^0 possibly mean? If your first instinct is 0, then I might agree with you — but we'd both be wrong! Let's make a list of known ***powers of* 2**, look for a pattern, and then extend that pattern to figure out what 2^0 is.

$2^4 = 16$
$2^3 = 8$
$2^2 = 4$
$2^1 = 2$

The exponents in the first column are clearly decreasing by 1 at each step, and each number in the second column is one-half of the number above it.

Look at the pattern in the powers of 2 in the first column: The exponents are decreasing by 1 each time — the next power of 2 in the sequence ought to be $\mathbf{2^0}$. Now look at the sequence of numbers 16, 8, 4, and 2 in the second column. Each number is one-half the preceding number; that is, we're dividing by 2 at each step. Therefore, the next number in the sequence should be **1** (which is half of 2). So, continuing the two patterns extends the list above to the following:

$2^4 = 16$
$2^3 = 8$
$2^2 = 4$
$2^1 = 2$
$\mathbf{2^0 = 1}$

This leads to the conclusion that $2^0 = 1$ — a very strange result, indeed.

$$2^0 = 1$$

We can now evaluate expressions like the following:

$$2^0[2^0 + (-2)^1 + 2^2 + (-2)^3 - (-2)^4]$$

$$= \quad 1\,[1 + (-2) + 4 + (-8) - (+16)]$$

$$= \quad 1[1 - 2 + 4 - 8 - 16]$$

$$= \quad 1[-21]$$

$$= \quad \mathbf{-21}$$

Homework

2. Evaluate each expression:

a. $2^0 + 2^0$

b. $2^0 \cdot 2^0$

c. $2^0 + 2^1 + 2^2 + 2^3 + 2^4$

d. $(1 + 1)^0$

e. $2^5 - 2^3 + 2^1 - 2^0$

f. $(8 - 6)^0 + (10 - 8)^1$

g. $2^0 \times 2^1 \times 2^2 \times 2^3 \times 2^4$

h. $\left(\frac{12}{6}\right)^0 + \left(\frac{100}{50}\right)^0 - (20 - 15 - 3)^0 + \left(3^2 - 7\right)^1$

Negative Exponents

Recall the powers of 2 we wrote above:

$$2^4 = 16$$

$$2^3 = 8$$

$$2^2 = 4$$

$$2^1 = 2$$

$$2^0 = 1$$

Remember the patterns we observed in the above list? The exponents decrease by 1 each time, and the answers are divided by 2 each time. The next exponent in the sequence is –1, and the next answer must be $1 \div 2$, which equals $\frac{1}{2}$. We've reached the conclusion that

$$\boxed{2^{-1} = \frac{1}{2}}$$

The next exponent would be –2, and the next answer would be $\frac{1}{2}$ divided by 2, which is $\frac{1}{2} \div 2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. So now we see that

$$2^{-2} = \frac{1}{4}$$

Similarly, $2^{-3} = \frac{1}{8}$, $2^{-4} = \frac{1}{16}$, and so on.

Do you see what's really going on here? Can you predict what 2^{-10} is? Let's look closely at the fact that $2^{-4} = \frac{1}{16}$. Note that 16 is 2^4, meaning that we can write

$$2^{-4} = \frac{1}{2^4}$$

From this observation, it now should be reasonable to say that

$$2^{-10} = \frac{1}{2^{10}} = \frac{1}{1024}$$

Let's put everything we've seen into one big list:

$$2^4 = 16$$
$$2^3 = 8$$
$$2^2 = 4$$
$$2^1 = 2$$
$$2^0 = 1$$
$$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$$
$$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$
$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$
$$2^{-4} = \frac{1}{2^4} = \frac{1}{16}$$
$$\vdots$$
$$2^{-10} = \frac{1}{2^{10}} = \frac{1}{1024}$$
$$\vdots$$

Powers of 2

We can try to describe the effect of a negative exponent this way:

2 raised to a negative exponent is the fraction:
"1 over 2 raised to the positive exponent."

As an algebraic formula, we may write

$$2^{-n} = \frac{1}{2^n}$$

where $n = 1, 2, 3, \ldots$

Note: The base 2 raised to a negative exponent is a positive number — it may be small, but it's positive. Therefore,

2 raised to <u>any</u> power is always positive.

Homework

3. Evaluate each power of 2:

a. 2^5	b. 2^1	c. 2^0	d. 2^7	e. 2^8
f. 2^{-5}	g. 2^{-8}	h. 2^{-3}	i. 2^{-7}	j. 2^{-9}
k. 2^6	l. 2^{-1}	m. 2^{-4}	n. $2^0 + 2^1 + 2^2 + 2^{-1} + 2^{-2}$	

4. a. Explain why 1 is actually a power of 2.
   b. T/F: Every power of 2 is even.
   c. T/F: Every even number is a power of 2.
   d. T/F: 64 is a power of 2.
   e. T/F: 50 is a power of 2.

❑ *LAUREN'S ALLOWANCE*

Day	# of Pennies
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1,024
⋮	⋮
30	1,073,741,824 (almost $11 million)



Lauren tells her father that she will give up her allowance for the rest of her life — and will pay for all of her college and graduate school costs herself — if he will agree to the following plan: He gives her 2¢ on the first day of the upcoming month, 4¢ on the second day of the month, 8¢ on the third day of the month, 16¢ on the fourth day of the month, and so on until the 30th day of the month. Dad (who was a history major) thinks this is a great money-saving idea (after just 30 days, he's off the hook for allowance and for college!), and accepts Lauren's proposal.

In the chart we have calculated Lauren's earnings in pennies for each of the first 10 days of the month; then we cut to the chase and calculated the number of pennies for the 30th day. Take your calculator and extend the table for days 11 through 29 and prove that the amount of money for day 30 is correct.

Now let's come up with a direct formula that computes the money earned from knowing <u>just</u> the day of the month, <u>without having to know all the previous days' amounts</u>. Notice that each amount of money is simply 2 raised to the power of the day. For example, consider the 9th day. If we raise 2 to the 9th power (use the exponent button on your calculator), we get 512. That is, $2^9 = 512$. Now calculate 2^{30} and you should get 1,073,741,824.

Who's smarter, Lauren or her father?

Homework

5. a. How many pennies will Lauren earn on the 15th day of the month?

   b. How many pennies will Lauren earn on the 31st of the month if Dad agrees to extend the plan that far?

   c. On which day of the month did Lauren earn <u>half</u> of what she earned on the 30th day of the month?

6. Now let's reverse the question. I'll give you a penny amount, and you tell me which day of the month Lauren earned that amount of money.

   a. 512¢ b. 4,096¢ c. 1,048,576¢ d. 33,554,432¢

Review Problems

7. Evaluate each expression:

   a. the square of 5 b. the cube of –5 c. the square of –6

   d. 4^5 e. $(-2)^6$ f. $(-3)^3$

   g. $2^1 - 2^0 + 2^3 - 2^2 - (10 - 8)^0$

   h. $2^1 \times 2^2 \times 2^0$

   i. $\left(\frac{18}{9}\right)^0 \times \left(\frac{-50}{-25}\right)^1 + (1024 - 1022)^0 - (10^2 - 98)^1$

   j. 2^{-6} k. 2^{-1} l. 2^{-9} m. $2^{-1} + 2^{-2} + 2^{-3}$

8. True/False:

   a. 1 is a power of 2.

   b. 128 is a power of 2.

   c. 24 is a power of 2.

Solutions

1. a. 49 b. 125 c. 27 d. 400
e. 9 f. –64 g. 1 h. 0
i. 1 j. 0 k. 1000 l. 100
m. 243 n. 49 o. 1,000,000 p. 16
q. –1 r. –32 s. 1 t. 1,000,000
u. 625 v. 1 w. 256 x. 0
y. 81 z. –243

2. a. 2 b. 1 c. 31 d. 1
e. 25 f. 3 g. 1024 h. 3

3. a. 32 b. 2 c. 1 d. 128 e. 256
f. $\frac{1}{32}$ g. $\frac{1}{256}$ h. $\frac{1}{8}$ i. $\frac{1}{128}$ j. $\frac{1}{512}$
k. 64 l. $\frac{1}{2}$ m. $\frac{1}{16}$ n. $7\frac{3}{4}$

4. a. Since $1 = 2^0$, 1 is a power of 2.
b. False, since 1 is a power of 2.
c. False, since 6 is even but it's not a power of 2.
d. True
e. False

5. a. 32,768 b. 2,147,483,648 c. 29th of the month

6. a. day 9 b. day 12 c. day 20 d. day 25

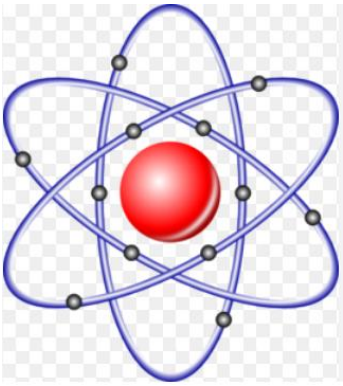
7. a. 25 b. –125 c. 36 d. 1024 e. 64
f. –27 g. 4 h. 8 i. 1 j. $\frac{1}{64}$
k. $\frac{1}{2}$ l. $\frac{1}{512}$ m. $\frac{7}{8}$

8. a. T b. T c. F

"Being defeated is often
a temporary condition.
Giving up is what
makes it permanent."

– *Marilyn Vos Savant*

THE FIVE LAWS OF EXPONENTS



Your chemistry teacher asks you to find the total mass of 3.01×10^{23} electrons assuming that the mass of one election is 9.1×10^{-31} kg. To solve this problem, you need to multiply the two numbers (which is explained in the chapter, *Scientific Notation*). Multiplying 3.01×9.1 is easy; just multiply the decimals. But how do we multiply $10^{23} \times 10^{-31}$? *What do we do with those exponents?* This chapter will answer that question.

❑ *EXPONENTS WITH NUMBERS*

Let's begin by avoiding variables for the moment and just stick to numbers so we can experiment with combining expressions containing exponents. We do this so that we understand, not merely memorize.

I. We want a way to simplify an expression like

$$2^3 \times 2^4$$

Maybe we multiply the 2's and multiply the 3 and the 4. Perhaps we don't multiply the 2's but still multiply the 3 and the 4. Or maybe we need to *add* the 3 and the 4. It's hard to know what to do unless you already know what to do! But we can figure out what to do if we just do the arithmetic:

$$2^3 \times 2^4 = 8 \times 16 = 128 = \mathbf{2^7}$$

From this example, it appears we simply keep the base (the 2) and *add* the exponents. Here's another example where this works:

$$3^2 \cdot 3^3 = 9 \cdot 27 = 243 = \mathbf{3^5}$$

I think we have a neat rule here: *To multiply powers of the same base, just keep the base and add the exponents.* This powerful rule means we can multiply powers without doing any heavy arithmetic; for example:

$$12^9 \times 12^{41} = \mathbf{12^{50}}$$ [Try that the long way!]

> **Important Note:** This shortcut is for multiplying powers of a base; it does not work if the operation is addition. Check this:
>
> $2^2 + 2^3 = 4 + 8 = 12$
>
> But 12 is not a nice power of 2 (since 2^3 is 8 and 2^4 is 16).
> So if you want to add exponents, be sure that the bases are the same and that the operation is multiplication.

II. Now we study what to do with a "power of a power." For example, we might come across $\left(2^2\right)^3$. Do we add the exponents, or maybe multiply them? Let's just work it out and see:

$$\left(2^2\right)^3 = 4^3 = 64 = 2^6$$

It appears we just multiply the exponents. This means we can simplify something like $\left(5^7\right)^{10}$ by just raising 5 to the product of 7 and 10:

$\left(5^7\right)^{10} = 5^{70}$, and we're done!

III. It's time for a **power of a product**. We can try this:

$$(2\times 3)^4 \;=\; 6^4 \;=\; 1{,}296 \qquad \text{(Order of Operations)}$$

But here's another way to get the same result:

$$2^4 \times 3^4 \;=\; 16\times 81 \;=\; 1{,}296$$

Do we raise *a product to a power* merely by raising each factor to the power?

IV. Let's try a **quotient of powers**. Here's an example:

$$\frac{10^6}{10^2} \;=\; \frac{1{,}000{,}000}{100} \;=\; 10{,}000 \qquad \text{(Order of Operations)}$$

Now watch this: If we subtract the exponents and keep the base of 10, we get 10^4, which is also equal to 10,000.

Let's do a second example where we put the bigger exponent on the bottom:

$$\frac{2^3}{2^7} \;=\; \frac{8}{128} \;=\; \frac{1}{16}$$

But $\frac{1}{16}$ can also be written as $\frac{1}{2^4}$, where the 4 results from subtracting the exponents. There's something going on here.

V. Our fifth example in this section will look at a **power of a quotient**.

$$\left(\frac{2}{3}\right)^4 \;=\; \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) \;=\; \frac{16}{81} \qquad \text{(the meaning of exponent)}$$

But if we just raise 2 to the 4th power, and then raise 3 to the 4th power, and write one over the other, we get the same result:

$$\frac{2^4}{3^4} \;=\; \frac{16}{81}$$

❑ *EXPONENTS WITH VARIABLES*



For each of the following five examples, we will "stretch-and-squish," and then we'll generalize what we observe to the official ***Five Laws of Exponents***.

I. We start by finding the product of x^3 and x^4:

$$x^3x^4 = (xxx)(xxxx) = xxxxxxx = x^7$$

Notice that the bases (the x's) are the same, and it's a multiplication problem. As long as the bases are the same, and it's a multiplication problem, it appears that we merely need to write down the base, and then add the exponents together to get the exponent of the answer. That is, $x^a x^b = x^{a+b}$.

$$x^3x^4 = x^7$$

[This law of exponents, where we multiply powers of the same base by adding the exponents, is almost always called the First Law of Exponents. The four laws that follow are not in any particular order.]

II. For our second example, let's raise a power to a power:

$$(x^4)^2 = (xxxx)^2 = (xxxx)(xxxx) = xxxxxxxx = x^8$$

We appear to have a shortcut at hand. Merely multiply the two exponents together and we're done. So, to raise a power to a power, we can write a general rule: $(x^a)^b = x^{ab}$.

$$(x^4)^2 = x^8$$

III. Now we're to try raising a product to a power; for instance,

$$(ab)^5 = (ab)(ab)(ab)(ab)(ab) = (aaaaa)(bbbbb) = a^5b^5$$

In general, when raising a product to a power, raise each factor to the power: $(xy)^n = x^n y^n$.

$$(ab)^5 = a^5b^5$$

Note that the quantity in the parentheses is a single term — there's no adding or subtracting in the parentheses. In fact, if there are two or more terms in the parentheses, this law of exponents does not apply.

IV. Next we divide powers of the same base. We'll need two examples for this law of exponents.

A. $\dfrac{x^6}{x^2} = \dfrac{xxxxxx}{xx} = \dfrac{\cancel{x}\,\cancel{x}\,xxxx}{\cancel{x}\,\cancel{x}} = x^4$

B. $\dfrac{y^4}{y^6} = \dfrac{yyyy}{yyyyyy} = \dfrac{\cancel{y}\,\cancel{y}\,\cancel{y}\,\cancel{y}}{\cancel{y}\,\cancel{y}\,\cancel{y}\,\cancel{y}\,yy} = \dfrac{1}{y^2}$

$$\frac{x^6}{x^2} = x^4 \qquad \frac{y^4}{y^6} = \frac{1}{y^2}$$

In general, when dividing powers of the same base, subtract the exponents, leaving the remaining factors on the top if the top exponent is bigger, and on the bottom if the bottom exponent is bigger.

V. Our last example in this section is the process of raising a quotient to a power. As usual, we stretch and squish; then we generalize to a law of exponents.

$$\left(\frac{a}{b}\right)^4 = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) = \frac{aaaa}{bbbb} = \frac{a^4}{b^4}$$

In general, we can raise a quotient to a power by raising both the top and bottom to the power: $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

$$\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$$

❑ *SUMMARY OF THE FIVE LAWS OF EXPONENTS*

Exponent Law	Example
$x^a x^b = x^{a+b}$	$x^2 x^6 = x^8$
$(x^a)^b = x^{ab}$	$(a^4)^3 = a^{12}$
$(xy)^n = x^n y^n$	$(wz)^7 = w^7 z^7$
For $a > b$, $\frac{x^a}{x^b} = x^{a-b}$ For $b > a$, $\frac{x^a}{x^b} = \frac{1}{x^{b-a}}$	$\frac{x^{10}}{x^2} = x^8$ $\frac{a^3}{a^7} = \frac{1}{a^4}$
$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	$\left(\frac{a}{b}\right)^5 = \frac{a^5}{b^5}$

❑ *SINGLE-STEP EXAMPLES*

EXAMPLE 1 [Let's call this law the *First Law of Exponents*.]

A. $A^7 A^5 = A^{7+5} = A^{12}$

The bases are the same, and it's a multiplication problem. So we repeat the base and add the exponents.

B. $x^2 x^3 x^4 = x^{2+3+4} = x^9$

All the bases are the same, and it's a multiplication problem, and so we add the exponents.

C. $(x+y)^4\,(x+y)^9 \;=\; (x+y)^{13}$

It doesn't matter what the base is, as long as we're multiplying powers of the same base.

EXAMPLE 2:

A. $(c^{10})^2 \;=\; c^{20}$

Raising a base to a power, and then raising that result to a further power requires simply that we multiply the exponents.

B. $\left((x^2)^3\right)^4 \;=\; x^{24}$

$(x^2)^3 \;=\; x^6$, and then $(x^6)^4 \;=\; x^{24}$
Shortcut: Just multiply all three exponents.

EXAMPLE 3:

A. $(ax)^5 \;=\; a^5x^5$

It's a power of a product (a single term). So just raise each factor to the 5th power.

B. $(abc)^7 \;=\; a^7b^7c^7$

Even a term with three factors can be raised to the 7th power by raising each factor to the 7th power.

EXAMPLE 4:

A. $\frac{x^7}{x^5} = x^2$

Since $7 > 5$, we divide powers of the same base by subtracting the exponents.

B. $\frac{w^{15}}{w^{25}} = \frac{1}{w^{10}}$

Since the bigger exponent is on the bottom, we subtract 15 from 25 and leave that power of w on the bottom.

EXAMPLE 5:

A. $\left[\frac{x}{z}\right]^7 = \frac{x^7}{z^7}$

To raise a quotient to a power, just raise both the top and bottom to the 7th power.

B. $\left(\frac{a+b}{u-w}\right)^{23} = \frac{(a+b)^{23}}{(u-w)^{23}}$

Just raise top and bottom to the 23rd power.

Homework

1. Use the Five Laws of Exponents to simplify each expression:

a. a^3a^4 b. $x^5x^6x^2$ c. y^3y^3 d. $z^{12}z$

e. $(x^3)^4$ f. $(z^8)^2$ g. $(n^{10})^{10}$ h. $(a^1)^7$

i. $(ab)^3$ j. $(xyz)^5$ k. $(RT)^1$ l. $(math)^5$

m. $\frac{a^8}{a^2}$ n. $\frac{b^3}{b^9}$ o. $\frac{w^5}{w^5}$ p. $\frac{Q^{100}}{Q^{50}}$

q. $\left(\frac{k}{w}\right)^4$ r. $\left(\frac{a}{b}\right)^{99}$ s. $\left(\frac{1}{m}\right)^{20}$ t. $a(bc)^2$

2. Use the Five Laws of Exponents to simplify each expression:

a. a^3a^5 b. $u^5u^7u^2$ c. $y^{30}y^{30}$ d. $z^{14}z$

e. $(x^4)^5$ f. $(z^9)^2$ g. $(n^{100})^{10}$ h. $(a^1)^9$

i. $(xy)^4$ j. $(abc)^{17}$ k. $(pn)^1$ l. $(love)^4$

m. $\frac{a^{10}}{a^2}$ n. $\frac{b^3}{b^{12}}$ o. $\frac{w^9}{w^9}$ p. $\frac{Q^{100}}{Q^{20}}$

q. $\left(\frac{x}{w}\right)^3$ r. $\left(\frac{a}{b}\right)^{999}$ s. $\left(\frac{1}{z}\right)^{22}$ t. $w(xy)^3$

❑ *WHEN NOT TO USE THE FIVE LAWS OF EXPONENTS*

$\boldsymbol{a^5b^6}$ cannot be simplified. Although the First Law of Exponents demands that the expressions be multiplied — and they are — it also requires that the bases be the same — and they aren't.

$\boldsymbol{x^3 + x^4}$ cannot be simplified. Even though the bases are the same, the First Law of Exponents requires that the two powers of x be <u>multiplied</u>.

$\boldsymbol{w^3 + w^3}$ can be simplified, but not by the First Law of Exponents, since the powers of w are not being multiplied. But the two terms are *like terms*, which means we add them together to get $\boldsymbol{2w^3}$.

$\boldsymbol{(a + b)^{23}}$ does not equal $a^{23} + b^{23}$. You may think that the third law of exponents, $(xy)^n = x^n y^n$, might apply, but it does not, and that's because xy is a single term, whereas $a + b$ consists of two terms. You'll have to wait until the chapter entitled *The Binomial Theorem* to learn a clever way to calculate the 23rd power of $a + b$. Also, you may have already learned in this class that $(a + b)^2$ is actually equal to $a^2 + 2ab + b^2$, and so again, $(a + b)^n \neq a^n + b^n$ (for $n \geq 2$).

Homework

3. Simplify each expression:

a. $y^4 y^4$	b. $a^3 b^4$	c. $x^4 x^3 x^2$	d. $p^3 t^2 p^2$
e. $a^3 + a^5$	f. $a^3 a^5$	g. $n^4 + n^4$	h. $x^3 - x^3$
i. $(x + y)^{55}$	j. $Q^2 + Q^2$	k. $u^5 w^6$	l. $h^6 - h^2$
m. $(a - b)^2$	n. $(ab)^2$	o. $(x^3)^3$	p. $x^4 + x^5$
q. $x^{14} + x^{14}$	r. $y^{12} - y^{12}$	s. $a^8 + a^9$	t. $a^{10} + a^{10}$
u. $(xy)^2$	v. $(x + y)^2$	w. $a^3 b^4$	x. $a^3 + b^4$
y. $a(a^2)(b^2)b$	z. $n^6 + n^6$		

❑ *MULTI-STEP EXAMPLES*

EXAMPLE 6:

A. $(-3x^2y^3)(-5xy^7) = (-3)(-5)\,(x^2x)(y^3y^7) = 15x^3y^{10}$

B. $-2x^2y(xy - 4x^3y^4) = -2x^3y^2 + 8x^5y^5$ (distribute)

C. $\left(2a^2b^3\right)^4 = 2^4(a^2)^4(b^3)^4 = 16a^8b^{12}$ (the 2 is in the parentheses)

D. $7\left(xy^{10}\right)^5 = 7x^5(y^{10})^5 = 7x^5y^{50}$ (the 7 is not in the parentheses)

E. $\left(\frac{a^2}{b^3}\right)^7 = \frac{(a^2)^7}{(b^3)^7} = \frac{a^{14}}{b^{21}}$

F. $\left(\frac{x^3y^9}{xy^{12}}\right)^5 = \left(\frac{x^2}{y^3}\right)^5 = \frac{(x^2)^5}{(y^3)^5} = \frac{x^{10}}{y^{15}}$ (simplify the inside first)

Homework

4. Simplify each expression:

a. $(-5a^3b^4)(-2a^2b)$

b. $(7xy)(-7x^2y^5)$

c. $(-2uw)(2uw)$

d. $x^3(2x^2 - x - 1)$

e. $3y^2(3y^2 - y + 3)$

f. $\left(a^2b^3\right)^4$

g. $\left(-5m^3n^{10}\right)^3$

h. $\left[-3p^3q^3\right]^4$

i. $4\left(xy^7\right)^{10}$

j. $-10(-2c^3y^4)^3$

k. $\left(\frac{a^3}{c^2}\right)^{10}$

l. $\left[\frac{2x^3}{3xy^4}\right]^3$

m. $\left(\dfrac{a^2b^3}{a^4b}\right)^5$ n. $2(3x^2y^3)^4$

❑ ZERO AS AN EXPONENT

You may have already learned that anything to the zero power is **1** (as long as it's not zero to the zero). Now that we have some laws of exponents at our disposal, we can actually prove this fact; that is, we can prove that $\boldsymbol{x^0 = 1}$.

Consider the fraction $\dfrac{x^6}{x^6}$, where $x \neq 0$. On the one hand, anything (except zero) divided by itself is 1:

$$\frac{x^6}{x^6} = \underline{1}$$

On the other hand, by one of the exponent laws, since it's division of powers of the same base, we can **subtract** the exponents:

$$\frac{x^6}{x^6} = \underline{x^0}$$

See the logic? Since $\dfrac{x^6}{x^6}$ is equal to 1 and it's also equal to x^0, it must follow that

$$\boldsymbol{x^0 = 1}$$

Any number (except 0) raised to the 0 power is 1.

EXAMPLE 7:

A. $(x - 3y + z)^0 = \mathbf{1}$ (any quantity ($\neq 0$) to the zero power is 1)

B. $(abc)^0 = \mathbf{1}$ (any quantity ($\neq 0$) to the zero power is 1)

C. $a + b^0 = \boldsymbol{a + 1}$ (the exponent is on the b only)

D. $uw^0 = u(1) = \boldsymbol{u}$ (the exponent is on the w only)

E. $(-187)^0 = \mathbf{1}$ (the exponent is on the –187)

F. $-14^0 = \mathbf{-1}$ (the exponent in on the 14, not on the minus sign)

Homework

5. Evaluate each expression:

a. $2^0 + 3^0$ b. $4^0 \cdot 5^0$ c. $6^0 + 2^1 + 2^2 + 2^3 + 2^4$

d. $(1 + 12)^0$ e. $2^5 - 2^3 + 2^1 - 2^0$ f. $(8 - 5)^0 + (10 - 8)^1$

g. $2^0 \times 2^1 \times 2^2 \times 2^3 \times 2^4$

h. $\left(\frac{12}{29}\right)^0 + \left(\frac{100}{77}\right)^0 - (20 - \pi - 3)^0 + \left(3^2 - 7\right)^1$

6. Simplify each expression:

a. x^0 b. xy^0 c. $x + y^0$ d. $(x + y)^0$

e. $(ab)^0$ f. $\left(\frac{a^2}{b^3}\right)^0$ g. $\frac{(x^2)^0}{y^3}$ h. $m^0 m$

i. $x^0 + x^0$ j. $Q^0 Q^0$ k. $a^0 - a^0$ l. $\left(\frac{2x^2y^0}{-3ab^{10}}\right)^0$

❑ *FINAL NOTE*

The Introduction to this chapter talked about working the following problem:

$$10^{23} \times 10^{-31}$$

The bases are the same (10) and the operation is multiplication. Thus, the First Law of Expoents says we can bring the 10 along and <u>add</u> the exponents:

10^{-8} (which equals 0.00000001)

Review Problems

7. Simplify: $\left(-3x^3y^4x^7\right)^3$

8. Simplify: $-3\left(x^5x^4x^8\right)^3$

9. Simplify: $\dfrac{a^2b^3c^9}{ab^4c^3}$

10. Simplify: $-2y^3(3y^4 - 2y^3 + 1)$

11. Simplify: $x^{12} + x^{14}$

12. Simplify: $u^{22} + u^{22}$

13. Simplify: $abcd^0e^0$

14. Simplify: $\left[\dfrac{10^0a^0b^{14}}{a^3b^7}\right]^5$

Solutions

1. a. a^7 b. x^{13} c. y^6 d. z^{13}
e. x^{12} f. z^{16} g. n^{100} h. a^7
i. a^3b^3 j. $x^5y^5z^5$ k. RT l. $m^5a^5t^5h^5$
m. a^6 n. $\frac{1}{b^6}$ o. 1 p. Q^{50}
q. $\frac{k^4}{w^4}$ r. $\frac{a^{99}}{b^{99}}$ s. $\frac{1}{m^{20}}$ t. ab^2c^2

2. a. a^8 b. u^{14} c. y^{60} d. z^{15}
e. x^{20} f. z^{18} g. n^{1000} h. a^9
i. x^4y^4 j. $a^{17}b^{17}c^{17}$ k. pn l. $l^4o^4v^4e^4$
m. a^8 n. $\frac{1}{b^9}$ o. 1 p. Q^{80}
q. $\frac{x^3}{w^3}$ r. $\frac{a^{999}}{b^{999}}$ s. $\frac{1}{z^{22}}$ t. wx^3y^3

3. a. y^8 b. As is c. x^9 d. p^5t^2
e. As is f. a^8 g. $2n^4$ h. 0
i. As is (for now) j. $2Q^2$ k. As is l. As is
m. $a^2 - 2ab + b^2$ n. a^2b^2 o. x^9 p. As is
q. $2x^{14}$ r. 0 s. As is t. $2a^{10}$
u. x^2y^2 v. $x^2 + 2xy + y^2$ w. As is
x. As is y. a^3b^3 z. $2n^6$

4. a. $10a^5b^5$ b. $-49x^3y^6$ c. $-4u^2w^2$ d. $2x^5 - x^4 - x^3$
e. $9y^4 - 3y^3 + 9y^2$ f. a^8b^{12} g. $-125m^9n^{30}$
h. $81p^{12}q^{12}$ i. $4x^{10}y^{70}$ j. $80c^9y^{12}$ k. $\frac{a^{30}}{c^{20}}$

l. $\frac{8x^6}{27y^{12}}$ m. $\frac{b^{10}}{a^{10}}$ n. $162x^8y^{12}$

5. a. 2 b. 1 c. 31 d. 1
e. 25 f. 3 g. 1024 h. 3

6. a. 1 b. x c. $x + 1$ d. 1 e. 1 f. 1
g. $\frac{1}{y^3}$ h. m i. 2 j. 1 k. 0 l. 1

7. $-27x^{30}y^{12}$ **8.** $-3x^{51}$ **9.** $\frac{ac^6}{b}$

10. $-6y^7 + 4y^6 - 2y^3$ **11.** As is **12.** $2u^{22}$

13. abc **14.** $\frac{b^{35}}{a^{15}}$

I have no particular talent.
I am merely inquisitive.

Albert Einstein

NEGATIVE EXPONENTS

The mass of a proton (the positively charged particle in the nucleus of every atom in the universe) is written in *scientific notation* as

$$\mathbf{1.67 \times 10^{-27}\ kilograms}$$

How do we interpret that? Is that a really big or a really small number? In this chapter, we will learn exactly what a negative exponent means.



❑ REVIEW OF THE FIVE LAWS OF EXPONENTS

Before we begin the discussion of negative exponents, it would be beneficial for us to review the meaning of an exponent and the Five Laws of exponents:

If n is a natural number (n = 1, 2, 3, . . .), then we know that the meaning of x^n is based upon repeated multiplication:

$$x^n = \underbrace{(x)(x)(x)\cdots(x)}_{n \text{ factors of } x}$$

For instance, we know that x^3 means $x \cdot x \cdot x$. We also learned that $x^0 = 1$ (as long as x itself isn't 0). But what does something like x^{-3} mean?

Now we recall the Five Laws of Exponents. The order doesn't make any difference, but I personally refer to the first one as the First Law of Exponents.

$$x^a x^b = x^{a+b} \qquad \frac{x^a}{x^b} = x^{a-b} \qquad (x^a)^b = x^{ab}$$

$$(xy)^a = x^a y^a \qquad \left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

❑ DEVELOPING THE MEANING OF A NEGATIVE EXPONENT

Consider the following power of x:

$$x^{-3}$$

What could it mean? To find out, let's multiply it by x^3 and see what happens:

$$x^{-3} \cdot \mathbf{x^3} = x^{-3+3} = x^0 = 1$$

which implies that

$$x^{-3} \cdot x^3 = 1$$

Now, x^{-3} is what we're analyzing, so let's "solve" for it (or isolate it) by dividing each side of the equation by x^3:

$$\frac{x^{-3} \cdot x^3}{x^3} = \frac{1}{x^3}$$

$$\Rightarrow \quad \boxed{x^{-3} = \frac{1}{x^3}}$$

We can do the same thing in general. Consider x^{-n}, where n is any number at all:

$$x^{-n} \cdot x^n = x^{-n+n} = x^0 = 1 \qquad \Rightarrow \quad x^{-n} = \frac{1}{x^n}$$

In short,

$$x^{-n} = \frac{1}{x^n}$$

for ANY number n, assuming $x \neq 0$.

This should convince you that a negative exponent means **reciprocal**. In fact, the exponent doesn't have to be a negative whole number. [Even things like $w^{-5/4}$ and $h^{-\sqrt{2\pi}}$ represent reciprocals; in fact, $w^{-5/4} = \frac{1}{w^{5/4}}$ and $h^{-\sqrt{2\pi}} = \frac{1}{h^{\sqrt{2\pi}}}$.]

This fact, together with our knowledge of fractions and the Laws of Exponents, is all we need to understand the examples in this chapter.

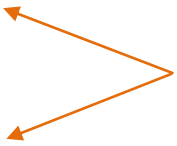
EXAMPLE 1:

A. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

B. $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$

C. $-3^{-4} = -\frac{1}{3^4} = -\frac{1}{81}$

D. $(-3)^{-4} = \frac{1}{(-3)^4} = \frac{1}{81}$



See the difference?

E. $\frac{1}{2^{-5}} = \frac{1}{\frac{1}{2^5}} = \frac{1}{\frac{1}{32}} = \frac{1}{1} \cdot \frac{32}{1} = 32$

F. $\dfrac{1}{10^{-3}} = \dfrac{1}{\frac{1}{10^3}} = \dfrac{1}{1} \cdot \dfrac{10^3}{1} = 10^3 = 1000$

G. $\left(\dfrac{2}{3}\right)^{-3} = \dfrac{1}{\left(\frac{2}{3}\right)^3} = \dfrac{1}{\frac{8}{27}} = \dfrac{1}{1} \cdot \dfrac{27}{8} = \dfrac{27}{8}$

H. $0^{-5} = \dfrac{1}{0^5} = \dfrac{1}{0} =$ **Undefined**

Homework

1. Evaluate each expression:

a. 3^{-2} b. 2^{-5} c. 10^{-3} d. 5^{-4} e. 1^{-10}

f. -2^{-3} g. $(-3)^{-3}$ h. -10^{-4} i. $(-5)^{-1}$ j. -6^{-2}

k. $\dfrac{1}{3^{-2}}$ l. $\dfrac{1}{10^{-4}}$ m. $\dfrac{1}{5^{-3}}$ n. $\left(\dfrac{3}{4}\right)^{-2}$ o. $\left(\dfrac{1}{6}\right)^{-3}$

2. Explain why $x = 0$ is NOT allowed in the expression x^{-8}.

EXAMPLE 2:

A. $n^{-1} = \dfrac{1}{n}$

B. $a^{-12} = \dfrac{1}{a^{12}}$

C. $\frac{x}{y^{-3}} = \frac{x}{\frac{1}{y^3}} = x \cdot \frac{y^3}{1} = xy^3$

D. $\frac{x^{-7}}{a^{12}} = \frac{\frac{1}{x^7}}{a^{12}} = \frac{1}{x^7} \cdot \frac{1}{a^{12}} = \frac{1}{a^{12}x^7}$

E. $a^2b^{-3} = a^2 \cdot \frac{1}{b^3} = \frac{a^2}{b^3}$

F. $R^{-1}T^{-7} = \frac{1}{R} \cdot \frac{1}{T^7} = \frac{1}{RT^7}$

EXAMPLE 3:

A. $x + y^{-5} = x + \frac{1}{y^5} = \underbrace{\left[\frac{y^5}{y^5}\right]\frac{x}{1} + \frac{1}{y^5}}_{\text{to make a common denominator}} = \frac{xy^5}{y^5} + \frac{1}{y^5} = \frac{xy^5 + 1}{y^5}$

B. $a^{-2} + b^{-7} = \frac{1}{a^2} + \frac{1}{b^7} = \frac{1}{a^2}\left[\frac{b^7}{b^7}\right] + \frac{1}{b^7}\left[\frac{a^2}{a^2}\right] = \frac{b^7 + a^2}{a^2b^7}$

Homework

3. Simplify each expression (no negative exponents in the answer):

a. x^{-9} b. a^{-30} c. $\frac{a}{b^{-2}}$ d. $\frac{x^{-4}}{y^2}$

e. $\frac{w^3}{x^{-3}}$ f. $\frac{m^{-3}}{n^{-4}}$ g. $a^{-4}b^{-5}$ h. k^3p^{-5}

i. $h^{-5}z^{7}$ j. $\dfrac{w^{-5}}{w^{-5}}$ k. $x+y^{-1}$ l. $a^{-3}+b^{2}$

m. $p^{-1}+r^{-1}$ n. $w^{-3}-z^{-2}$ o. $a^{-3}-a^{-3}$ p. $a^{-1}b^{-2}c^{-3}$

EXAMPLE 4: **Using the Five Laws of Exponents:**

A. $x^{-6}x^{-8} = x^{-14} = \boldsymbol{\dfrac{1}{x^{14}}}$

Add the exponents

B. $\left(N^{-5}\right)^{-7} = \boldsymbol{N^{35}}$

Multiply the exponents

C. $(abc)^{-4} = a^{-4}b^{-4}c^{-4} = \dfrac{1}{a^4}\cdot\dfrac{1}{b^4}\cdot\dfrac{1}{c^4} = \boldsymbol{\dfrac{1}{a^4b^4c^4}}$

Apply exponent to
each factor

D. $\dfrac{w^{5}}{w^{15}} = w^{5-15} = w^{-10} = \boldsymbol{\dfrac{1}{w^{10}}}$

Subtract the exponents

E. $\dfrac{x^{-17}}{x^{-12}} = x^{-17-(-12)} = x^{-17+12} = x^{-5} = \boldsymbol{\dfrac{1}{x^{5}}}$

Subtract the exponents

F. $\left(\dfrac{a}{b}\right)^{-3} = \dfrac{a^{-3}}{b^{-3}} = \dfrac{\frac{1}{a^3}}{\frac{1}{b^3}} = \dfrac{1}{a^3}\cdot\dfrac{b^3}{1} = \boldsymbol{\dfrac{b^3}{a^3}}$

Apply exponent to
top and bottom

Another way: $\left(\frac{a}{b}\right)^{-3} = \frac{1}{\left(\frac{a}{b}\right)^{3}} = \frac{1}{\frac{a^3}{b^3}} = \frac{b^3}{a^3}$

G. $(x+y)^{-2}$

Be careful; no law of exponents applies here since the quantity $x + y$ consists of <u>two</u> terms — it is not a product — so do NOT apply the exponent to each term of the sum. Begin the problem by dealing with the meaning of the negative exponent.

$$(x+y)^{-2} = \frac{1}{(x+y)^2} = \frac{1}{(x+y)(x+y)} = \frac{\mathbf{1}}{\boldsymbol{x^2 + 2xy + y^2}}$$

Homework

4. Simplify each expression (no negative exponents in the answer):

a. $a^{-3}a^{7}$ b. $x^{-5}x^{-10}$ c. $k^{-10}k^{10}$ d. $m^{-5}w^{-3}$

e. $(x^2)^{-5}$ f. $(a^{-5})^{3}$ g. $(c^{-3})^{-4}$ h. $(b^2)^{-1}$

i. $(ax)^{-5}$ j. $(xyz)^{-3}$ k. $(x+y)^{-1}$ l. $(a-b)^{-2}$

m. $\frac{x^{-5}}{x^3}$ n. $\frac{z^5}{z^{-4}}$ o. $\frac{a^{-5}}{a^{-8}}$ p. $\frac{y^{-12}}{y^{-7}}$

q. $\left(\frac{a}{b}\right)^{-4}$ r. $\left(\frac{y}{x}\right)^{-1}$ s. $\left(\frac{a}{b+c}\right)^{-2}$ t. $\left(\frac{a+b}{xy+wz}\right)^{0}$

EXAMPLE 5: **Simplify each expression:**

A. $(2x^{-8})(-5x^{6}) = (2)(-5)x^{-8}x^{6} = -10x^{-2} = -10\left(\frac{1}{x^{2}}\right) = -\frac{\mathbf{10}}{\boldsymbol{x^{2}}}$

B. $\left(x^{2}y^{-3}\right)^{4} = \left(x^{2}\right)^{4}\left(y^{-3}\right)^{4} = x^{8}y^{-12} = x^{8}\left(\frac{1}{y^{12}}\right) = \frac{\boldsymbol{x^{8}}}{\boldsymbol{y^{12}}}$

C. $\left(\frac{a^{-3}}{b^{4}}\right)^{-5} = \frac{\left(a^{-3}\right)^{-5}}{\left(b^{4}\right)^{-5}} = \frac{a^{15}}{b^{-20}} = \frac{a^{15}}{\frac{1}{b^{20}}} = \frac{a^{15}}{1}\cdot\frac{b^{20}}{1} = \boldsymbol{a^{15}b^{20}}$

Final Note: In the Introduction, we learned about the mass of a proton. We're now in a position to see just what that number is:

$1.67\times\mathbf{10^{-27}} = 1.67\times\frac{\mathbf{1}}{\mathbf{10^{27}}} = \frac{1.67}{1}\times\frac{1}{10^{27}} = \frac{1.67}{10^{27}}$

$= \frac{1.67}{1{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000}$

$=$ 0.0000000000000000000000000167 kilograms

Shortcut: Move the decimal point 27 places to the left.

[If available, check out the chapter on *Scientific Notation* for all the details and shortcuts regarding numbers like 1.67×10^{-27}.]

A proton certainly doesn't weigh very much! In addition, I hope you can appreciate why we use scientific notation (1.67×10^{-27}) rather than the number in the box.

Homework

5. Simplify each expression (no negative exponents in the answer):

a. $(2x^{-4})(3x^{7})$ b. $(-3y^{-3})(2y^{-5})$ c. $(-4a^{3})(a^{-7})$

d. $\left(u^{3}u^{-5}\right)^{-2}$ e. $\left(w^{-5}w^{-1}\right)^{7}$ f. $\left(a^{-2}b^{-3}\right)^{-5}$

g. $\left(t^{3}u^{-4}\right)^{3}$ h. $\left(\frac{x^{2}}{x^{5}}\right)^{-2}$ i. $\left(\frac{y^{-3}}{y^{-4}}\right)^{5}$

j. $\left(\frac{a^{-2}}{b^{5}}\right)^{-3}$ k. $\left(\frac{c^{3}}{d^{-4}}\right)^{-5}$ l. $\left(\frac{a^{3}b^{-4}}{x^{-12}z^{-2}}\right)^{0}$

6. Express each scientific notation number as a regular number:

a. 2.3×10^{7} b. 7.11×10^{-5} c. 5.09×10^{-10}

Review Problems

7. Evaluate: a. -2^{-4} b. $(-2)^{-4}$

8. Evaluate: a. $\left(\frac{4}{5}\right)^{-3}$ b. $\left(\frac{1}{9}\right)^{-1}$

9. Simplify: a. $\frac{a^{-2}}{b^{-3}}$ b. $\frac{x^{-9}}{x^{9}}$

10. Simplify: a. $a^{-2}a^{-3}$ b. $a^{-2} - a^{-3}$

11. Simplify: a. $(x^{-10})^5$ b. $(z^{-5})^{-5}$

12. Simplify: $(3x^{-3}y^{-2})^{-1}(-4x^3y^{10})^{-2}$

13. Simplify: a. $\left(\frac{x^{-4}}{x^{-6}}\right)^{-2}$ b. $\left(\frac{a^3}{b^{-4}}\right)^5$

14. Simplify: a. $(uw)^{-2}$ b. $(u + w)^{-2}$

15. Explain why $x = 0$ is NOT allowed in the expression x^{-2}.

16. Express as a regular number: 4.9×10^{-13}

Solutions

1. a. $\frac{1}{9}$ b. $\frac{1}{32}$ c. $\frac{1}{1000}$ d. $\frac{1}{625}$ e. 1 f. $-\frac{1}{8}$
g. $-\frac{1}{27}$ h. $-\frac{1}{10{,}000}$ i. $-\frac{1}{5}$ j. $-\frac{1}{36}$ k. 9 l. 10,000
m. 125 n. $\frac{16}{9}$ o. 216

2. If x were 0 in the expression x^{-8}, we would have
$x^{-8} = 0^{-8} = \frac{1}{0^8} = \frac{1}{0}$, which is Undefined.

3. a. $\frac{1}{x^9}$ b. $\frac{1}{a^{30}}$ c. ab^2 d. $\frac{1}{x^4y^2}$ e. x^3w^3 f. $\frac{n^4}{m^3}$
g. $\frac{1}{a^4b^5}$ h. $\frac{k^3}{p^5}$ i. $\frac{z^7}{h^5}$ j. 1 k. $\frac{xy+1}{y}$ l. $\frac{1+a^3b^2}{a^3}$

m. $\frac{r+p}{pr}$ n. $\frac{z^2-w^3}{w^3z^2}$ o. 0 p. $\frac{1}{ab^2c^3}$

4. a. a^4 b. $\frac{1}{x^{15}}$ c. 1 d. $\frac{1}{m^5w^3}$ e. $\frac{1}{x^{10}}$

f. $\frac{1}{a^{15}}$ g. c^{12} h. $\frac{1}{b^2}$ i. $\frac{1}{a^5x^5}$ j. $\frac{1}{x^3y^3z^3}$

k. $\frac{1}{x+y}$ l. $\frac{1}{a^2-2ab+b^2}$ m. $\frac{1}{x^8}$ n. z^9

o. a^3 p. $\frac{1}{y^5}$ q. $\frac{b^4}{a^4}$ r. $\frac{x}{y}$ s. $\frac{b^2+2bc+c^2}{a^2}$

t. 1

5. a. $6x^3$ b. $-\frac{6}{y^8}$ c. $-\frac{4}{a^4}$ d. u^4 e. $\frac{1}{w^{42}}$ f. $a^{10}b^{15}$

g. $\frac{t^9}{u^{12}}$ h. x^6 i. y^5 j. a^6b^{15} k. $\frac{1}{c^{15}d^{20}}$ l. 1

6. a. 23,000,000 b. 0.0000711 c. 0.0000000000509

7. a. $-\frac{1}{16}$ b. $\frac{1}{16}$ **8**. a. $\frac{125}{64}$ b. 9

9. a. $\frac{b^3}{a^2}$ b. $\frac{1}{x^{18}}$ **10**. a. $\frac{1}{a^5}$ b. $\frac{a-1}{a^3}$

11. a. $\frac{1}{x^{50}}$ b. z^{25} **12**. $\frac{1}{48x^3y^{18}}$

13. a. $\frac{1}{x^4}$ b. $a^{15}b^{20}$ **14**. a. $\frac{1}{u^2w^2}$ b. $\frac{1}{u^2+2uw+w^2}$

15. Because $0^{-2} = \frac{1}{0^2} = \frac{1}{0}$ which is Undefined.

16. 0.00000000000049

"A life spent making mistakes is not only more honorable, but more useful than a life spent doing nothing."

– ***George Bernard Shaw***

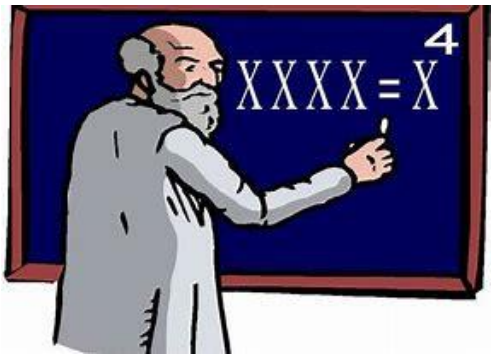
FRACTIONAL EXPONENTS

We know what n^5 means $(n \cdot n \cdot n \cdot n \cdot n)$, and we might know what x^{-3} means $(1/x^3)$. You should also commit to memory that $w^1 = w$ for any number w, and that $y^0 = 1$ for any number y (that isn't zero). But what could something like $a^{5/3}$ mean . . . an exponent that's a fraction?? Stick around and you'll find out.



❑ *REVIEW OF EXPONENT LAWS*

Let's begin by reviewing The Five Laws of Exponents:



$$x^a x^b = x^{a+b}$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$(x^a)^b = x^{ab}$$

$$(x\,y)^a = x^a\,y^a$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

Homework

1. Simplify each expression (assumes you've studied *Negative Exponents*):

a. x^3x^7	b. $\dfrac{a^{10}}{a^5}$	c. $(z^3)^5$	d. $\left(\dfrac{x}{y}\right)^9$
e. $(mn)^6$	f. r^4t^5	g. $a^2 + a^4$	h. $g^6 + g^6$
i. $(x+y)^2$	j. x^4x^{-3}	k. $(c^{-3})^{-5}$	l. $(ab)^{-4}$
m. $\dfrac{x^{-8}}{x^{-6}}$	n. $\dfrac{a^{-4}}{a^4}$	o. $\left(\dfrac{u}{w}\right)^{-3}$	p. $(a-b)^2$

❑ *THE MEANING OF A FRACTIONAL EXPONENT*

Now for a new kind of exponent: <u>fractional</u>. For example, is there any reasonable meaning for the expression "9 to the 1/2 power": $9^{1/2}$? To determine the meaning of this number, we can give $9^{1/2}$ a name; call it R. Now check out the following:

$$R = 9^{1/2}$$

$$\Rightarrow \quad R^{\mathbf{2}} = (9^{1/2})^{\mathbf{2}} \qquad \text{(square each side, just to see what happens)}$$

$$\Rightarrow \quad R^2 = 9^{\frac{1}{2}\cdot 2} \qquad \text{(law of exponents)}$$

$$\Rightarrow \quad R^2 = 9^1 \qquad \text{(1/2 and 2 are reciprocals)}$$

$$\Rightarrow \quad R^2 = 9$$

$$\Rightarrow \quad R = \pm 3 \qquad \text{(don't forget that there are 2 solutions)}$$

Since R is equal to both $9^{1/2}$ and ± 3, it follows that

$$9^{1/2} = \pm 3$$

But we can't have two answers to one question, so we'll keep the positive 3 and ignore the negative 3, and declare that

$$9^{1/2} = 3$$

What are we saying here? Since we know that $\sqrt{9} = 3$, we're saying that

$9^{1/2}$ simply means the **positive <u>square root</u> of 9.**

Without any more gory details, we could prove that all fractional exponents translate to radicals:

$$x^{1/2} = \sqrt{x} \qquad x^{1/3} = \sqrt[3]{x} \qquad x^{1/4} = \sqrt[4]{x}$$

Examples:

$9^{1/2} = 3$ $\quad 100^{1/2} = 10$ $\quad 25^{1/2} = 5$

$64^{1/3} = 4$ $\quad 8^{1/3} = 2$ $\quad 125^{1/3} = 5$

$16^{1/4} = 2$ $\quad 81^{1/4} = 3$ $\quad 256^{1/4} = 4$

Summary:

Assuming n is a natural number bigger than 1:

$$x^{1/n} = \sqrt[n]{x} \qquad \text{for } n = 2, 3, 4, \ldots$$

EXAMPLE 1: **Evaluate each expression with a fractional exponent:**

A. $225^{1/2} = \sqrt{225} = \mathbf{15}$

B. $125^{1/3} = \sqrt[3]{125} = \mathbf{5}$

C. $81^{1/4} = \sqrt[4]{81} = \mathbf{3}$

D. $(-32)^{1/5} = \sqrt[5]{-32} = \mathbf{-2}$

E. $-16^{1/2} = -\sqrt{16} = \mathbf{-4}$

F. $(-16)^{1/2} = \sqrt{-16} = \mathbf{Not\ a\ real\ number}$

G. $-81^{1/4} = -\sqrt[4]{81} = \mathbf{-3}$

H. $(-81)^{1/4} = \sqrt[4]{-81} = \mathbf{Not\ a\ real\ number}$

Homework

2. Explain why $25^{1/2}$ is a real number, but $(-25)^{1/2}$ is not.

3. Evaluate each expression:

a. $36^{1/2}$	b. $8^{1/3}$	c. $16^{1/4}$	d. $32^{1/5}$
e. $625^{1/2}$	f. $1^{1/3}$	g. $0^{1/4}$	h. $243^{1/5}$
i. $-25^{1/2}$	j. $-49^{1/2}$	k. $(-64)^{1/2}$	l. $(-16)^{1/4}$
m. $(-64)^{1/3}$	n. $(-1)^{1/5}$	o. $(-32)^{1/5}$	p. $(-1)^{1/4}$

4. Convert each expression to radical form:

a. $x^{1/2}$	b. $y^{1/3}$	c. $z^{1/4}$	d. $w^{1/5}$

e. $(ab)^{1/2}$ f. $ab^{1/2}$ g. $xy^{1/3}$ h. $(xy)^{1/3}$

i. $y+z^{1/2}$ j. $(y+z)^{1/2}$ k. $(a-b)^{1/3}$ l. $(Q+R-T)^{1/4}$

5. Convert each expression to exponent form:

a. $\sqrt{x}$ b. $\sqrt[4]{y}$ c. $\sqrt[3]{z}$ d. $\sqrt[5]{n}$

e. $a\sqrt{b}$ f. $\sqrt{ab}$ g. $x\sqrt[4]{y}$ h. $\sqrt[3]{tw}$

i. $\sqrt{x+y}$ j. $\sqrt[3]{p-q}$ k. $\sqrt[4]{a+n}$ l. $\sqrt[6]{x-x}$

6. True/False:

a. The expression $x^{1/2}$ is always defined.

b. The expression $x^{1/3}$ is always defined.

❑ *MORE FRACTIONAL EXPONENTS*

The previous problems each had a numerator of 1 in the fractional exponent. What about an expression like $27^{2/3}$? What could this mean? Let's dissect $27^{2/3}$ using our laws of exponents to determine the value of this number.

	$27^{2/3}$	(the power of 27 we're analyzing)
=	$27^{\frac{1}{3}\cdot 2}$	(certainly $\frac{1}{3}\cdot 2 = \frac{2}{3}$)
=	$\left(27^{1/3}\right)^2$	(law of exponents: $(x^a)^b = x^{ab}$)
=	$\left(\sqrt[3]{27}\right)^2$	(a 1/3 exponent indicates cube root)
=	3^2	(the cube root of 27 is 3)
=	**9**	(3 squared is 9)

Here's another example. Let's calculate $16^{5/4}$:

$$\mathbf{16^{5/4}} = 16^{\frac{1}{4}\cdot 5} = \left(16^{1/4}\right)^5 = \left(\sqrt[4]{16}\right)^5 = 2^5 = \mathbf{32}$$

And a third example:

$$\mathbf{243^{2/5}} = 243^{\frac{1}{5}\cdot 2} = \left(243^{1/5}\right)^2 = \left(\sqrt[5]{243}\right)^2 = 3^2 = \mathbf{9}$$

Homework

7. Using the three examples above as a guide, find the value of each fractional power:

a. $8^{2/3}$ b. $4^{3/2}$ c. $9^{3/2}$ d. $16^{5/2}$

e. $27^{4/3}$ f. $27^{2/3}$ g. $16^{3/4}$ h. $32^{7/5}$

❑ *THE BOTTOM LINE FOR FRACTIONAL EXPONENTS*

What's really going on here? There must be a simpler way to view a fractional exponent, and thus a simpler way to calculate one. If you look at the above examples and the homework you just completed, you may have noticed that the denominator of the fractional exponent indicated a root, while the numerator indicated a power. For example, in the first example above, we showed that $27^{2/3}$ was calculated by first taking the cube root of 27, which is 3, and then squaring that result to get **9**.

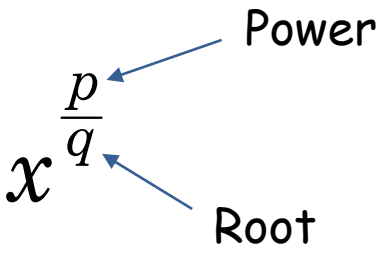
Thus, a problem like $16^{5/2}$ is calculated quickly as the square root of 16, raised to the fifth power, which is 4^5, which is **1,024**.

In summary,

$$x^{p/q} = \left(\sqrt[q]{x}\right)^{p}$$

EXAMPLE:

$$8^{5/3} = \left(\sqrt[3]{8}\right)^{5} = 2^{5} = 32$$



One more example for this section: To calculate $\mathbf{64^{2/3}}$, think cube root of 64, raised to the second power, which is 4^2, which is **16**.

Homework

8. Using the power-root idea, find the value of each fractional power:

a. $8^{4/3}$	b. $4^{1/2}$	c. $9^{5/2}$	d. $16^{3/2}$
e. $27^{2/3}$	f. $27^{4/3}$	g. $16^{5/4}$	h. $32^{6/5}$
i. $8^{2/3}$	j. $4^{3/2}$	k. $9^{3/2}$	l. $16^{5/2}$
m. $27^{4/3}$	n. $125^{2/3}$	o. $16^{3/4}$	p. $32^{7/5}$

❑ *NEGATIVE FRACTIONAL EXPONENTS*

Since a negative exponent indicates a reciprocal, we can combine negative exponents (assuming you've studied them) with the fractional exponents we're learning now. Recalling that $x^{-n} = \frac{1}{x^n}$, we can work the following examples.

EXAMPLE 2: **Evaluate each expression:**

A. $9^{-1/2} = \frac{1}{9^{1/2}} = \frac{1}{\sqrt{9}} = \mathbf{\frac{1}{3}}$

B. $-27^{-1/3} = -\frac{1}{27^{1/3}} = -\frac{1}{\sqrt[3]{27}} = \mathbf{-\frac{1}{3}}$

C. $8^{-2/3} = \frac{1}{8^{2/3}} = \frac{1}{\left(\sqrt[3]{8}\right)^2} = \frac{1}{2^2} = \mathbf{\frac{1}{4}}$

D. $9^{-5/2} = \frac{1}{9^{5/2}} = \frac{1}{\left(\sqrt{9}\right)^5} = \frac{1}{3^5} = \mathbf{\frac{1}{243}}$

E. $-16^{-3/4} = -\frac{1}{16^{3/4}} = -\frac{1}{\left(\sqrt[4]{16}\right)^3} = -\frac{1}{2^3} = \mathbf{-\frac{1}{8}}$

F. $(-16)^{-3/2} = \frac{1}{(-16)^{3/2}} = \frac{1}{\left(\sqrt{-16}\right)^3} =$ **Not a real number**

G. $(-8)^{-2/3} = \frac{1}{(-8)^{2/3}} = \frac{1}{\left(\sqrt[3]{-8}\right)^2} = \frac{1}{(-2)^2} = \mathbf{\frac{1}{4}}$

Homework

9. Evaluate each expression:

a. $9^{-3/2}$ b. $27^{-2/3}$ c. $8^{-4/3}$ d. $8^{-1/3}$

e. $16^{-5/4}$ f. $81^{-1/4}$ g. $81^{-3/4}$ h. $32^{-1/5}$

i. $-8^{-4/3}$ j. $(-25)^{-3/2}$ k. $(-64)^{-2/3}$ l. $(-32)^{-6/5}$

10. Convert each expression to radical form:

a. $x^{2/3}$ b. $y^{5/4}$ c. $z^{1/7}$

d. $(a+b)^{3/2}$ e. $(x-y)^{4/3}$ f. $(xy)^{4/5}$

g. $uw^{2/3}$ h. $x+y^{4/5}$ i. $(ab+c)^{5/6}$

11. Convert each expression to exponent form:

a. $\sqrt[3]{t}$ b. $\sqrt[5]{z}$ c. $\sqrt[4]{ab}$

d. $x\sqrt[3]{y}$ e. $\left(\sqrt{a}\right)^3$ f. $\left(\sqrt[3]{p}\right)^2$

g. $\left(\sqrt[4]{w}\right)^5$ h. $\left(\sqrt[7]{a+b}\right)^3$ i. $\left(\sqrt{x-y}\right)^{10}$

❑ *THE LAWS OF EXPONENTS*

The same five laws of exponents we've used with all the previous exponents still work just fine with fractional exponents.

EXAMPLE 3: **Simplify each expression:**

A. $a^{1/2}a^{1/3} = a^{1/2+1/3} = a^{3/6+2/6} = \boldsymbol{a^{5/6}}$

(add the exponents)

B. $y^{1/2}y^{1/2} = y^{1/2+1/2} = y^1 = \boldsymbol{y}$

(add the exponents)

C. $\left(x^{3/4}\right)^{7/2} = x^{(3/4)(7/2)} = \boldsymbol{x^{21/8}}$

(multiply the exponents)

D. $\dfrac{n^{5/6}}{n^{2/3}} = n^{5/6-2/3} = n^{5/6-4/6} = \boldsymbol{n^{1/6}}$

(subtract the exponents)

E. $(ab)^{4/5} = \boldsymbol{a^{4/5}b^{4/5}}$

(raise each factor to the 4/5 power)

F. $\left(\dfrac{a}{b}\right)^{5/3} = \dfrac{\boldsymbol{a^{5/3}}}{\boldsymbol{b^{5/3}}}$

(raise top and bottom to the 5/3 power)

G. $\dfrac{w^{1/2}}{w^{4/5}} = w^{1/2-4/5} = w^{5/10-8/10} = w^{-3/10} = \dfrac{\boldsymbol{1}}{\boldsymbol{w^{3/10}}}$

(subtract the exponents) (LCD) (a negative exponent means reciprocal)

Homework

12. Simplify each expression:

a. $x^{1/2}x^{2/3}$ b. $\dfrac{y^{4/5}}{y^{1/5}}$ c. $\dfrac{w^{1/2}}{w^{2/3}}$ d. $\left(a^{2/7}\right)^7$

e. $\left(\dfrac{p}{q}\right)^{3/8}$ f. $t^{4/5}t^{1/3}$ g. $\left(k^{5/2}\right)^{2/5}$ h. $(wz)^{2/3}$

EXAMPLE 4: **Simplify each expression:**

A. $x^{-1/2}x^{-2/3} = x^{-1/2-2/3} = x^{-3/6-4/6} = x^{-7/6} = \mathbf{\frac{1}{x^{7/6}}}$

B. $\frac{n^{1/2}}{n^{-4/5}} = n^{1/2-(-4/5)} = n^{1/2+4/5} = n^{5/10+8/10} = \mathbf{n^{13/10}}$

C. $\left(x^{2/3}\right)^{-1/4} = x^{(2/3)(-1/4)} = x^{-1/6} = \mathbf{\frac{1}{x^{1/6}}}$

D. $(xy)^{-2/3} = x^{-2/3}y^{-2/3} = \frac{1}{x^{2/3}}\cdot\frac{1}{y^{2/3}} = \mathbf{\frac{1}{x^{2/3}y^{2/3}}}$

E. $\left(\frac{g}{h}\right)^{-4/3} = \frac{g^{-4/3}}{h^{-4/3}} = \frac{\frac{1}{g^{4/3}}}{\frac{1}{h^{4/3}}} = \frac{1}{g^{4/3}}\times\frac{h^{4/3}}{1} = \mathbf{\frac{h^{4/3}}{g^{4/3}}}$

Homework

13. Simplify each expression:

a. $x^{4/5}x^{-3/5}$ b. $y^{1/3}y^{-5/3}$ c. $\left(a^{-1/2}\right)^{-2/3}$

d. $(abc)^{-3/4}$ e. $\left(\frac{w}{z}\right)^{-2/5}$ f. $\frac{n^{-1/2}}{n^{-2/3}}$

g. $\frac{a^{-3}}{a^{5/2}}$ h. $\frac{x}{x^{-2/3}}$ i. $\left(\left(c^{1/2}\right)^{-4/3}\right)^{-3/2}$

Review Problems

14. Convert $\sqrt[4]{x+y}$ to exponent form.

15. Convert $ab^{2/3}$ to radical form.

16. Convert $(a+b)^{5/2}$ to radical form.

17. Convert $\sqrt{a^3-x^3}$ to exponent form.

18. Evaluate: a. $9^{1/2}$ b. $64^{1/3}$ c. $81^{1/4}$ d. $32^{1/5}$

19. Evaluate: a. $8^{2/3}$ b. $27^{4/3}$ c. $32^{2/5}$ d. $16^{3/4}$

20. Evaluate: a. $-9^{1/2}$ b. $(-9)^{1/2}$ c. $(-8)^{1/3}$ d. $(-16)^{1/4}$

21. Evaluate: a. $9^{-3/2}$ b. $8^{-4/3}$ c. $125^{-4/3}$ d. $-81^{-1/4}$

22. Simplify: a. $x^{1/2}x^{4/5}$ b. $\dfrac{a^{1/3}}{a^{2/5}}$ c. $(ab)^{4/7}$

23. Simplify: a. $\left(\dfrac{a}{b}\right)^{2/7}$ b. $\left(x^{2/3}\right)^{3/5}$ c. $a^{1/2}a^{1/3}a^{1/4}$

24. Simplify: a. $y^{1/2}y^{-1/2}$ b. $\dfrac{n^{1/3}}{n^{-4/3}}$ c. $(PQ)^{-2/3}$

25. Simplify: a. $\left(\dfrac{x}{w}\right)^{-7/10}$ b. $\left(p^{-2/3}\right)^{5/6}$ c. $x^{2/3}+x^{1/3}$

Solutions

1. a. x^{10} b. a^5 c. z^{15} d. $\frac{x^9}{y^9}$ e. m^6n^6

f. As is g. As is h. $2g^6$ i. $x^2 + 2xy + y^2$

j. x k. c^{15} l. $\frac{1}{a^4b^4}$ m. $\frac{1}{x^2}$ n. $\frac{1}{a^8}$

o. $\frac{w^3}{u^3}$ p. Assuming you've learned how to multiply binomials, the answer is $a^2 - 2ab + b^2$

2. $25^{1/2} = \sqrt{25} = 5$, a real number. But $(-25)^{1/2} = \sqrt{-25}$, not a real number.

3. a. 6 b. 2 c. 2 d. 2 e. 25 f. 1 g. 0 h. 3 i. –5
j. –7 k. Not real l. Not real m. –4 n. –1 o. –2
p. Not real

4. a. $\sqrt{x}$ b. $\sqrt[3]{y}$ c. $\sqrt[4]{z}$ d. $\sqrt[5]{w}$ e. $\sqrt{ab}$
f. $a\sqrt{b}$ g. $x\sqrt[3]{y}$ h. $\sqrt[3]{xy}$ i. $y+\sqrt{z}$ j. $\sqrt{y+z}$
k. $\sqrt[3]{a-b}$ l. $\sqrt[4]{Q+R-T}$

5. a. $x^{1/2}$ b. $y^{1/4}$ c. $z^{1/3}$ d. $n^{1/5}$ e. $ab^{1/2}$ f. $(ab)^{1/2}$
g. $xy^{1/4}$ h. $(tw)^{1/3}$ i. $(x+y)^{1/2}$ j. $(p-q)^{1/3}$ k. $(a+n)^{1/4}$ l. 0

6. a. False; if $x = -9$, for instance, then $(-9)^{1/2} = \sqrt{-9}$ which is not a real number, which means that $x^{1/2}$ is undefined in this class when $x = -9$. In fact, $x^{1/2}$ is undefined whenever x is a negative number.

b. True; since $x^{1/3} = \sqrt[3]{x}$, and since the cube root is defined whether x is positive, zero, or negative, $x^{1/3}$ is always defined.

7. a. $8^{2/3} = 8^{\frac{1}{3}\cdot 2} = \left(8^{1/3}\right)^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

b. 8 c. 27 d. 1024 e. 81 f. 9 g. 8

h. $32^{7/5} = 32^{\frac{1}{5}\cdot 7} = \left(32^{1/5}\right)^7 = \left(\sqrt[5]{32}\right)^7 = 2^7 = 128$

8. a. $8^{4/3}$ is the cube root of 8, raised to the 4th power: $2^4 = 16$

b. 2

c. $9^{5/2}$ is the square root of 9, raised to the 5th power: $3^5 = 243$

d. 64 e. 9 f. 81 g. 32

h. $32^{6/5}$ is the fifth root of 32, raised to the 6th power: $2^6 = 64$

i. 4 j. 8 k. 27 l. 1024 m. 81 n. 25 o. 8 p. 128

9. a. $\frac{1}{27}$ b. $\frac{1}{9}$ c. $\frac{1}{16}$ d. $\frac{1}{2}$ e. $\frac{1}{32}$ f. $\frac{1}{3}$

g. $\frac{1}{27}$ h. $\frac{1}{2}$ i. $-\frac{1}{16}$ j. Not real k. $\frac{1}{16}$ l. $\frac{1}{64}$

10. a. $\left(\sqrt[3]{x}\right)^2$ b. $\left(\sqrt[4]{y}\right)^5$ c. $\sqrt[7]{z}$ d. $\left(\sqrt{a+b}\right)^3$ e. $\left(\sqrt[3]{x-y}\right)^4$

f. $\left(\sqrt[5]{xy}\right)^4$ g. $u\left(\sqrt[3]{w}\right)^2$ h. $x+\sqrt[5]{y^4}$ i. $\left(\sqrt[6]{ab+c}\right)^5$

11. a. $t^{1/3}$ b. $z^{1/5}$ c. $(ab)^{1/4}$ d. $xy^{1/3}$ e. $a^{3/2}$

f. $p^{2/3}$ g. $w^{5/4}$ h. $(a+b)^{3/7}$ i. $(x-y)^{10/2} = (x-y)^5$

12. a. $x^{7/6}$ b. $y^{3/5}$ c. $\frac{1}{w^{1/6}}$ d. a^2

e. $\frac{p^{3/8}}{q^{3/8}}$ f. $t^{17/15}$ g. k h. $w^{2/3}z^{2/3}$

13. a. $x^{1/5}$ b. $\frac{1}{y^{4/3}}$ c. $a^{1/3}$ d. $\frac{1}{a^{3/4}b^{3/4}c^{3/4}}$

e. $\frac{z^{2/5}}{w^{2/5}}$ f. $n^{1/6}$ g. $\frac{1}{a^{11/2}}$ h. $x^{5/3}$

i. c

14. $(x+y)^{1/4}$ **15**. $a\left(\sqrt[3]{b}\right)^2$

16. $\left(\sqrt{a+b}\right)^5$ or $\sqrt{(a+b)^5}$

17. $(a^3-x^3)^{1/2}$

18. a. 3 b. 4 c. 3 d. 2

19. a. 4 b. 81 c. 4 d. 8

20. a. –3 b. Not real c. –2 d. Not real

21. a. $\frac{1}{27}$ b. $\frac{1}{16}$ c. $\frac{1}{625}$ d. $-\frac{1}{3}$

22. a. $x^{13/10}$ b. $\frac{1}{a^{1/15}}$ c. $a^{4/7}b^{4/7}$

23. a. $\frac{a^{2/7}}{b^{2/7}}$ b. $x^{2/5}$ c. $a^{13/12}$

24. a. 1 b. $n^{5/3}$ c. $\frac{1}{P^{2/3}Q^{2/3}}$

25. a. $\frac{w^{7/10}}{x^{7/10}}$ b. $\frac{1}{p^{5/9}}$ c. As is

"If you limit your choices only to what seems possible or reasonable, you disconnect yourself from what you truly want, and all that is left is a compromise."

- Robert Fritz

MORE EXPONENTS

❑ INTRODUCTION

This chapter is a continuation of the exponent ideas we've used many times before. Our goal is to combine expressions with exponents in them. First, a quick review of exponents:

$2^3 = 8$ $\quad -3^4 = -81$ $\quad 1^{124} = 1$ $\quad 0^{433} = 0$

$(-4)^3 = -64$ $\quad (-2)^6 = 64$ $\quad 2^0 = 1$ $\quad (8-6)^0 = 1$

❑ REVIEW OF STRANGE EXPONENTS

We've worked with base 2 extensively, analyzing what happens when we apply various exponents to it. First we observed that $2^1 = 2$. It's reasonable to conclude that if x is any number,

$$x^1 = x$$

Next we encountered the exponent 0. 2^0 equaled 1, so we take a leap of faith and generalize that for any x (that's not zero),

$$x^0 = 1$$

[0^0 has no meaning until calculus]

Then came the strange negative exponent. We saw, for example, that $2^{-10} = \frac{1}{2^{10}}$. It's also the case that for any non-zero base x,

$$x^{-n} = \frac{1}{x^n}$$ where $n = 1, 2, 3, \ldots$

Zero raised to a negative power will be dealt with in the homework.

Homework

1. Evaluate each power:

a. 17^1	b. 999^1	c. $(3.8)^1$	d. $(-7)^1$
e. $\left(\frac{2}{5}\right)^1$	f. 19^0	g. $(-2.3)^0$	h. $\left(\frac{1}{4}\right)^0$
i. 3^{-1}	j. 4^{-2}	k. 5^{-3}	l. 6^{-4}

2. Find the value of the expression $\left(\dfrac{7x^3 - \sqrt{a-b} - \int_0^{\pi/2} \sin x \, dx}{\ln\left(\frac{1}{\tan\theta}\right)^{e^{i\pi}}} \right)^0$.

3. Assuming all variables are not zero, simplify each expression:

a. x^1	b. y^0	c. n^{-3}	d. z^{-5}
e. $(abc + 1)^0$	f. a^1	g. b^0	h. $(xy)^1$
i. t^{-1}	j. $(xy)^0 + 1$	k. $(xy)^0$	l. xy^0
m. $x^0 y$	n. $(x + y)^0$	o. $x + y^0$	p. $a^0 - b$
q. $(c - d)^0$	r. $10 \cdot 2^0$	s. w^{-2}	t. z^{-1}
u. a^{-5}	v. u^{-8}	w. n^{-4}	x. w^{-6}
y. a^{-7}	z. k^{-8}		

Hint: Exactly what is being raised to the 0 power?

4. Prove that 0^{-2} is undefined.

❑ *COMBINING THINGS WITH EXPONENTS*

EXAMPLE 1: **Simplify each exponential expression using the stretch-and-squish technique.**

Stretch-and-squish means to expand the bases in the problem using the definition of exponents, then do some kind of simplifying using previous techniques, and then squish it back to exponent form.

A. $\boldsymbol{n^3 \times n^5}$

$$n^3 \times n^5 = (n \cdot n \cdot n) \times (n \cdot n \cdot n \cdot n \cdot n) = nnnnnnnn = \boldsymbol{n^8}$$

B. $\boldsymbol{\frac{x^6}{x^2}}$

$$\frac{x^6}{x^2} = \frac{x\,x\,x\,x\,x\,x}{x\,x} = \frac{\cancel{x\,x}\,x\,x\,x\,x}{\cancel{x\,x}} = \boldsymbol{x^4}$$

C. $\boldsymbol{\frac{n^3}{n^7}}$

$$\frac{n^3}{n^7} = \frac{nnn}{nnnnnnn} = \frac{\cancel{nnn}}{\cancel{nnn}nnnn} = \frac{1}{nnnn} = \boldsymbol{\frac{1}{n^4}}$$

D. $\boldsymbol{(ab)^3}$

	$(ab)^3$	
=	$(ab)(ab)(ab)$	(definition of cubing)
=	$ababab$	(parentheses not necessary)
=	$(aaa)(bbb)$	(rearrange the factors)
=	$\boldsymbol{a^3b^3}$	(rewrite with exponents)

E. $\boldsymbol{(uwz)^2}$

$(uwz)^2 = (uwz)(uwz) = (uu)(ww)(zz) = \boldsymbol{u^2w^2z^2}$

F. $\boldsymbol{\left(\frac{x}{y}\right)^2}$

$\left(\frac{x}{y}\right)^2 = \left(\frac{x}{y}\right)\left(\frac{x}{y}\right) = \frac{xx}{yy} = \boldsymbol{\frac{x^2}{y^2}}$

G. $\boldsymbol{\left(\frac{a}{b}\right)^3}$

$\left(\frac{a}{b}\right)^3 = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) = \frac{aaa}{bbb} = \boldsymbol{\frac{a^3}{b^3}}$

H. $\boldsymbol{(x^4)^2}$

$(x^4)^2 = (x^4)(x^4) = (xxxx)(xxxx) = xxxxxxxx = \boldsymbol{x^8}$

EXAMPLE 2: **Simplify each expression:**

A. $x^2x^5 = (xx)(xxxxx) = xxxxxxx = \boldsymbol{x^7}$

B. $n^3 + n^7 = nnn + nnnnnnn =$???

Note: We don't have a simple product of n's (due to the plus sign), so we can't write the sum using a single exponent. Besides, n^3 and n^7 are unlike terms, and therefore cannot be added. However we look at it, this problem cannot be simplified. On a test you can write your answer either as $\boldsymbol{n^3 + n^7}$ or **As is**.

C. $a^4 + a^4 = \boldsymbol{2a^4}$ These are like terms, so they add up.

D. u^3w^7 cannot be simplified, since $(uuu)(wwwwwww)$ is just what it is, 3 factors of u multiplied by 7 factors of w.

E. $y^3y^4 - y^7 = (yyy)(yyyy) - y^7 = y^7 - y^7 = \mathbf{0}$

F. $(-2c^3)(-10c^5) = (-2)(-10)(ccc)(ccccc) = 20(cccccccc) = \mathbf{20c^8}$

G. $(aqt)^3 = (aqt)(aqt)(aqt) = (aaa)(qqq)(ttt) = \boldsymbol{a^3q^3t^3}$

H. $3(a^3b^4)^2$

$= 3(a^3b^4)(a^3b^4)$ (the 3 is not being squared)

$= 3(aaa)(bbbb)(aaa)(bbbb)$ (stretch)

$= 3(aaaaaa)(bbbbbbbb)$ (rearrange factors)

$= \mathbf{3a^6b^8}$ (squish)

I. $(-2xy^2)^4$

$= (-2xy^2)\,(-2xy^2)\,(-2xy^2)\,(-2xy^2)$ (stretch)

$= (-2)(-2)(-2)(-2)(x)(x)(x)(x)(yy)(yy)(yy)(yy)$ (stretch)

$= 16(xxxx)(yyyyyyyy)$ (rearrange)

$= \mathbf{16x^4y^8}$ (squish)

J. $\left(\dfrac{g^4}{h^3}\right)^2$

$= \left(\dfrac{g^4}{h^3}\right)\left(\dfrac{g^4}{h^3}\right) = \dfrac{g^4 \cdot g^4}{h^3 \cdot h^3} = \dfrac{gggg \cdot gggg}{hhh \cdot hhh} = \boldsymbol{\dfrac{g^8}{h^6}}$

Homework

Use the stretch-and-squish technique (where appropriate) to simplify each expression:

5. a. $n^3 n^3$ b. xx^4 c. $z^4 z^4$ d. $a^4 a^5 a^3$
   e. $u^3 + u^4$ f. $w^9 + w^9$ g. $v^4 - v^3$ h. $c^{12} - c^{12}$

6. a. $\frac{x^8}{x^2}$ b. $\frac{y^9}{y^9}$ c. $\frac{z^7}{z^8}$ d. $\frac{a^7}{b^4}$

7. a. $(uv)^3$ b. $(abc)^2$ c. $(xy)^4$
   d. $(mnpq)^2$ e. $(jk)^2$ f. $(axy)^1$
   g. $(ax + b)^0$

8. True/False: $(x + y)^2 = x^2 + y^2$
   Check it out by using numbers for x and y.

9. a. $\left(\frac{a}{b}\right)^2$ b. $\left(\frac{w}{z}\right)^3$ c. $\left(\frac{x}{y}\right)^4$ d. $\left(\frac{1}{n}\right)^5$
   e. $\left(\frac{u}{v}\right)^1$ f. $\left(\frac{f}{g}\right)^0$ g. $\left(\frac{x}{2}\right)^5$ h. $(a - 5)^2$

10. a. $(2^3)^2$ b. $(3^2)^3$ c. $(1^4)^3$ d. $(0^2)^{44}$
    e. $(x^4)^2$ f. $(a^0)^4$ g. $(n^{33})^0$ h. $(m^0)^0$

11. a. $(2x)(3x)$ b. $(3a^2)(-2a)$ c. $5y^3(2y^3)$
    d. $(-x^2)(-x^4)$ e. $a^5 + a^3$ f. $9n^5 + 8n^5$
    g. $10q^3 - 10q^3$ h. $(2xy)(-3y)$ i. $(-5x^2y)(-3xy^2)$
    j. $(3g^3)(4g^4)$ k. $(9x^5)\,(9x^5)$ l. $4t^3 + 3t^4$

12. a. $(a^2b)^3$ b. $(xy^3)^2$ c. $(-2x)^4$ d. $(3a^2n^3)^3$

e. $(-3z^4)^3$ f. $2(a^2b)^3$ g. $-3(m^2n^2)^2$ h. $(-3m^2n^2)^2$

i. $(-cd^4)^3$ j. $(-x^2y)^4$ k. $\left(\frac{c^2}{d^3}\right)^2$ l. $\left(\frac{x}{y^4}\right)^2$

m. $\left(\frac{2x}{y^3}\right)^2$ n. $\left(\frac{-5x^2}{y^3}\right)^3$ o. $\left(\frac{-2u^3}{w}\right)^4$ p. $\left(\frac{(2x)^3}{x^3}\right)^4$

Review Problems

13. Evaluate each expression:

a. 2^4 b. 2^9 c. 2^{-2} d. 2^{-6}

e. 2^{-10} f. 10^3 g. 10^6 h. 10^{-4}

i. 10^{-7} j. 2^0 k. 10^0 l. $10^1 + 2^1$

m. $2^0 - 10^0$ n. $(2^3)^2$ o. $(10^4)^3$ p. $2^3 \cdot 2^4$

q. $10^2 \cdot 10^5$ r. $2^3 \cdot 10^2$ s. $2^3 + 10^2$ t. $2^{-1} + 2^{-2}$

u. $10^{-3} + 10^2$ v. $(10^{-2})^3$ w. $(2^{-3})^2$ x. $(10^{19})^0$

y. $(5^3)^0$ z. $(17^0)^{14}$

14. Simplify each expression:

a. $x^0 + x^1$ b. $x^2 \cdot x^0$ c. x^{-3} d. w^{-1}

e. $(ab)^1$ f. $(a + b)^0$ g. x^2x^4 h. $x^2 + x^4$

i. $x^4 - x$ j. x^3y^7 k. $a^5 + a^5$ l. a^5a^5

m. $\frac{x^3}{x}$ n. $\frac{a^6}{a^{10}}$ o. $(ab)^3$ p. $(xyz)^2$

q. $\left(\frac{x}{y}\right)^5$ r. $(x^2)^3$ s. $(u^3)^2$ t. $(w^3)^3$

u. $(A^{44})^0$ v. $(Q^0)^9$ w. $(2^5)^2$ x. $(2^2)^5$

y. $\frac{2^{10}}{2^9}$ z. $\frac{x^{10}}{x^9}$

15. Simplify each expression:

a. $(-3x^2)(4xy)$ b. $(-x^3w)(-2xw^3)$ c. $(2ab^4)^3$

d. $2(ab^4)^3$ e. $\left(\frac{x}{k^3}\right)^2$ f. $\frac{x^2}{k^3}$

g. $(2^{-1})(10^{-1})$ h. $\left(\frac{x^3}{2^{-5}}\right)^0$ i. $10^0 - 10^{-1}$

j. $2^0 + 3^0 + 4^0$ k. $2^{-1} + 2^{-2} + 2^{-3}$ l. $\frac{x^{100}}{x^{20}}$

Solutions

1. a. 17 b. 999 c. 3.8 d. −7 e. $\frac{2}{5}$ f. 1

g. 1 h. 1 i. $\frac{1}{3}$ j. $\frac{1}{16}$ k. $\frac{1}{125}$ l. $\frac{1}{1296}$

2. 1

3. a. x b. 1 c. $\frac{1}{n^3}$ d. $\frac{1}{z^5}$ e. 1 f. a

g. 1 h. xy i. $\frac{1}{t}$ j. 2 k. 1 l. x

m. y n. 1 o. $x + 1$ p. $1 - b$ q. 1 r. 10

s. $\frac{1}{w^2}$ t. $\frac{1}{z}$ u. $\frac{1}{a^5}$ v. $\frac{1}{u^8}$ w. $\frac{1}{n^4}$ x. $\frac{1}{w^6}$

y. $\frac{1}{a^7}$ z. $\frac{1}{k^8}$

4. $0^{-2} = \frac{1}{0^2} = \frac{1}{0}$, which is undefined.

5. a. $n^3n^3 = (nnn)(nnn) = nnnnnn = n^6$

b. $xx^4 = x(xxxx) = xxxxx = x^5$

c. z^8 d. a^{12} e. As is f. $2w^9$ g. As is h. 0

6. a. $\frac{x^8}{x^2} = \frac{xxxxxxxx}{xx} = \frac{\cancel{xx}xxxxxx}{\cancel{xx}} = x^6$ b. 1

c. $\frac{z^7}{z^8} = \frac{zzzzzzz}{zzzzzzzz} = \frac{\cancel{zzzzzzz}}{\cancel{zzzzzzz}z} = \frac{1}{z}$ d. As is

7. a. $(uv)^3 = (uv)(uv)(uv) = uuuvvv = u^3v^3$

b. $a^2b^2c^2$ c. x^4y^4 d. $m^2n^2p^2q^2$

e. j^2k^2 f. axy g. 1

8. The statement is false; pick some numbers for x and y, plug them into each side of the statement, and you'll see why. [See Chapter 2.]

9. a. $\left(\frac{a}{b}\right)^2 = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right) = \frac{aa}{bb} = \frac{a^2}{b^2}$ b. $\frac{w^3}{z^3}$ c. $\frac{x^4}{y^4}$

d. $\left(\frac{1}{n}\right)^5 = \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)\left(\frac{1}{n}\right) = \frac{1\cdot1\cdot1\cdot1\cdot1}{nnnnn} = \frac{1}{n^5}$

e. $\frac{u}{v}$ f. 1 g. $\frac{x^5}{32}$ h. $a^2 - 10a + 25$

10. a. $(2^3)^2 = (2 \cdot 2 \cdot 2)^2 = (2 \cdot 2 \cdot 2)\,(2 \cdot 2 \cdot 2) = 64$

b. 729 c. 1 d. 0 e. x^8

f. $(a^0)^4 = 1^4 = 1$ g. 1 h. 1

11. a. $6x^2$ b. $-6a^3$ c. $10y^6$ d. x^6

e. As is f. $17n^5$ g. 0 h. $-6xy^2$

i. $15x^3y^3$ j. $12g^7$ k. $81x^{10}$ l. As is

12. a. a^6b^3 b. x^2y^6 c. $16x^4$ d. $27a^6n^9$

e. $-27z^{12}$ f. $2a^6b^3$ g. $-3m^4n^4$ h. $9m^4n^4$

i. $-c^3d^{12}$ j. x^8y^4 k. $\frac{c^4}{d^6}$ l. $\frac{x^2}{y^8}$

m. $\frac{4x^2}{y^6}$ n. $\frac{-125x^6}{y^9}$ o. $\frac{16u^{12}}{w^4}$ p. 4096

13. a. 16 b. 512 c. $\frac{1}{4}$ d. $\frac{1}{64}$

e. $\frac{1}{1024}$ f. 1000 g. 1,000,000 h. $\frac{1}{10{,}000}$

i. $\frac{1}{10{,}000{,}000}$ j. 1 k. 1 l. 12

m. 0 n. 64 o. 1,000,000,000,000

p. 128 q. 10,000,000 r. 800 s. 108

t. $\frac{3}{4}$ u. 100.001 v. $\frac{1}{1{,}000{,}000}$ w. $\frac{1}{64}$

x. 1 y. 1 z. 1

14. a. $1 + x$ b. x^2 c. $\frac{1}{x^3}$ d. $\frac{1}{w}$

e. ab f. 1 g. x^6 h. $x^2 + x^4$

i. $x^4 - x$ j. x^3y^7 k. $2a^5$ l. a^{10}

m. x^2 n. $\frac{1}{a^4}$ o. a^3b^3 p. $x^2y^2z^2$

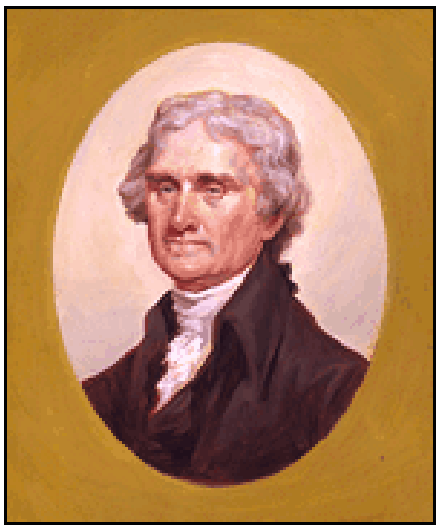
	q.	$\frac{x^5}{y^5}$	r.	x^6	s.	u^6	t.	w^9
	u.	1	v.	1	w.	1024	x.	1024
	y.	2	z.	x				
15.	a.	$-12x^3y$	b.	$2x^4w^4$	c.	$8a^3b^{12}$	d.	$2a^3b^{12}$
	e.	$\frac{x^2}{k^6}$	f.	As is	g.	$\frac{1}{20}$	h.	1
	i.	$\frac{9}{10}$	j.	3	k.	$\frac{7}{8}$	l.	x^{80}

❑ *TO ∞ AND BEYOND!*

Do some research to determine the meaning of $9^{1/2}$.

"Nothing can stop the man with the right mental attitude from achieving his goal; nothing on earth can help the man with the wrong mental attitude."

Thomas Jefferson



ADVANCED FACTORING

We can now factor lots of quadratic binomials (like $4x^2 - 9$) and trinomials (like $n^2 + 10n + 21$). Sorry to tell you this, but we're not done with factoring just yet. In this chapter, we learn how to factor 4th degree expressions (an exponent of 4), and expressions containing GCFs you might never have seen before.



❑ *FACTORING QUARTICS*

EXAMPLE 1: **Factor each quartic (4th degree) polynomial:**

A. $\boldsymbol{c^4 - 256}$

$= (c^2 + 16)(c^2 - 16)$ (difference of squares)

$= \boxed{(c^2 + 16)(c + 4)(c - 4)}$ (difference of squares again)

Note: $c^2 + 16$ cannot be factored any further.

B. $\boldsymbol{9a^4 - 37a^2 + 4}$

$= (9a^2 - 1)(a^2 - 4)$ (factor trinomial)

Now we notice that each factor is quadratic and is the difference of two squares. Therefore, each factor can be factored further to get a final answer consisting of four factors:

$$(3a + 1)(3a - 1)(a + 2)(a - 2)$$

Homework

1. Factor each quartic polynomial:

a. $x^4 - 1$

b. $x^4 - x^2 - 6$

c. $n^4 - 10n^2 + 9$

d. $a^4 - 81$

e. $36w^4 - 25w^2 + 4$

f. $9x^4 - 34x^2 + 25$

g. $c^4 - 16$

h. $x^4 - 8x^2 - 9$

i. $x^4 - 3x^2 - 10$

j. $g^4 - 256$

k. $36u^4 - 85u^2 + 9$

l. $y^4 + 81$

❑ *THE GCF REVISITED*

EXAMPLE 2: **Factor: $(a + b)^2 + 4(a + b)$**

Solution: This problem consists of two terms: $(a + b)^2$ and $4(a + b)$. Notice that each of these two terms contains the same factor, namely $a + b$. In other words, the GCF of the two terms is $a + b$. Factoring out this GCF gives us the final factored form, a single term consisting of two factors:

$$(a + b)(a + b + 4)$$

It's a single term because the final operation is mulitplication.

NOTE: The thing not to do in this kind of problem is to distribute the original expression; if you do, you'll be going in the wrong direction. Check it out:

$$(a + b)^2 + 4(a + b) = a^2 + 2ab + b^2 + 4a + 4b$$

Do you really want to try to factor that last expression?

So, when you see an expression, like $a + b$ in this problem, that occurs multiple times in an expression, it's usually best to leave it intact. Also, notice that we have converted a 2-termed expression into 1 term — we have factored.

Alternate Method: Let's try a substitution method. We might be able to better see the essence of the problem if we replace $a + b$ with a simpler symbol — for example, x can represent $a + b$. Then the original expression

$$(a + b)^2 + 4(a + b)$$

is transformed into

$$x^2 + 4x$$

The GCF in this form is clearly x, so we pull it out in front:

$$x(x + 4)$$

Now substitute in the reverse direction to get $a + b$ back in the problem:

$$\boldsymbol{(a + b)(a + b + 4)}$$ (the same answer as before)

EXAMPLE 3: **Factor: $x^2(u - w) - 100(u - w)$**

Solution: The two given terms have a GCF of $u - w$. Factoring this GCF out gives

$$(u - w)(x^2 - 100)$$

But we're not done yet. The second factor is a difference of squares. Factoring that part gives us our final factorization:

$$\boxed{(u - w)(x + 10)(x - 10)}$$

EXAMPLE 4: **Factor: $w^2(x + z) - 4w(x + z) + 3(x + z)$**

Solution: Let's use substitution to make this expression appear a little less intimidating; we'll convert every occurrence of $x + z$ to the symbol A:

$$w^2A - 4wA + 3A$$

Pulling out the GCF of A, we get

$$A(w^2 - 4w + 3)$$

Factor the trinomial in the usual way:

$$A(w - 3)(w - 1)$$

Last, replace the A with its original definition of $x + z$:

$$\boxed{(x + z)(w - 3)(w - 1)}$$

Homework

2. Factor each expression:

a. $(x + y)^2 + 7(x + y)$

b. $(a - b)^2 - c(a - b)$

c. $x^2(c + d) + 5(c + d)$

d. $n^2(a - b) - 9(a - b)$

e. $x^2(a + 4) + 5x(a + 4) + 6(a + 4)$

f. $y^2(m + n) + 7y(m + n)$

g. $2x^2(a + b) + 3x(a + b) - 5(a + b)$

h. $4x^2(w + z) - 9(w + z)$

i. $(u - w)^2 - 9(u - w)$

j. $n^2(a + b) - 9n(a + b)$

k. $(t + r)y^2 - 100(t + r)$

l. $3ax^2 - 20ax - 7a$

❑ *GROUPING WITH FOUR TERMS*

EXAMPLE 5: **Factor:** $a^2 + ac + ab + bc$

Solution: Group the first two terms and the last two terms:

$$(a^2 + ac) + (ab + bc)$$

Now factor each pair of grouped terms separately (using the GCF in each pair) :

$a(a + c) + b(a + c)$ (notice that the GCF is $a + c$)

Even though we've grouped and factored, we can't be done because there are still two terms, and we need one term in the final answer to a factoring question. So we continue — using our knowledge of the previous section — and factor out the GCF, which is $a + c$:

$$\boxed{(a + c)(a + b)}$$

By the commutative property of multiplication ($xy = yx$), the final answer could also be written $(a + b)(a + c)$. Also, to check our answer, just double distribute the answer and you should get the original expression.

EXAMPLE 6: **Factor: $x^3 - 7x^2 - 9x + 63$**

Solution: Group the first two terms and the last two terms:

$$(x^3 - 7x^2) + (-9x + 63)$$

Now factor the GCF in each pair of grouped terms. The first GCF is obvious: x^2. Choosing the GCF in the second grouping is a little trickier — should we choose 9 or –9? Ultimately, it's a trial-and-error process. Watch what happens if we choose –9 for the GCF:

$$x^2(x - 7) - 9(x - 7)$$ (check the signs carefully)

We now see two terms whose GCF is $x - 7$. Pull it to the front:

$$(x - 7)(x^2 - 9)$$

All this, and we're still not done. The second factor is the difference of two squares — now we're done:

$$\boxed{(x - 7)(x + 3)(x - 3)}$$

EXAMPLE 7: **Factor: $ab + cd + ad + bc$**

Solution: Group the first two terms and the last two terms (after all, this technique worked quite well in the previous two examples):

$$(ab + cd) + (ad + bc)$$

We're stuck; there's no way to factor either pair of terms (the GCF = 1 in each case), so let's swap the two middle terms of the original problem and again group in pairs:

$$(ab + ad) + (cd + bc)$$

Pull out the GCF from each set of parentheses:

$$a(b + d) + c(d + b)$$

Do we have a common factor in these two terms? Well, does $b + d = d + b$? Since addition is commutative, they are equal. So the GCF is $b + d$, and when we pull it out in front, we're done:

$$\boxed{(b + d)(a + c)}$$

EXAMPLE 8: **Factor: $2ax - bx - 2ay + by$**

Solution: Group in pairs, as usual:

$$(2ax - bx) + (-2ay + by)$$

Pull out the GCF in each grouping:

$$x(2a - b) + y(-2a + b)$$

Problem: There's no common factor; however, the factors $2a - b$ and $-2a + b$ are opposites of each other, and that gives us a clue. Let's go back to our first step and factor out $-y$ rather than y:

$$x(2a - b) - y(2a - b)$$ (distribute to make sure we're right)

Now we see a good GCF, so we pull it out in front, and we're done:

$$\boxed{(2a - b)(x - y)}$$ [Check by multiplying out]

Homework

3. Factor each expression:

a. $xw + xz + wy + yz$

b. $a^2 + ac + ab + bc$

c. $x^3 - 4x^2 + 3x - 12$

d. $n^3 - n^2 - 5n + 5$

e. $x^3 + x^2 - 9x - 9$

f. $ac - bd + bc - ad$

g. $xw + yz - xz - wy$

h. $2ac - 2ad + bc - bd$

i. $6xw - yz + 3xz - 2wy$

j. $hj - j^2 - hk + jk$

k. $ax + ay - bx - by$

l. $x^3 - 2x^2 - 25x + 50$

m. $xw + 2wy - xz - 2yz$

n. $a^3 - a^2 - 5a + 5$

o. $4tw - 2tx + 2w^2 - wx$

p. $6x^3 + 2x^2 - 9x - 3$

q. Not factorable

r. $6a^3 - 15a^2 + 10a - 25$

❑ *MORE GROUPING AND SUBSTITUTION PROBLEMS*

EXAMPLE 9: **Factor: $(w + z)^2 - a^2$**

Solution: After some practice, you might not need a substitution for this kind of problem, but we'll use one for this problem. Let $n = w + z$. The starting problem then becomes

$$n^2 - a^2$$

This is just a standard difference of squares:

$$(n + a)(n - a)$$

Now substitute in the other direction:

$$\boxed{(w + z + a)(w + z - a)}$$

EXAMPLE 10: **Factor: $x^2 + 6x + 9 - y^2$**

Solution: Grouping in pairs has worked quite well so far, so let's try it again:

$$(x^2 + 6x) + (9 - y^2)$$

We see that the first pair of terms has a nice GCF of x, and the second is the difference of squares:

$x(x + 6) + (3 + y)(3 - y)$

Good try, but there's no common factor in these two terms. In fact, no grouping into pairs will result in a common factor -- a dead end. Let's go back to the original problem and regroup so that the first three terms are together:

$(x^2 + 6x + 9) - y^2$

The first set of three terms is a perfect square trinomial, and factors into the square of a binomial:

$(x + 3)^2 - y^2$

leaving us with another difference of squares (just like the previous example), which factors to

$$\boxed{(x + 3 + y)(x + 3 - y)}$$

Homework

4. Factor each expression:

a. $(x + y)^2 - z^2$

b. $(a - b)^2 - c^2$

c. $x^2 + 4x + 4 - y^2$

d. $n^2 - 6n + 9 - Q^2$

e. $(u + w)^2 - T^2$

f. $y^2 + 10y + 25 - x^2$

g. $a^2 + 2ab + b^2 - c^2$

h. $w^2 - 2wy + y^2 - 49$

i. $4x^2 + 4x + 1 - t^2$

j. $9x^2 - 12x + 4 - y^2$

❑ *FACTORING CUBICS USING THE GCF*

EXAMPLE 11: **Factor each cubic (3rd degree) polynomial:**

A. $\mathbf{5q^3 + 10q^2 + 5q}$

This is not as bad as it looks, if we remember to start with the GCF:

$5q^3 + 10q^2 + 5q$ (the polynomial to factor)

$= 5q(q^2 + 2q + 1)$ (factor out $5q$, the GCF)

$= 5q(q + 1)(q + 1)$ (factor the trinomial)

$= \boxed{5q(q + 1)^2}$ (write it more simply)

B. $\mathbf{4x^3 - x}$

$= x(4x^2 - 1)$ (factor out x, the GCF)

$= \boxed{x(2x + 1)(2x - 1)}$ (difference of squares)

Homework

5. Factor each cubic polynomial:

a. $x^3 - x$

b. $2n^3 + 6n^2 + 4n$

c. $10a^3 - 5a^2 - 5a$

d. $7y^3 + 70y^2 + 175y$

e. $36w^3 - 9w$

f. $24z^3 - 20z^2 - 24z$

Review Problems

6. Factor each expression:

a. $10ax^4 - 160a$

b. $Z^2(P - Q) - 144(P - Q)$

c. $50x^3 - 75x^2 - 2x + 3$

d. $12ac - 10bd + 8bc - 15ad$

e. $a^2 - 2ab + b^2 - c^2$

f. $x^2 + 2xy + y^2 - 144$

g. $x^4 - 34x^2 + 225$

h. $x^4 - 8x^2 - 9$

i. $x^3 - 7x^2 + 9x - 63$

j. $n^3 + 3n^2 - 16n - 48$

k. $(a + b)^2 - 5(a + b) + 6$

l. $(x - y)^2 + 7(x - y) + 6$

m. $(a - b)^2 + 6(a - b) - 16$

n. $hm - hn + km - kn$

Solutions

1. a. $(x^2 + 1)(x + 1)(x - 1)$

b. $(x^2 + 2)(x^2 - 3)$

c. $(n + 1)(n - 1)(n + 3)(n - 3)$

d. $(a^2 + 9)(a + 3)(a - 3)$

e. $(2w + 1)(2w - 1)(3w + 2)(3w - 2)$

f. $(x + 1)(x - 1)(3x + 5)(3x - 5)$

g. $(c^2 + 4)(c + 2)(c - 2)$

h. $(x^2 + 1)(x + 3)(x - 3)$

i. $(x^2 + 2)(x^2 - 5)$

j. $(g^2 + 16)(g + 4)(g - 4)$

k. $(2u + 3)(2u - 3)(3u + 1)(3u - 1)$

l. Not factorable

2. a. $(x + y)(x + y + 7)$

b. $(a - b)(a - b - c)$

c. $(c + d)(x^2 + 5)$

d. $(a - b)(n + 3)(n - 3)$

e. $(a + 4)(x + 3)(x + 2)$

f. $y(m + n)(y + 7)$

g. $(a + b)(2x + 5)(x - 1)$

h. $(w + z)(2x + 3)(2x - 3)$

i. $(u - w)(u - w - 9)$

j. $n(a + b)(n - 9)$

k. $(t + r)(y + 10)(y - 10)$

l. $a(3x + 1)(x - 7)$

3.

a.	$(x + y)(w + z)$	b.	$(a + b)(a + c)$	c.	$(x^2 + 3)(x - 4)$
d.	$(n^2 - 5)(n - 1)$	e.	$(x + 1)(x + 3)(x - 3)$	f.	$(a + b)(c - d)$
g.	$(x - y)(w - z)$	h.	$(2a + b)(c - d)$	i.	$(3x - y)(2w + z)$
j.	$(h - j)(j - k)$	k.	$(a - b)(x + y)$	l.	$(x - 2)(x + 5)(x - 5)$
m.	$(x + 2y)(w - z)$	n.	$(a^2 - 5)(a - 1)$	o.	$(2t + w)(2w - x)$
p.	$(2x^2 - 3)(3x + 1)$	q.	Not factorable	r.	$(3a^2 + 5)(2a - 5)$

4.

a.	$(x + y + z)(x + y - z)$	b.	$(a - b + c)(a - b - c)$
c.	$(x + 2 + y)(x + 2 - y)$	d.	$(n - 3 + Q)(n - 3 - Q)$
e.	$(u + w + T)(u + w - T)$	f.	$(y + 5 + x)(y + 5 - x)$
g.	$(a + b + c)(a + b - c)$	h.	$(w - y + 7)(w - y - 7)$
i.	$(2x + 1 + t)\ (2x + 1 - t)$	j.	$(3x - 2 + y)(3x - 2 - y)$

5.

a.	$x(x + 1)(x - 1)$	b.	$2n(n + 1)(n + 2)$
c.	$5a(2a + 1)(a - 1)$	d.	$7y(y + 5)^2$
e.	$9w(2w + 1)(2w - 1)$	f.	$4z(3z + 2)(2z - 3)$

6.

a.	$10a(x^2 + 4)(x + 2)(x - 2)$	b.	$(P - Q)(Z + 12)(Z - 12)$
c.	$(2x - 3)(5x + 1)(5x - 1)$	d.	$(3a + 2b)(4c - 5d)$
e.	$(a - b + c)(a - b - c)$	f.	$(x + y + 12)(x + y - 12)$
g.	$(x + 5)(x - 5)(x + 3)(x - 3)$	h.	$(x^2 + 1)(x + 3)(x - 3)$
i.	$(x^2 + 9)(x - 7)$	j.	$(n + 4)(n - 4)(n + 3)$
k.	$(a + b - 3)(a + b - 2)$	l.	$(x - y + 6)(x - y + 1)$
m.	$(a - b + 8)(a - b - 2)$	n.	$(m - n)(h + k)$

"A college degree is not a sign that one is a finished product, but an indication a person is prepared for life."

Reverend Edward A. Malloy, *Monk's Reflections*

COMPLETE FACTORING

Consider the problem of factoring **$10x^2 + 50x + 60$.** Look at the 10. Its factor pairs are 1 and 10, or 2 and 5. Now take a look at the 60. It's downright scary to consider all the pairs of factors of that number. But watch what happens in our first example if we deal with the GCF first, and then worry about the rest later.

❑ *COMPLETE FACTORING*

EXAMPLE 1: **Factor completely: $10x^2 + 50x + 60$**

Solution: As mentioned above, the secret to factoring this quadratic expression is to utilize the **GCF**. The variable x is not common to all three terms, so we'll ignore it. However, each of the three terms does contain a factor of 10 (since 10 divides evenly into each of the coefficients 10, 50, and 60). Thus,

	$10x^2 + 50x + 60$	(the given expression)
=	$10(x^2 + 5x + 6)$	(pull out the GCF of 10)
=	$10(x + 3)(x + 2)$	(factor the quadratic)

Not so difficult, after all. Therefore, the complete factorization of $10x^2 + 50x + 60$ is

$$\boxed{10(x + 3)(x + 2)}$$

The key to complete factoring is to FIRST pull out the GCF!

Homework

1. Factor each expression completely:

a. $7x^2 - 35x + 42$

b. $10n^2 - 10$

c. $5a^2 - 30a + 45$

d. $50u^2 - 25u - 25$

e. $7w^2 - 700$

f. $9n^2 + 9$

g. $5y^2 - 125$

h. $3x^2 + 15x + 12$

i. $14x^2 - 7x - 7$

j. $13t^2 + 117$

k. $48z^2 - 28z + 4$

l. $24a^2 - 120a + 150$

❑ *Reducing Fractions*

EXAMPLE 2: **Reduce to lowest terms:** $\dfrac{5a^2 - 45}{a^2 + 6a + 9}$

Solution: We need to factor the numerator and the denominator. If we then see any common factors, we can divide them out.

$$\frac{5a^2 - 45}{a^2 + 6a + 9} = \frac{5(a^2 - 9)}{a^2 + 6a + 9} = \frac{5(a+3)(a-3)}{(a+3)(a+3)} = \frac{5\cancel{(a+3)}(a-3)}{\cancel{(a+3)}(a+3)}$$

Notice that factoring the numerator required two steps: pulling out the GCF of 5, followed by factoring the $a^2 - 9$. If we hadn't factored out the 5, we would never have been able to divide out anything, and we would have reached the false conclusion that the fraction is not reducible. So we can write the reduced fraction as $\frac{5(a-3)}{a+3}$. But, assuming it's not too much work, it's customary to remove parentheses. Thus, the final answer (after distributing the 5 to the $a - 3$) is

$$\boxed{\frac{5a-15}{a+3}}$$

Homework

2. Reduce each fraction to lowest terms:

a. $\frac{n^2-4}{n^2-4n+4}$

b. $\frac{2x^2+8x+6}{6x^2+18x+12}$

c. $\frac{x^2-4x+1}{x^2-4x+1}$

d. $\frac{10y^2-30y+20}{5y^2-15y+10}$

e. $\frac{2x^2-2}{4x-4}$

f. $\frac{3n^2-3n-90}{3n^2+30n+75}$

g. $\frac{14x+98}{21x^2-63x-1470}$

h. $\frac{5a^2-30a-135}{10a^2-60a-270}$

❑ *Additional Quadratic Equations*

Now we'll use the GCF factoring method to solve more quadratic equations. The following example should convince you that factoring out a simple number first makes the rest of the factoring — and thus solving the equation — vastly easier.

EXAMPLE 3: **Solve for k: $16k^2 = 40k + 24$**

Solution: Solving any quadratic equation by factoring requires making one side of the equation zero. To this end, we will first bring the $40k$ and the 24 to the left side, factor in two steps, divide each side by the greatest common factor, set each factor to 0, and then solve each resulting equation. (That's a lot of steps!)

	$16k^2 = 40k + 24$	(the original equation)
$\Rightarrow$	$16k^2 - 40k - 24 = 0$	(subtract $40k$ and 24)
$\Rightarrow$	$8(2k^2 - 5k - 3) = 0$	(factor out 8, the GCF)
$\Rightarrow$	$\frac{\cancel{8}(2k^2 - 5k - 3)}{\cancel{8}} = \frac{0}{8}$	(divide each side by 8)
$\Rightarrow$	$2k^2 - 5k - 3 = 0$	(simplify)
$\Rightarrow$	$(2k + 1)(k - 3) = 0$	(factor)
$\Rightarrow$	$2k + 1 = 0$ or $k - 3 = 0$	(set each factor to 0)
$\Rightarrow$	$\boxed{k = -\frac{1}{2} \text{ or } k = 3}$	(solve each linear equation)

This was a lot of work — you might be thinking that the Quadratic Formula (assuming you've studied it) would have been easier to use than factoring, and you're probably right! Generally, you can use whichever method you prefer (assuming it works), but pay attention to the method your instructor may require on an exam.

❑ *CAREFUL ! ! !*

Do you see the step in the preceding example where we divided both sides of the equation by 8? This was legal because we did the same thing to both sides of the equation, and we did NOT divide by zero. Do not ever fall into the trap of dividing each side of an equation by something with the variable in it — you may unknowingly be dividing by zero, resulting in a loss of a solution to the equation.



To illustrate this warning , the correct way to solve the quadratic equation $x^2 + x = 0$ is the following:

$$x^2 + x = 0$$

$$\Rightarrow \quad x(x+1) = 0$$

$$\Rightarrow \quad x = 0 \text{ or } x + 1 = 0$$

$$\Rightarrow \quad x = 0 \text{ or } x = -1$$

We have two solutions: 0 and –1, no doubt about it.

Check: $x = 0$: $\mathbf{0}^2 + \mathbf{0} = 0 + 0 = 0$ ✓

$x = -1$: $(\mathbf{-1})^2 + (\mathbf{-1}) = 1 + (-1) = 1 - 1 = 0$ ✓

Now let's do it the wrong way, by dividing by the variable:

$$x^2 + x = 0 \Rightarrow x(x+1) = 0 \Rightarrow \frac{x(x+1)}{x} = \frac{0}{x} \Rightarrow x + 1 = 0$$

$\Rightarrow$ $x = -1$, which is merely one of the two solutions. That is, *we lost a solution when we divided by the variable*. Since the purpose of algebra is to obtain solutions — not throw them away — we see that dividing by the variable was a really bad idea.

Homework

3. Solve each quadratic equation:

a. $7x^2 - 35x + 42 = 0$

b. $10n^2 = 10$

c. $5a^2 + 45 = 30a$

d. $50u^2 = 25u + 25$

e. $7w^2 - 700 = 0$

f. $180z^2 - 30z - 60 = 0$

g. $4x^2 + 4x - 24 = 0$

h. $16x^2 = 6 - 4x$

i. $10x^2 - 490 = 0$

j. $75w^2 + 48 = 120w$

Review Problems

4. Factor completely: $30q^2 + 68q + 30$

5. Reduce: $\dfrac{18x+18}{14x^2+42x+28}$

6. Solve for x by factoring: $30x^2 = 190x + 140$

❑ *TO ∞ AND BEYOND*

A. Factor: $x^2 - 363x + 21{,}600$

Hint: Use the Quadratic Formula (and your calculator) to solve the equation $x^2 - 363x + 21{,}600 = 0$, and then use the solutions to write the factorization.

B. Solve for x: $x^4 - 13x^2 + 36 = 0$

Hint: Factoring will take two steps, producing four solutions.

Solutions

1. a. $7(x-3)(x-2)$ b. $10(n+1)(n-1)$ c. $5(a-3)^2$
d. $25(2u+1)(u-1)$ e. $7(w+10)(w-10)$ f. $9(n^2+1)$
g. $5(y+5)(y-5)$ h. $3(x+1)(x+4)$ i. $7(2x+1)(x-1)$
j. $13(t^2+9)$ k. $4(4z-1)(3z-1)$ l. $6(2a-5)^2$

2. a. $\frac{n+2}{n-2}$ b. $\frac{x+3}{3x+6}$ c. 1 d. 2
e. $\frac{x+1}{2}$ f. $\frac{n-6}{n+5}$ g. $\frac{2}{3x-30}$ h. $\frac{1}{2}$

3. a. $2, 3$ b. ± 1 c. 3 d. $1, -\frac{1}{2}$ e. ± 10
f. $\frac{2}{3}, -\frac{1}{2}$ g. $2, -3$ h. $\frac{1}{2}, -\frac{3}{4}$ i. ± 7 j. $\frac{4}{5}$

4. $2(5x+3)(3x+5)$ **5**. $\frac{9}{7x+14}$ **6**. $7, -\frac{2}{3}$

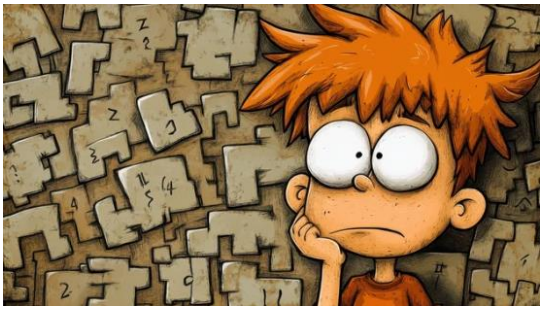
– Definition of ***happiness*** by John F. Kennedy (1917-1963)

"The full use of your powers along lines of excellence."



FACTORING THE SUM AND DIFFERENCE OF CUBES

We've learned that we can factor the *difference of squares* $x^2 - y^2$ into $(x + y)(x - y)$. We've also determined that the *sum of squares* $x^2 + y^2$ cannot be factored.



Now we're about to show that the ***difference of cubes*** $x^3 - y^3$ can also be factored — and perhaps surprisingly — even the ***sum of cubes*** $x^3 + y^3$ can be factored.

❑ *DIVISION, REMAINDERS, AND FACTORS*

Is 6 a **factor** of 161? **No** — divide 161 by 6 and you'll get 26 remainder **5**. Since the remainder is not 0, 6 is not a factor of 161. In other words, 6 does not go into 161 "evenly."

Is 7 a **factor** of 161? **Yes** — divide 161 by 7 and you'll get 23, remainder **0**. Thus, 7 divides into 161 exactly 23 times. And therefore, $161 = 7 \times 23$. We have factored 161 into 7×23 by showing that the factor 7 divides into 161 without remainder.

These observations are the key to factoring the sum and difference of cubes.

❑ *PERFECT CUBES*

We know that $2^3 = 8$. Since the cube of 2 is 8, we say that 8 is a ***perfect cube***. Here are some more examples of perfect cubes:

125 is a *perfect cube* because it's the cube of 5:

$$125 = 5^3$$

1 is a perfect cube because it's the cube of 1:

$$1 = 1^3$$

–27 is a perfect cube because it's the cube of –3:

$$27 = (-3)^3$$

0 is a perfect cube because it's the cube of 0:

$$0 = 0^3$$

$\boldsymbol{x^3}$ is a perfect cube because it's the cube of x:

$$x^3 = x^3$$

$\boldsymbol{27y^3}$ is a perfect cube because it's the cube of $3y$:

$$27y^3 = (3y)^3$$

$\boldsymbol{8n^6}$ is a perfect cube because it's the cube of $2n^2$:

$$8n^6 = (2n^2)^3$$

$\boldsymbol{64z^{12}}$ is a perfect cube because it's the cube of $4z^4$:

$$64z^{12} = (4z^4)^3$$

$\boldsymbol{(a-b)^3}$ is a perfect cube because it's the cube of $a - b$.

Homework

1. a. $64m^3$ is a perfect cube because it's the cube of _____.

b. $216n^3$ is a perfect cube because it's the cube of _____.

c. $27A^6$ is a perfect cube because it's the cube of _____.

d. _____ is a perfect cube because it's the cube of $7z^2$.

e. _____ is a perfect cube because it's the cube of $-3a^3$.

❑ *Factoring a Sum of Cubes — Long Division*

We're now ready to try to factor a sum of cubes; for example, what is the factorization of

$$x^3 + 8\,?$$ [What should we multiply to get $x^3 + 8$?]

To answer this question, we should try to divide $x^3 + 8$ by something that goes into it evenly; that is, divide $x^3 + 8$ by something that will leave a remainder of 0. But what should we divide by? Since both terms of $x^3 + 8$ are perfect cubes, let's divide it by the binomial $x + 2$, since these two terms are the cube roots of x^3 and 8. Maybe this will work and maybe it won't, but we've got to try something.

$$\begin{array}{r} x^2 - 2x + 4 \\ x+2 \overline{\big|\, x^3 + 0x^2 + 0x + 8} \\ \underline{x^3 + 2x^2} \\ -2x^2 + 0x \\ \underline{-2x^2 - 4x} \\ 4x + 8 \\ \underline{4x + 8} \\ 0 \end{array}$$

Here's the division of $x^3 + 8$ by $x + 2$. Note that the dividend has two zeros placed in it to account for the missing terms.

Also note that the remainder is 0. This means that $x + 2$ is a factor of $x^3 + 8$ and therefore, that $x^2 - 2x + 4$ is the other factor.

Now we write the results of our long division as a multiplication problem, giving us the factorization of $x^3 + 8$:

$$x^3 + 8 = (x+2)(x^2 - 2x + 4)$$

Check by multiplying

❑ *FACTORING A DIFFERENCE OF CUBES — LONG DIVISION*

For our difference of cubes, let's try to factor $n^3 - 27$. What do you think one of the factors will be? Consider the binomial consisting of the individual cube roots of n^3 and –27, namely n and – 3. This time it's your turn to carry out the long division. Here's what you should end up with:

$$\begin{array}{r} n^2 + 3n + 9 \\ n - 3 \overline{\big| \; n^3 + 0n^2 + 0n - 27} \end{array}$$

We now have our factorization:

$$n^3 - 27 = (n-3)(n^2 + 3n + 9)$$

Again, check by multiplying

EXAMPLE 1:

A. Factor: $N^3 - 1$

Divide $N^3 - 1$ by $N - 1$ and you should get the factorization

$$\boldsymbol{N^3 - 1 = (N-1)(N^2 + N + 1)}$$

B. Factor: $8p^3 + 27$

Divide $8p^3 + 27$ by $2p + 3$. It should divide evenly, giving

$$\boldsymbol{8p^3 + 27 = (2p + 3)(4p^2 - 6p + 9)}$$

C. Factor: $(a + b)^3 - 125$.

This is tricky, and it will be much easier to perform the long division if we make a substitution first. If we let $x = a + b$, then the expression to factor becomes $x^3 - 125$. The appropriate quantity to divide this by would be $x - 5$. When the long division is finished, the quotient is $x^2 + 5x + 25$ with remainder 0. We therefore get the factorization

$$x^3 - 125 = (x - 5)(x^2 + 5x + 25)$$

But the original problem didn't have any x's in it. So we need to substitute back the other way — when we convert each x back into $a + b$, we get the factorization

$$(a + b)^3 - 125 = \big((a + b) - 5\big)\big((a + b)^2 + 5(a + b) + 25\big),$$

which can be simplified to the final answer of

$$\mathbf{(a + b)^3 - 125 = (a + b - 5)(a^2 + 2ab + b^2 + 5a + 5b + 25)}$$

Homework

2. In the discussion above, we arrived at the following factorizations:

   a. $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$

   b. $n^3 - 27 = (n - 3)(n^2 + 3n + 9)$

   Verify each result by simplifying the right side of the statement so that it becomes the left side.

3. Factor each expression:

   a. $x^3 - 8$ b. $n^3 + 27$ c. $z^3 + 1$

   d. $8x^3 - 27$ e. $27y^3 + 125$ f. $64a^3 - 1$

4. Factor each expression:

   a. $(x+y)^3+8$ b. $(a-b)^3-27$ c. $(p+q)^3+1$

5. Factor x^5+1. Hint: Divide by $x+1$.

6. Factor A^5-32. Hint: Divide by $A-2$.

7. Factor each expression:

   a. w^5+1 b. c^5-1 c. y^5-32

   d. z^5+32 e. n^5+243 f. m^5-243

8. Factor each expression:

   a. x^7-1 b. y^7+1 c. u^7-128 d. z^7+128

❑ *CREATING FORMULAS FOR FACTORING CUBES*

Factoring the sum (or difference) of cubes using long division is kind of tedious (and prone to errors). I suggest we do long division on a "generic" sum of cubes and see if we can come up with a formula that will allow us to factor the sum of cubes more efficiently.

Let's start by considering the factorization of x^3+y^3. If you've understood this chapter so far, you'll understand what I'm about to do: Divide x^3+y^3 by $x+y$, and confirm that the remainder comes out to be 0. Then we'll have our factorization.

$$x+y\overline{\left|x^3+y^3\right.}$$

We first have to pad up the dividend with some placeholders. What terms to include may not be obvious, but there's a nice pattern here: The powers on the x's go down while the powers on the y's go up.

$$
\begin{array}{r}
x^2 - xy + y^2 \\
x + y \;\overline{\left| \; x^3 + 0x^2y + 0xy^2 + y^3 \right.} \\
\underline{x^3 + x^2y} \\
-x^2y + 0xy^2 \\
\underline{-x^2y - xy^2} \\
xy^2 + y^3 \\
\underline{xy^2 + y^3} \\
0 \quad \checkmark
\end{array}
$$

Since the remainder is 0, we have our factorization:

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

EXAMPLE 2: **Factor: $a^3 + 8$**

$$
\begin{aligned}
& a^3 + 8 \\
= \quad & a^3 + 2^3 && \text{[Here } x = a \text{ and } y = 2\text{]} \\
= \quad & (a + 2)(a^2 - a \cdot 2 + 2^2) && \text{[From the formula for } x^3 + y^3\text{]} \\
= \quad & \boxed{(a + 2)(a^2 - 2a + 4)} && \text{[Multiply out to confirm]}
\end{aligned}
$$

EXAMPLE 3: **Factor: $n^6 + 125w^3$**

$$
\begin{aligned}
& n^6 + 125w^3 \\
= \quad & (n^2)^3 + (5w)^3 && \text{[Here } x = n^2 \text{ and } y = 5w\text{]} \\
= \quad & (n^2 + 5w)\left((n^2)^2 - n^2(5w) + (5w)^2\right) \\
= \quad & \boxed{(n^2 + 5w)(n^4 - 5n^2w + 25w^2)}
\end{aligned}
$$

Homework

9. Find the factorization of $x^3 - y^3$ by dividing it by $x - y$.

10. Redo Problem #3 using our two new formulas for factoring the sum and difference of cubes.

❑ *To ∞ and Beyond*

A. Consider the expression $\boldsymbol{x^6 - y^6}$. Factor it in two ways:

1) Factor it as a difference of squares, followed by using the sum and difference of cubes.

2) Factor it as a difference of cubes, followed by a difference of squares.

Which method do you think is better, and why?

B. If I asked you for the *cube root* of 8, you'd probably say 2. And you're right, $2^3 = 8$. ✓ But if you have studied imaginary numbers (involving the number $\boldsymbol{i}$, defined by $\boldsymbol{i} = \sqrt{-1}$), then the fact is that 8 actually has <u>three</u> cube roots, the real root 2 we saw above, and two *non-real* roots (roots containing the imaginary number $\boldsymbol{i}$).

Here's how we can find them. Let x represent a cube root of 8. This implies that $x^3 = 8$, which implies that $x^3 - 8 = 0$. We now have a cubic equation that just might produce three solutions (just like a quadratic equation (degree = 2) might yield two solutions). These three potential solutions will all be cube roots of 8. Here's where the factoring in the chapter comes in:

$$x^3 - 8 = 0$$

$\Rightarrow \quad (x-2)(x^2+2x+4) = 0$ (difference of cubes)

$\Rightarrow \quad x-2=0 \text{ OR } x^2+2x+4 = 0$ ($AB=0 \Rightarrow A=0$ OR $B=0$)

The first equation (linear) is easy to solve: $\boldsymbol{x = 2}$, which we already knew was a cube root of 8.

To solve the second equation (quadratic) we will apply the Quadratic Formula (or we could complete the square). Since $a = 1$, $b = 2$, and $c = 4$, we can solve for x:

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}\,i}{2}$$ (the i is <u>not</u> in the radical sign)

$$= \frac{2\left(-1 \pm \sqrt{3}\,i\right)}{2}$$ (factor the numerator)

$$= -1 \pm \sqrt{3}\,i$$ (divide out the 2's, and notice that we have two solutions)

And we now have our three solutions to the equation $x^3 - 8 = 0$, which is equivalent to having our three cube roots of 8, as promised:

$$2,\ -1+\sqrt{3}\,i, \text{ and } -1-\sqrt{3}\,i$$

If we allow ourselves to venture beyond the real numbers, we can find <u>three</u> cube roots of 8.

If you're up to it, verify the two complex cube roots of 8 by cubing each one and verifying that the final result will be 8 in both cases.

Solutions

1. a. $4m$ b. $6n$ c. $3A^2$ d. $343z^6$ e. $-27a^9$

2. a. $(x+2)(x^2-2x+4) = x^3-2x^2+4x+2x^2-4x+8 = x^3+8$ ✓

b. You try it.

3. a. $(x-2)(x^2+2x+4)$ b. $(n+3)(n^2-3n+9)$

c. $(z+1)(z^2-z+1)$ d. $(2x-3)(4x^2+6x+9)$

e. $(3y+5)(9y^2-15y+25)$ f. $(4a-1)(16a^2+4a+1)$

4. a. $(x+y+2)(x^2+2xy+y^2-2x-2y+4)$

b. $(a-b-3)(a^2-2ab+b^2+3a-3b+9)$

c. $(p+q+1)(p^2+2pq+q^2-p-q+1)$

5. $(x+1)(x^4-x^3+x^2-x+1)$

6. $(A-2)(A^4+2A^3+4A^2+8A+16)$

7. a. $(w+1)(w^4-w^3+w^2-w+1)$

b. $(c-1)(c^4+c^3+c^2+c+1)$

c. $(y-2)(y^4+2y^3+4y^2+8y+16)$

d. $(z+2)(z^4-2z^3+4z^2-8z+16)$

e. $(n+3)(n^4-3n^3+9n^2-27n+81)$

f. $(m-3)(m^4+3m^3+9m^2+27m+81)$

8. a. $(x-1)(x^6+x^5+x^4+x^3+x^2+x+1)$

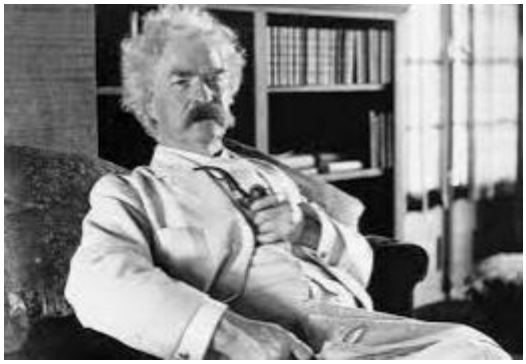
b. $(y+1)(y^6-y^5+y^4-y^3+y^2-y+1)$

c. $(u-2)(u^6+2u^5+4u^4+8u^3+16u^2+32u+64)$

d. $(z+2)(z^6-2z^5+4z^4-8z^3+16z^2-32z+64)$

9. $x^3-y^3 = (x-y)(x^2+xy+y^2)$

"Keep away from people who try to belittle your ambitions. Small people always do that, but the really great make you feel that you, too, can become great."



Mark Twain

FACTORING QUADRATICS – AN INTRODUCTION

By now, you should know how to factor $x^2 + 9x$ into $x(x + 9)$. That's just the GCF (Greatest Common Factor) at work. But what about the prospect of factoring $2x^2 + 13x + 15$? This chapter illustrates the first steps needed to solve this problem.



❑ *RECALLING HOW WE MULTIPLY BINOMIALS*

To set the stage for factoring binomials (two terms) and trinomials (three terms), we'll do two more examples of double distributing to analyze its inner workings:

$$(2x + 3)(x - 5)$$

i) Multiply the FIRST terms in each set of parentheses: $2x$ and x. The product is $\underline{2x^2}$.

ii) Multiply the OUTER terms: $2x$ and -5. The product is $\underline{-10x}$.

iii) Multiply the INNER terms: 3 and x. The product is $\underline{3x}$.

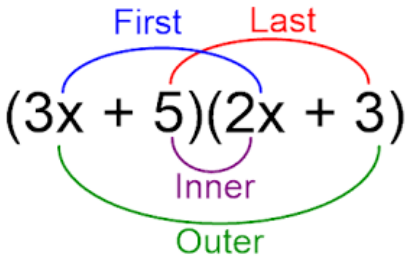
iv) Multiply the LAST terms in each set of parentheses: 3 and -5. The product is $\underline{-15}$.

v) Writing out these four products, followed by combining like terms, gives us

$2x^2 - 10x + 3x - 15$

First, Outer, Inner, Last

$= \mathbf{2x^2 - 7x - 15}$



For a second example,

$$(3x + 5)(2x + 3)$$

$$= 6x^2 + 9x + 10x + 15$$

- $6x^2$: product of the **First** terms
- $9x$: product of the **Outer** terms
- $10x$: product of the **Inner** terms
- 15: product of the **Last** terms

$$= \underline{\mathbf{6x^2 + 19x + 15}}$$

The key idea to absorb here is that the $6x^2$ in the answer is the product of the first terms, while the 15 is the product of the last terms. And even more importantly, the middle term in the answer, $19x$, is the sum of the outer and inner products.

Homework

1. Find the following products — do all the work in your head:

   a. $(x + 2)(x + 3)$
   b. $(n - 3)(n - 5)$
   c. $(a + 7)(a - 5)$
   d. $(b - 3)(b + 10)$
   e. $(y + 6)(y - 6)$
   f. $(z + 3)(z + 3)$
   g. $(u + 1)(u + 7)$
   h. $(v + 7)(v - 7)$
   i. $(x + 8)(x + 8)$
   j. $(k - 10)(k - 10)$

2. Find the following products — do all the work in your head:

   a. $(2x - 1)(x + 4)$
   b. $(3n - 3)(2n - 5)$
   c. $(a + 9)(3a - 1)$
   d. $(6y - 1)(2y + 7)$

e. $(2m - 7)(2m + 7)$

f. $(4w + 5)(4w + 5)$

g. $(5x + 1)(5x - 1)$

h. $(2n + 3)(7n - 10)$

i. $(7u - 3)(7u - 3)$

j. $(12a + 13)(12a - 13)$

❑ *UNSCRAMBLING THE EGG*

Now suppose that we've been handed the quadratic expression

$$2x^2 + 13x + 15$$

to factor. [In other words, *what* times *what* equals $2x^2 + 13x + 15$?] We've done enough double distributing (we hope) to realize that the factorization probably looks like this, the product of two binomials:

$$(\underline{\text{slot 1}} + \underline{\text{slot 2}})(\underline{\text{slot 3}} + \underline{\text{slot 4}})$$

where the contents of the four slots have to be determined.

We know that the product of slot 1 and slot 3 must be $2x^2$. It's logical to assume that slot 1 is $2x$ and slot 3 is x. Now we have

$$(2x + \underline{\text{slot 2}})(x + \underline{\text{slot 4}})$$

We also know that the product of slot 2 and slot 4 must be 15. Now we're at a crossroads: The product of 3 and 5 is 15, but the product of 1 and 15 is also 15. It gets worse — even when we select a pair of factors of 15 to try, the <u>order</u> in which we place those factors in the slots might be crucial. As we try various pairs of numbers, keep in mind that we're trying to factor $2x^2 + 13x + 15$.



Let's try 1 and 15 (in that order):

$(2x + \underline{\mathbf{1}})(x + \underline{\mathbf{15}}) = 2x^2 + 30x + x + 15 = 2x^2 + 31x + 15$ ☹

Flip around the 1 and the 15:

$(2x + \underline{\mathbf{15}})(x + \underline{\mathbf{1}}) = 2x^2 + 2x + 15x + 15 = 2x^2 + 17x + 15$ ☹

That rules out using 1 and 15. Let's use the 5 and the 3 (in that order):

$(2x + \underline{\mathbf{5}})(x + \underline{\mathbf{3}}) = 2x^2 + 6x + 5x + 15 = 2x^2 + 11x + 15$ ☹

Flip the 5 and 3:

$(2x + \underline{\mathbf{3}})(x + \underline{\mathbf{5}}) = 2x^2 + 10x + 3x + 15 = 2x^2 + 13x + 15$ ☺

And so the factorization of $2x^2 + 13x + 15$ is **$(2x + 3)(x + 5)$**.

> Recall that we can summarize the process we just completed in two ways: In changing
>
> $2x^2 + 13x + 15$ to $(2x + 3)(x + 5)$
>
> we have
>
> 1) converted an expression whose final operation is addition into an expression whose final operation is multiplication, and we have
> 2) converted an expression with three terms into an expression with one term.
>
> Either way, we've accomplished the goal of **factoring**.

Homework

3. True/False:

   a. $x^2 + 5x + 6$ factors into $(x + 3)(x + 2)$.

   b. $y^2 - 16$ factors into $(y + 4)(y - 4)$.

   c. $x^2 + 9x + 15$ factors into $(x + 3)(x + 5)$.

   d. $n^2 - 6n + 9$ factors into $(n - 3)(n - 3)$.

   e. $2a^2 - 11a - 6$ factors into $(2a + 1)(a - 6)$.

   f. $u^2 + 25$ factors into $(u + 5)(u + 5)$.

g. $a^2 + 10a + 25$ factors into $(a + 5)^2$.

h. $y^2 - 2y + 4$ factors into $(y - 2)^2$.

4. Matching:

a. ___	$x^2 + 4x + 3$	1. $(2x + 1)(x - 6)$
b. ___	$x^2 - 9$	2. $(x + 5)(x - 5)$
c. ___	$x^2 + 10x + 25$	3. $(x + 3)(x + 1)$
d. ___	$x^2 + 16$	4. $(x + 3)(x - 3)$
e. ___	$x^2 + 4x + 4$	5. $(2x - 3)(x + 2)$
f. ___	$2x^2 - 11x - 6$	6. $(x + 5)(x + 5)$
g. ___	$x^2 - 25$	7. $(x + 2)(x + 2)$
h. ___	$2x^2 + x - 6$	8. None of them

5. Finish the factorization of each expression:

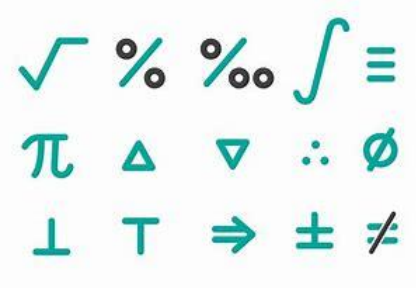
a. $x^2 - 10x + 16 = (x - 8)(\quad)$

b. $n^2 + 5n - 14 = (n + 7)(\quad)$

c. $a^2 - 17a + 72 = (a - 9)(\quad)$

d. $q^2 - 49 = (q + 7)(\quad)$

e. $c^2 + 6c + 9 = (c + 3)(\quad)$



6. Finish the factorization of each expression:

a. $12n^2 + 8n - 15 = (6n - \quad)(2n + \quad)$

b. $16x^2 - 9 = (4x + \quad)(4x - \quad)$

c. $21z^2 - 4z - 1 = (7z + \quad)(3z - \quad)$

d. $9a^2 + 24a + 16 = (3a + \quad)(3a + \quad)$

e. $6x^2 - 23x + 7 = (2x - \quad)(3x - \quad)$

f. $25t^2 - 49 = (5t + \quad)(5t - \quad)$

g. $16c^2 - 24c + 9 = (4c - \quad)(4c - \quad)$

h. $14x^2 - 58x + 8 = (7x - \quad)(2x - \quad)$

7. Which of the following is the factorization of $x^2 - 2x - 8$?

   a. $(x + 8)(x - 1)$ b. $(x - 8)(x + 1)$

   c. $(x + 2)(x - 4)$ d. $(x - 2)(x + 4)$

8. Which of the following is the factorization of $3n^2 + 7n - 10$?

   a. $(3n - 1)(n + 10)$ b. $(3n + 5)(n - 2)$ c. $(3n - 5)(n + 2)$

   d. $(3n - 10)(n + 1)$ e. $(3n + 2)(n - 5)$ f. $(3n + 10)(n - 1)$

9. Your friend shows you her attempt at the following factorization:

   $$6x^2 + 17x - 14 = (2x + 1)(3x - 14)$$

   You seriously doubt the accuracy of her answer and suggest that perhaps she's made a mistake. She gets a little miffed and tells you, "Look! $2x$ times $3x$ is $6x^2$ and 1 times -14 is -14. See? It all works out." How might you gently show her the error of her ways?



"I'll have the math homework."

Review Problems

10. Find the product in your head:
    $(3n - 7)(4n + 7)$

11. True/False: $16x^2 - 2x - 3$ factors into $(8x + 3)(2x - 1)$.

12. Finish the factorization of $14a^2 - 11a - 3$: $(a - 1)(\quad)$

13. Finish the factorization of $10n^2 - 31n + 15$: $(5n \quad)(2n \quad)$

Solutions

1. a. $x^2 + 5x + 6$ b. $n^2 - 8n + 15$ c. $a^2 + 2a - 35$
d. $b^2 + 7b - 30$ e. $y^2 - 36$ f. $z^2 + 6z + 9$
g. $u^2 + 8u + 7$ h. $v^2 - 49$ i. $x^2 + 16x + 64$
j. $k^2 - 20k + 100$

2. a. $2x^2 + 7x - 4$ b. $6n^2 - 21n + 15$ c. $3a^2 + 26a - 9$
d. $12y^2 + 40y - 7$ e. $4m^2 - 49$ f. $16w^2 + 40w + 25$
g. $25x^2 - 1$ h. $14n^2 + n - 30$ i. $49u^2 - 42u + 9$
j. $144a^2 - 169$

3. a. T b. T c. F d. T
e. T f. F g. T h. F

4. a. 3 b. 4 c. 6 d. 8
e. 7 f. 1 g. 2 h. 5

5. a. $x - 2$ b. $n - 2$ c. $a - 8$ d. $q - 7$ e. $c + 3$

6. a. 5, 3 b. 3, 3 c. 1, 1 d. 4, 4 e. 7, 1
f. 7, 7 g. 3, 3 h. 1, 8

7. c.

8. f.

9. Everything she said was right — she just didn't complete the process. You would point out (gently, remember) that the product of the outer terms, the $2x$ and the -14, is $-28x$, while the product of the inner terms, the 1 and the $3x$, is $3x$. Combining these two like terms produces $-25x$ for the middle term, <u>not</u> the required $17x$ in the original problem.

10. $12n^2 - 7n - 49$

11. True

12. $14a + 3$

13. $-3 \quad -5$

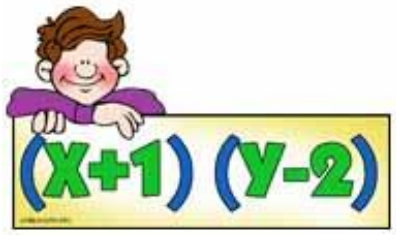


"The best thing for being sad," replied Merlin, beginning to puff and blow, "is to learn something. That's the only thing that never fails. You may grow old and trembling in your anatomies, you may lie awake at night listening to the disorder of your veins, you may miss your only love, you may see the world about you devastated by evil lunatics, or know your honour trampled in the sewers of baser minds. There is only one thing for it then — to *learn*. Learn why the world wags and what wags it. That is the only thing which the mind can never exhaust, never alienate, never be tortured by, never fear or distrust, and never dream of regretting. Learning is the only thing for you. Look what a lot of things there are to learn."



– T.H. White, *The Once and Future King* –

FACTORING QUADRATICS, THE REAL DEAL



Hopefully you have grasped the concept of factoring. We now present a series of examples that try to turn the random method into something a little more systematic. But no matter what method is used, it's essentially a matter of *trial and error*. (Some teachers call it *guess and check*.)

❑ *EXAMPLES*

EXAMPLE 1: **Factor: $6x^2 - 7x - 5$**

Solution: We first consider how $6x^2$ can be broken up using multiplication. We see $6x \cdot x$ and $3x \cdot 2x$.

Using $6x$ and x, we start with

$$(6x ____)(x ____)$$

Now put numbers whose product is -5 in the slots:

$(6x + 5)(x - 1) = 6x^2 - 6x + 5x - 5 = 6x^2 - x - 5$ ☹

$(6x - 5)(x + 1) = 6x^2 + 6x - 5x - 5 = 6x^2 + x - 5$ ☹

$(6x + 1)(x - 5) = 6x^2 - 30x + x - 5 = 6x^2 - 29x - 5$ ☹

$(6x - 1)(x + 5) = 6x^2 + 30x - x - 5 = 6x^2 + 29x - 5$ ☹

That's it for the -5; we've tried everything.

Now we'll start again, using $3x$ and $2x$ in the front slots:

$$(3x ____)(2x ____)$$

Now try the same combinations as above for the –5:

$(3x + 5)(2x - 1) = 6x^2 - 3x + 10x - 5 = 6x^2 + 7x - 5$ ☹

$(3x - 5)(2x + 1) = 6x^2 + 3x - 10x - 5 = \underline{6x^2 - 7x - 5}$ ☺

Although there are other ways to arrange the 5 and the 1, there's no need to go further; we've found what we were looking for.

Therefore, the final factorization of $6x^2 - 7x - 5$ is

$$\boxed{(3x - 5)(2x + 1)}$$

EXAMPLE 2: **Factor: $n^2 - 5n + 6$**

Solution: As before, we focus on the first and last terms. There's only one way to break up the n^2, namely $n \cdot n$. So we start with

$$(n ____)(n ____)$$

But there are a few ways to break up the 6. Noting that 6 is positive — and therefore the signs of its factors must be the same — let's begin experimenting:

$(n + 6)(n + 1) = n^2 + n + 6n + 6 = n^2 + 7n + 6$ ☹

$(n + 1)(n + 6) = n^2 + 6n + n + 6 = n^2 + 7n + 6$ ☹

$(n - 6)(n - 1) = n^2 - n - 6n + 6 = n^2 - 7n + 6$ ☹

$(n - 1)(n - 6) = n^2 - 6n - n + 6 = n^2 - 7n + 6$ ☹

$(n + 3)(n + 2) = n^2 + 2n + 3n + 6 = n^2 + 5n + 6$ ☹

$(n + 2)(n + 3) = n^2 + 3n + 2n + 6 = n^2 + 5n + 6$ ☹

$(n - 3)(n - 2) = n^2 - 2n - 3n + 6 = \underline{n^2 - 5n + 6}$ ☺

and thus our final factorization is

$$\boxed{(n-3)(n-2)}$$

Did we really need all those attempts to find the answer? NO.

Isn't it possible to see that the factors of 6 must be negative, considering the $-5n$ term? OF COURSE.

You only need to use as many options as needed until you reach the required goal. Some students can just "see" what combination works, while others need a lot of experimenting. The more you practice, the quicker you'll find the right factors.

EXAMPLE 3: **Factor: $y^2 + 7y + 14$**

Solution: There's really only one way to begin; we split the y^2 term into y and y:

$$(y ____)(y ____)$$

Now let's work on the 14:

$(y + 14)(y + 1) = y^2 + y + 14y + 14 = y^2 + 15y + 14$ ☹

$(y + 1)(y + 14) = y^2 + 14y + y + 14 = y^2 + 15y + 14$ ☹

$(y - 14)(y - 1) = y^2 - y - 14y + 14 = y^2 - 15y + 14$ ☹

$(y - 1)(y - 14) = y^2 - 14y - y + 14 = y^2 - 15y + 14$ ☹

$(y + 7)(y + 2) = y^2 + 2y + 7y + 14 = y^2 + 9y + 14$ ☹

$(y + 2)(y + 7) = y^2 + 7y + 2y + 14 = y^2 + 9y + 14$ ☹

$(y - 7)(y - 2) = y^2 - 2y - 7y + 14 = y^2 - 9y + 14$ ☹

$(y - 2)(y - 7) = y^2 - 7y - 2y + 14 = y^2 - 9y + 14$ ☹

Nothing but sad faces, and we've tried every possible arrangement. This can mean only one thing: There's no way to factor the expression $y^2 + 7y + 14$. The expression is

$y^2 + 7y + 14$ is called **prime** because, like the prime number 13, it can't be factored.

Not factorable

EXAMPLE 4: **Factor: $w^2 - 25$**

Solution: This quadratic expression has only two terms, but which one is missing? When a quadratic starts with w^2, we expect the next term to contain a w. So it's the middle term that is missing. But the middle term is not the one we focus on anyway, so let's begin the usual process:

The binomial $w^2 - 25$ is called a **difference of squares**, because it's w-squared minus 5-squared.

$$(w + 25)(w - 1) = w^2 - w + 25w - 25 = w^2 + 24w - 25$$ ☹

$$(w - 25)(w + 1) = w^2 + w - 25w - 25 = w^2 - 24w - 25$$ ☹

$$(w + 1)(w - 25) = w^2 - 25w + w - 25 = w^2 - 24w - 25$$ ☹

$$(w - 1)(w + 25) = w^2 + 25w - w - 25 = w^2 + 24w - 25$$ ☹

$$(w + 5)(w - 5) = w^2 - 5w + 5w - 25 = \underline{w^2 - 25}$$ ☺

Therefore, the expression $w^2 - 25$ factors into

$(w + 5)(w - 5)$

EXAMPLE 5: **Factor: $9u^2 + 12u + 4$**

Solution: The quadratic term, $9u^2$, can be broken down two ways, $9u \times u$ and $3u \times 3u$.

Let's start the multiplications using $9u$ and u:

$(9u + 4)(u + 1) = 9u^2 + 9u + 4u + 4 = 9u^2 + 13u + 4$ ☹

$(9u + 1)(u + 4) = 9u^2 + 36u + u + 4 = 9u^2 + 37u + 4$ ☹

$(9u + 2)(u + 2) = 9u^2 + 18u + 2u + 4 = 9u^2 + 20u + 4$ ☹

That's about it for the $9u$ and u combination. Now for $3u$ and $3u$:

$(3u + 4)(3u + 1) = 9u^2 + 3u + 12u + 4 = 9u^2 + 15u + 4$ ☹

$(3u + 1)(3u + 4) = 9u^2 + 12u + 3u + 4 = 9u^2 + 15u + 4$ ☹

$(3u + 2)(3u + 2) = 9u^2 + 6u + 6u + 4 = \underline{9u^2 + 12u + 4}$ ☺

Eureka! The factorization of $9u^2 + 12u + 4$ is $(3u + 2)(3u + 2)$, which we can write more succinctly as

$$\boxed{(3u+2)^2}$$

EXAMPLE 6: **Factor: $a^2 + 49$**

Solution: Let's start experimenting right away:

$(a + 7)(a + 7) = a^2 + 7a + 7a + 49 = a^2 + 14a + 49$ ☹

$(a + 49)(a + 1) = a^2 + a + 49a + 49 = a^2 + 50a + 49$ ☹

$(a + 1)(a + 49) = a^2 + 49a + a + 49 = a^2 + 50a + 49$ ☹

$(a - 7)(a - 7) = a^2 - 7a - 7a + 49 = a^2 - 14a + 49$ ☹

$(a - 49)(a - 1) = a^2 - a - 49a + 49 = a^2 - 50a + 49$ ☹

$(a - 1)(a - 49) = a^2 - 49a - a + 49 = a^2 - 50a + 49$ ☹

Have we tried everything? It appears we have, and no smiley face! (As we saw before, not every expression can be factored.) So we say that $a^2 + 49$ is

Not factorable

First Notice: Even though $a^2 + 49$ is not factorable, $a^2 - 49$ is factorable, since $a^2 - 49 = (a + 7)(a - 7)$. The difference between a plus sign and a minus sign makes all the difference in the world — so be careful!

Second Notice: Some students jump to the conclusion that when two terms are connected by a plus sign, the expression is not factorable. But consider $4x^2 + 16$. It may not factor with two sets of parentheses like we're learning in this chapter, but it does have a greatest common factor (GCF) of 4, which can be factored out to produce $4(x^2 + 4)$. Thus, $4x^2 + 16$ is factorable.

Homework

1. Factor each expression:

a. $2x^2 + 3x + 1$	b. $3n^2 - 7n + 2$	c. $5a^2 + 3a - 2$
d. $3m^2 - 11m - 20$	e. $4x^2 - 3x - 1$	f. $6u^2 + 7u - 10$
g. $4z^2 - 4z - 3$	h. $6y^2 - 5y - 6$	i. $7n^2 - 45n + 18$
j. $a^2 + 16a + 63$	k. $2x^2 + x - 3$	l. $7z^2 - 12z + 5$

2. Factor each expression:

a. $x^2 + 5x + 6$	b. $x^2 - 5x + 6$	c. $x^2 - 5x - 6$
d. $x^2 + 5x - 6$	e. $n^2 + 10n + 9$	f. $z^2 - 4z - 5$
g. $t^2 - 20t + 96$	h. $u^2 - 6u - 16$	i. $Q^2 + 34Q - 72$
j. $x^2 + 25x + 156$	k. $a^2 - 17a + 70$	l. $t^2 + t - 110$

3. Factor each expression:

a. $x^2 + 8x + 16$ b. $y^2 - 10y + 25$ c. $a^2 + 18a + 81$

d. $b^2 - 20b + 100$ e. $4z^2 + 4z + 1$ f. $9n^2 - 24n + 16$

g. $25x^2 - 30x + 9$ h. $x^2 + 6x + 36$ i. $2t^2 + 33t + 100$

j. $16a^2 + 8a + 1$ k. $9u^2 - 12u + 4$ l. $w^2 + 50w + 625$

4. Factor each expression:

a. $p^2 - 1$ b. $c^2 - 4$ c. $R^2 - 16$ d. $z^2 - 36$

e. $x^2 - 25$ f. $y^2 - 81$ g. $n^2 - 10$ h. $w^2 + 16$

i. $a^2 - 144$ j. $e^2 - 72$ k. $m^2 + 100$ l. $W^2 - 1$

m. $x^2 - 9$ n. $a^2 - 49$ o. $g^2 - 64$ p. $y^2 + 4$

5. Factor each expression:

a. $4x^2 - 9$ b. $9y^2 - 49$ c. $u^2 - 2$ d. $v^2 + 1$

e. $16z^2 - 49$ f. $49w^2 - 16$ g. $49a^2 - 144$ h. $121b^2 - 64$

i. $9x^2 + 2$ j. $1 - x^2$ k. $16 - n^2$ l. $25 - 4g^2$

m. $9 + t^2$ n. $144N^2 - 169$ o. $225a^2 - 1$

6. Factor each expression:

a. $3x^2 + 10x - 8$ b. $t^2 - 121$ c. $y^2 + 10y + 25$

d. $16a^2 - 121$ e. $b^2 - 20$ f. $n^2 + 121$

g. $x^2 + 3x + 1$ h. $12q^2 - 23q + 5$ i. $6a^2 - 13a + 6$

j. $x^2 + 14x + 13$ k. $4y^2 - 49$ l. $9Q^2 + 12Q + 4$

m. $25z^2 - 10z + 1$ n. $16x^2 + 34x - 15$ o. $16x^2 + 118x - 15$

p. $16x^2 - 77x - 15$ q. $16x^2 - 72x + 45$ r. $16a^2 - 8a + 1$

s. $x^2 + 7x + 5$ t. $8c^2 + 2c - 21$ u. $8c^2 - 13c - 21$

v. $3a^2 - 5a - 12$ w. $6x^2 + 17x - 14$ x. $2y^2 + 7y - 9$

Review Problems

7. Factor each expression:

a. $x^2 + 17x + 72$ b. $y^2 - 9y + 8$ c. $N^2 + 100$

d. $N^2 - 100$ e. $x^2 - 18x + 81$ f. $a^2 + 10a + 25$

g. $t^2 + 4t - 45$ h. $a^2 - 21a - 22$ i. $a^2 - 9a - 22$

j. $2x^2 - 9x - 5$ k. $6x^2 + x - 40$ l. $6x^2 - 13x - 15$

m. $6x^2 + 11x - 17$ n. $R^2 - 144$ o. $25n^2 - 30n + 9$

p. $T^2 + 144$ q. $49w^2 + 70w + 25$ r. $9a^2 - 9a + 2$

s. $9a^2 - 19a + 2$ t. $x^2 + 16x - 36$ u. $x^2 + 37x + 36$

v. $30c^2 - 11c + 1$ w. $16a^2 - 6a - 1$ x. $36q^2 + 60q + 25$

y. $x^2 + 8x + 18$ z. $32n^2 - 66n + 27$

Solutions

1. a. $(2x + 1)(x + 1)$ b. $(3n - 1)(n - 2)$ c. $(5a - 2)(a + 1)$

d. $(3m + 4)(m - 5)$ e. $(4x + 1)(x - 1)$ f. $(6u - 5)(u + 2)$

g. $(2z + 1)(2z - 3)$ h. $(2y - 3)(3y + 2)$ i. $(7n - 3)(n - 6)$

j. $(a + 9)(a + 7)$ k. $(2x + 3)(x - 1)$ l. $(7z - 5)(z - 1)$

2. a. $(x + 3)(x + 2)$ b. $(x - 2)(x - 3)$ c. $(x - 6)(x + 1)$
d. $(x + 6)(x - 1)$ e. $(n + 9)(n + 1)$ f. $(z - 5)(z + 1)$
g. $(t - 12)(t - 8)$ h. $(u - 8)(u + 2)$ i. $(Q + 36)(Q - 2)$
j. $(x + 13)(x + 12)$ k. $(a - 10)(a - 7)$ l. $(t + 11)(t - 10)$

3. a. $(x + 4)^2$ b. $(y - 5)^2$ c. $(a + 9)^2$
d. $(b - 10)^2$ e. $(2z + 1)^2$ f. $(3n - 4)^2$
g. $(5x - 3)^2$ h. Not factorable i. $(2t + 25)(t + 4)$
j. $(4a + 1)^2$ k. $(3u - 2)^2$ l. $(w + 25)^2$

4. a. $(p + 1)(p - 1)$ b. $(c + 2)(c - 2)$ c. $(R + 4)(R - 4)$
d. $(z + 6)(z - 6)$ e. $(x + 5)(x - 5)$ f. $(y + 9)(y - 9)$
g. Not factorable h. Not factorable i. $(a + 12)(a - 12)$
j. Not factorable k. Not factorable l. $(W + 1)(W - 1)$
m. $(x + 3)(x - 3)$ n. $(a + 7)(a - 7)$ o. $(g + 8)(g - 8)$
p. Not factorable

5. a. $(2x + 3)(2x - 3)$ b. $(3y + 7)(3y - 7)$ c. Not factorable
d. Not factorable e. $(4z + 7)(4z - 7)$ f. $(7w + 4)(7w - 4)$
g. $(7a + 12)(7a - 12)$ h. $(11b + 8)(11b - 8)$ i. Not factorable
j. $(1 + x)(1 - x)$ k. $(4 + n)(4 - n)$ l. $(5 + 2g)(5 - 2g)$
m. Not factorable n. $(12N + 13)(12N - 13)$ o. $(15a + 1)(15a - 1)$

6. a. $(3x - 2)(x + 4)$ b. $(t + 11)(t - 11)$ c. $(y + 5)^2$
d. $(4a + 11)(4a - 11)$ e. Not factorable f. Not factorable
g. Not factorable h. $(3q - 5)(4q - 1)$ i. $(2a - 3)(3a - 2)$
j. $(x + 1)(x + 13)$ k. $(2y + 7)(2y - 7)$ l. $(3Q + 2)^2$
m. $(5z - 1)^2$ n. $(8x - 3)(2x + 5)$ o. $(8x - 1)(2x + 15)$
p. $(16x + 3)(x - 5)$ q. $(4x - 3)(4x - 15)$ r. $(4a - 1)^2$

s. Not factorable t. $(4c + 7)(2c - 3)$ u. $(8c - 21)(c + 1)$

v. $(3a + 4)(a - 3)$ w. $(2x + 7)(3x - 2)$ x. $(2y + 9)(y - 1)$

7. a. $(x + 9)(x + 8)$ b. $(y - 1)(y - 8)$ c. Not factorable

d. $(N + 10)(N - 10)$ e. $(x - 9)^2$ f. $(a + 5)^2$

g. $(t + 9)(t - 5)$ h. $(a - 22)(a + 1)$ i. $(a - 11)(a + 2)$

j. $(2x + 1)(x - 5)$ k. $(3x + 8)(2x - 5)$ l. $(6x + 5)(x - 3)$

m. $(6x + 17)(x - 1)$ n. $(R + 12)(R - 12)$ o. $(5n - 3)^2$

p. Not factorable q. $(7w + 5)^2$ r. $(3a - 1)(3a - 2)$

s. $(9a - 1)(a - 2)$ t. $(x + 18)(x - 2)$ u. $(x + 36)(x + 1)$

v. $(5c - 1)(6c - 1)$ w. $(8a + 1)(2a - 1)$ x. $(6q + 5)^2$

y. Not factorable z. $(16n - 9)(2n - 3)$

"The school is the last expenditure on which America should be willing to economize."

Franklin D. Roosevelt



FACTORING – THE GREATEST COMMON FACTOR

❑ *INTRODUCTION*

Do you remember the concept of a **prime number**? Here's a hint: 13 is prime, but 15 is not prime and 22 is not a prime.

A number is ***prime*** if it has exactly two factors.

For example, 13 is prime because 13 has exactly two factors (divisors), just 1 and 13. The number 2 is also prime, because its only factors are 1 and 2. But 15 is not prime — this is because 15 has more than two factors; in fact, it has four factors: 1, 3, 5, and 15. The number 1 is also not prime, since it does not have two factors; its only factor is 1. Thus, the first few primes are

> The number *a* is a ***factor*** of *b* if *a* divides into *b* evenly (without remainder). For example, 10 is a factor of 30, but 8 is not a factor of 30.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, and so on, . . . ***forever***!

Believe it or not, every number (bigger than 3) that isn't prime can be written as a product of primes. For example, 90 is not prime, but 90 can be written as a product of primes:

$$90 = 2 \times 3 \times 3 \times 5$$

which we can write as

$$90 = 2 \times 3^2 \times 5$$

> The largest known prime number (as of Oct, 2024) is
>
> $2^{136,279,841} - 1$
>
> and contains over 41 million digits.
>
> If you wrote this prime number down, writing one digit per second, 24/7, it would take you well over a year, filling thousands of pages.

Notice that all the factors on the right side of the equality are primes, and that their product is surely 90.

This product of primes is called the ***prime factorization*** of 90. Writing 90 as $2 \times 3 \times 15$ is not its prime factorization because 15 is not prime. Basically, then,

Factoring *is the art of expressing something as a product of things — things which cannot be broken down any further.*

Homework

1. a. How many primes are there?
   b. What is the smallest prime?
   c. What is the only even prime?
   d. How many primes are there less than 100?
   e. Is the number $123 \times 56{,}893$ prime? Prove your answer.

2. a. Find the prime factorization of 306.
   b. Find the prime factorization of 1,024.

❑ *A Different View of the Distributive Property*

We've generally viewed the distributive property in a form like this:

$$A(B + C) = AB + AC \qquad \text{DISTRIBUTING}$$

and we saw the power of such a law in simplifying expressions and solving complicated equations. But the distributive property is a statement of equality — we might find it useful to flip it around the equals sign and write it as

$$AB + AC = A(B + C) \qquad \text{FACTORING}$$

This provides a whole new perspective. It allows us to take a pair of terms, the sum $AB + AC$, find the **common factor** A (it's in both terms), and "pull" the A out in front, and write the sum $AB + AC$ as the product $A(B + C)$.

This use of the distributive property in reverse is called ***factoring***. Notice that using the distributive property in reverse converts the two terms $AB + AC$, into one term, $A(B + C)$.

For example, suppose we want to factor $9a + 9b$; that is, we want to convert $9a + 9b$ from a sum to a product. We notice that 9 is a common factor of the two terms. We pull the 9 away from both terms, and put it out in front to get $9(a + b)$, and we're done factoring:

$9a + 9b$ factors into $9(a + b)$.

To check this answer, distribute $9(a + b)$ and you'll get the original $9a + 9b$.

RECAP:

$9(a + b)$ → Distribute → $9a + 9b$ One term to two terms

$9a + 9b$ → Factor → $9(a + b)$ Two terms to one term

❑ *FACTORING*

EXAMPLE 1: **Factor each expression:**

A. $7x + 7y = 7(x + y)$

B. $3x + 12 = 3(x + 4)$

C. $ax + bx = x(a + b)$

D. $Rw - Ew = w(R - E)$

E. $9z + 9 = 9(z + 1)$

F. $mn - m = m(n - 1)$

G. $-6R + 8 = -2(3R - 4)$

Alternatively, we could pull out a positive 2, yielding $2(-3R + 4)$.

H. $-ax - at = -a(x + t)$ I. $-x + 5 = -(x - 5)$

J. $-n - 9 = -(n + 9)$

K. $6r + 8s - 10t = 2(3r + 4s - 5t)$

Note that every problem in the preceding example can be checked by distributing the answer. Our next example shows how we can factor out a variable in a quadratic expression.

EXAMPLE 2: **Factor each expression:**

A. $x^2 + 3x = x(x + 3)$ B. $n^2 - 7n = n(n - 7)$

C. $t^2 + t = t(t + 1)$ D. $y^2 - y = y(y - 1)$

E. $m^2 - 10m = m(m - 10)$ F. $a^2 + 40a = a(a + 40)$

Sometimes we can pull out a number and a variable.

EXAMPLE 3: **Factor: $2a^2 - 8a$**

Solution: What common factor can be pulled out in front? Since 2 is a factor of both terms, it can be pulled out. But a is a common factor, so it needs to come out in front, also. In other words, the quantity $2a$ is common to both terms (and it's the largest quantity that is common to both terms). So we factor it out and leave inside the parentheses what must be left. Thus, $2a^2 - 8a$ factors into

$$\boxed{2a(a - 4)}$$

Check by distributing

EXAMPLE 4: **Factor: $20n + 50$**

Solution: What factor is common to both terms that can then be pulled out in front? We have a little dilemma here. There are three numbers we could factor out: 2, 5, and 10. Let's agree to pull out the 10, since it's the largest factor that is common to both terms. Therefore, $2n + 50$ factors into

$$10(2n + 5)$$

New Terminology: In Example 3 we factored out the quantity $2a$ because it was common to both terms, and it was the greatest common factor. In Example 4 we factored out the number 10 because it was the greatest factor that was common to both terms. Each quantity, the $2a$ and the 10, is called the ***greatest common factor***, or **GCF**.

Homework

3. How would you convince your buddy that factoring $20x + 30y$ produces a result of $10(2x + 3y)$?

4. Your stubborn friend believes that $6w + 9z$ factors to $6(w + 3z)$. Prove her wrong.

5. Finish the factorization of each expression:

   a. $wx + wz = w(\quad)$
   b. $4P - 4Q = 4(\quad)$
   c. $9x - 36 = 9(\quad)$
   d. $8y - 12t = 4(\quad)$
   e. $7u + 7 = 7(\quad)$
   f. $-2n + 8 = -2(\quad)$
   g. $-a + b = -(\quad)$
   h. $-c - d = -(\quad)$
   i. $2x + 4y - 8z = 2(\quad)$
   j. $aw - au + az = a(\quad)$

k. $14x^2 - 21x = 7x(\quad)$ l. $20a^2 + 30a - 40 = 10(\quad)$

6. Factor each expression:

a. $3P + 3Q$ b. $9n - 27$ c. $cn + dn$

d. $wx - xy$ e. $7t - 7$ f. $x + xy$

g. $-8L + 10$ h. $-ab - bc$ i. $-u - 5$

j. $-z - x + 10$ k. $2x + 2y + 2z$ l. $5a - 10b + 15c$

7. Finish the factorization of each expression:

a. $4a + 8b = 4(\quad)$ b. $9u^2 - 3u = 3u(\quad)$

c. $15Q - 45R = 15(\quad)$ d. $18x^2 + 12x = 6x(\quad)$

e. $10y^2 - 20y = 10y(\quad)$ f. $50a + 75b = 25(\quad)$

g. $7t^2 + 28t = 7t(\quad)$ h. $48w - 64z = 16(\quad)$

i. $100a^2 - 80a = 20a(\quad)$ j. $47y^2 + 47y = 47y(\quad)$

8. Factor each expression:

a. $2x^2 - 16x$ b. $3n^2 + 9n$ c. $4a^2 + 4$

d. $7u^2 + 9u$ e. $2a^2 - 16$ f. $4x^2 + 8y$

g. $5b^2 - 10b$ h. $7w^2 + 21w$ i. $7x^2 + 8x$

j. $7x + 9y$ k. $12x^2 - 12x$ l. $17Q + 17R$

m. $15g - 45h$ n. $ab + ac$ o. $xy - yz$

p. $-2x + 8$ q. $-10n - 15m$ r. $6e - 19f$

Review Problems

9. Factor each expression:

a. $3x - 12$ b. $9x + 9$ c. $7y^2 - 14y$

d. $2n^2 - 10n$ e. $10w^2 + 45w$ f. $8x + 13$

g. $-x - 4$ h. $14n^2 - 21n + 35$ i. $20n^2 - 20n$



Solutions

1. a. Infinitely many b. 2 c. 2 d. 25
e. NO — Whatever that number actually is, it's certainly divisible by both 123 and 56,893. That's too many factors to be prime.

2. a. $306 = 2 \times 3^2 \times 17$ b. 2^{10}

3. Here's what I would do. First, the given expression, $20x + 30y$, consists of two terms, and the result, $10(2x + 3y)$, consists of one term. Since factoring is the process of converting two or more terms into a single

term, so far so good. Moreover, if I take my answer, $10(2x + 3y)$, and distribute to remove the parentheses, I will get $20x + 30y$, the original problem. I hope your buddy is now convinced.

4. While it may be true that the original expression consists of two terms, and her answer consists of one term, there's still one big problem. Ask her to take her answer, $6(w + 3z)$, and distribute it to remove the parentheses. She will get $6w + 18z$, which is <u>not</u> equal to the original problem. Therefore, her factorization can't possibly be right.

5.

a. $x + z$	b. $P - Q$	c. $x - 4$	d. $2y - 3t$
e. $u + 1$	f. $n - 4$	g. $a - b$	h. $c + d$
i. $x + 2y - 4z$	j. $w - u + z$	k. $2x - 3$	l. $2a^2 + 3a - 4$

6.

a. $3(P + Q)$	b. $9(n - 3)$	c. $n(c + d)$
d. $x(w - y)$	e. $7(t - 1)$	f. $x(1 + y)$
g. $-2(4L - 5)$	h. $-b(a + c)$	i. $-(u + 5)$
j. $-(z + x - 10)$	k. $2(x + y + z)$	l. $5(a - 2b + 3c)$

7.

a. $a + 2b$	b. $3u - 1$	c. $Q - 3R$	d. $3x + 2$
e. $y - 2$	f. $2a + 3b$	g. $t + 4$	h. $3w - 4z$
i. $5a - 4$	j. $y + 1$		

8.

a. $2x(x - 8)$	b. $3n(n + 3)$	c. $4(a^2 + 1)$
d. $u(7u + 9)$	e. $2(a^2 - 8)$	f. $4(x^2 + 2y)$
g. $5b(b - 2)$	h. $7w(w + 3)$	i. $x(7x + 8)$
j. Not factorable	k. $12x(x - 1)$	l. $17(Q + R)$
m. $15(g - 3h)$	n. $a(b + c)$	o. $y(x - z)$
p. $-2(x - 4)$	q. $-5(2n + 3m)$	r. Not factorable

9.

a. $3(x - 4)$	b. $9(x + 1)$	c. $7y(y - 2)$
d. $2n(n - 5)$	e. $5w(2w + 9)$	f. Not factorable
g. $-(x + 4)$	h. $7(2n^2 - 3n + 5)$	i. $20n(n - 1)$

❑ *TO ∞ AND BEYOND*

1. Factor: $\pi^4 x^2 + \pi^3 x^3 - \pi^2 x^4$

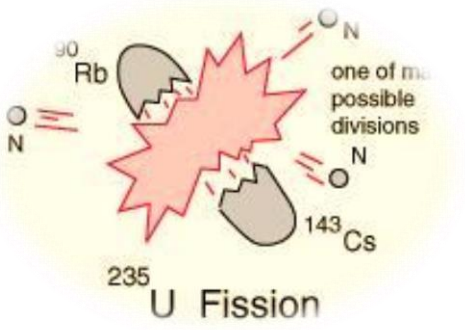
2. The price of a can of root beer is more than \$0.20. How many cans of root beer could you buy for exactly \$4.37?

"Give a man a fish
and you feed him for a day.

Teach a man to fish
and you feed him for a lifetime."

Chinese Proverb

CREATING FORMULAS



Many times in your math, science, computer, statistics, and business classes, formulas will be given to you. For example, you may know that the Einstein formula that relates mass and energy is $E = mc^2$ (where c is the speed of light). But these formulas have to come from somewhere, don't they?

In the formulas below, do not worry that you may not know the meanings of some of the terms used. Your job is to see how the "English" formula is translated into an "Algebra" formula.

❑ *CREATING FORMULAS FROM DESCRIPTIONS*

A. Assets is the sum of liabilities and capital. $A = L + C$

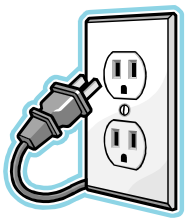
B. Profit is the difference of revenue and expenses. $P = R - E$

C. Distance is the product of rate and time. $d = rt$

D. Density is the quotient of mass and volume. $D = \frac{m}{V}$

E. The average is the sum divided by the number of items. $A = \frac{S}{n}$

F. Voltage is the product of current (I) and resistance. $V = IR$



G. Potential energy is the product of the mass, the acceleration of gravity (g), and the height. $P = mgh$

H. Energy is mass times the square of the speed of light (c). $E = mc^2$

I. Kinetic energy is one-half the product of mass and the square of the velocity. $K = \frac{1}{2}mv^2$

J. The frequency of genotypes is the square of the sum of the allele frequencies. $f = (A_1 + A_2)^2$

K. The square of the hypotenuse is equal to the sum of the squares of the legs. $h^2 = a^2 + b^2$

L. The area of a rectangle is the product of its length and its width. $A = lw$

M. The perimeter of a square is four times the length of its side. $P = 4s$

N. The number of bits (b) is found by multiplying the number of bytes (B) by 8. $b = 8B$

Homework

Create a formula from the given description. It's fine if the variables you choose don't match the variables in the solutions.



1. Liabilities is the difference between assets and capital.

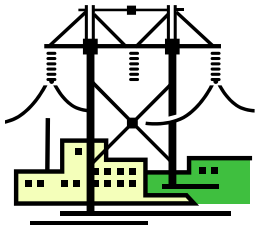
2. Revenue is the sum of profit and expenses.

3. Rate is the quotient of distance and time.

4. Mass is the product of density and volume.

5. The number of items is the quotient of the sum and the average.

6. Current is the quotient of voltage and resistance.



7. Mass is found by dividing potential energy by the product of gravity and height.

8. Mass is the quotient of energy and the square of the speed of light.

9. Mass is twice the kinetic energy divided by the square of the velocity.

10. The width of a rectangle is the quotient of the area and the length.

11. The side of a square is one-fourth of the perimeter.

12. The number of bytes is found by dividing the number of bits by 8.

13. The sum of the squares of the legs is equal to the square of the hypotenuse.

14. The width is equal to the square of the difference of x and y.

Review Problems

15. Create a formula from the given description:

    a. Capital is the difference between assets and liabilities.

    b. Expense is the difference between revenue and profit.

    c. Time is the quotient of distance and rate.

d. Volume is the quotient of mass and density.

e. The sum is the product of the average and the number of items.

f. Resistance is the quotient of voltage and current.

g. Length is the quotient of the area and the width.

h. Energy is the product of the squares of a and b.

i. Mass is the square of the product of u and w.

Solutions

1. $L = A - C$

2. $R = P + E$

3. $r = \frac{d}{t}$

4. $m = dv$

5. $n = \frac{S}{A}$

6. $I = \frac{V}{R}$

7. $m = \frac{P}{gh}$

8. $m = \frac{E}{c^2}$

9. $m = \frac{2K}{v^2}$

10. $w = \frac{A}{l}$

11. $s = \frac{P}{4}$ or $\frac{1}{4}P$

12. $B = \frac{b}{8}$

13. $a^2 + b^2 = h^2$

14. $w = (x - y)^2$

15. a. $C = A - L$ b. $E = R - P$ c. $t = \frac{d}{r}$ d. $V = \frac{m}{d}$

e. $S = An$ f. $R = \frac{V}{i}$ g. $l = \frac{A}{w}$ h. $E = a^2b^2$

i. $m = (uw)^2$

❑ *To ∞ and Beyond*

A. Express as a formula:

The hypotenuse, h, is the square root of the sum of the squares of the legs, a and b.

B. Express as a formula:

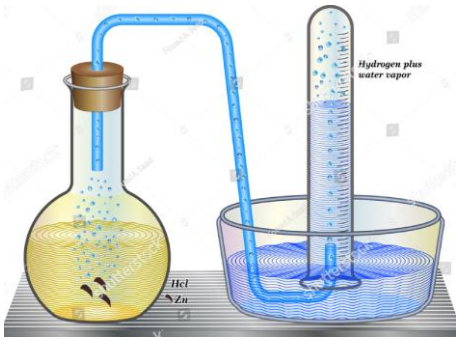
The force of gravity, F, between two objects is the product of the gravitational constant, G, the mass, m_1, of the first object, and the mass, m_2, of the second object — all divided by the square of the distance, r, between the objects.

"The only good is knowledge, and the only evil is ignorance."

- Σοχρατεσ (Socrates)

SOLVING FORMULAS

Solving equations such as $2x + 7 = 20$ is obviously the cornerstone of algebra. But in science, business, and computer programming it is also necessary to solve equations that might have a variety of letters in them. For example, if you wanted to find the temperature in the chemistry gas law $PV = nRT$, you might have to solve this equation for the variable T.



A ***formula*** (called a *literal equation* by many) is just an equation with two or more variables designed to accomplish something specific. The equation for the perimeter of a rectangle, $P = 2L + 2W$, is an example of a formula. If we needed to solve for the width in this rectangle formula, we would have to isolate the variable W using the same equation-solving techniques you already know.

❑ *REVIEW OF SOLVING EQUATIONS*

Some students find the solving of formulas to be very abstract, while others consider it much easier than solving regular equations. Let's begin this chapter with a quick review of solving one-step equations.

To solve the equation $n + 3 = 10$, we recognize that 3 has been added to the unknown n. To remove the 3 — that is, to undo the addition — we subtract 3 from each side of the equation. This isolates the n, and we get $n = 7$.

Similarly, to solve the equation $y - 10 = 27$, we add 10 to each side of the equation, resulting in a solution of $y = 37$.

To isolate the variable in the equation $9w = 45$, we notice that multiplication binds the 9 with the w. To remove the 9, therefore, we divide each side of the equation by 9, yielding a value of 5 for w.

For our last review example in this section, to solve the equation $\frac{x}{12} = 24$ we are obliged to multiply each side of the equation by 12, since multiplying undoes dividing. And so $x = 288$.

In the following sections we also "isolate" variables, but now we have to remove other variables (not just numbers) in order to isolate the one we want. Since there's no arithmetic involved — just algebra concepts — you can see why some students think this is not so bad, after all. Perhaps with a little practice you'll feel the same way.

❑ *ONE-STEP EXAMPLES OF SOLVING FORMULAS*

EXAMPLE 1: **Solve for x: $x + a = b$**

Solution: To isolate the x we need to remove the a. Since the a is being added to the x, we remove it by subtracting it from each side of the equation, thus isolating the x:

	$x + a = b$	(the original formula)
$\Rightarrow$	$x + a - \boldsymbol{a} = b - \boldsymbol{a}$	(subtract a from each side)
$\Rightarrow$	$\boxed{x = b - a}$	(simplify)

EXAMPLE 2: **Solve for u: $u - w = A$**

Solution: To isolate the u, we need to remove the w. Since the w is being subtracted from the u, we remove it by adding w to each side of the equation:

$u - w = A$ (the original formula)

$\Rightarrow \quad u - w + \boldsymbol{w} = A + \boldsymbol{w}$ (add w to each side)

$\Rightarrow \quad \boxed{u = A + w}$ (simplify)

EXAMPLE 3: **Solve for n: $cn = R$**

Solution: To isolate the n, we need to remove the c. Since the operation between the c and the n is multiplication, we remove the c by dividing each side of the equation by c:

$cn = R$ (the original formula)

$\Rightarrow \quad \frac{cn}{\boldsymbol{c}} = \frac{R}{\boldsymbol{c}}$ (divide each side by c)

$\Rightarrow \quad \boxed{n = \frac{R}{c}}$ (simplify)

EXAMPLE 4: **Solve for W: $\frac{W}{\boldsymbol{a}} = -\boldsymbol{x}$**

Solution: To isolate the W, we need to remove the a. Since W is being divided by a, we can remove the a by multiplying each side of the equation by a:

$\frac{W}{a} = -x$ (the original formula)

$\Rightarrow \quad \frac{W}{a}\left[\boldsymbol{a}\right] = -x\left[\boldsymbol{a}\right]$ (multiply each side by a)

$\Rightarrow \quad \frac{W}{\cancel{a}}\left[\cancel{a}\right] = -ax$ (cross-cancel the a's)

$\Rightarrow \quad \boxed{W = -ax}$ (simplify)

EXAMPLE 5: **Solve for x: $x(y - z) = Q$**

Solution: To isolate the x, we need to remove the quantity $y - z$. What is the operation connecting the x with the y (z)? It's multiplication, so we use division to reverse the operation and thus isolate the x:

$x(y - z) = Q$ (the original formula)

$\Rightarrow \quad \frac{x(y-z)}{\boldsymbol{y-z}} = \frac{Q}{\boldsymbol{y-z}}$ (divide each side by $y - z$)

$\Rightarrow \quad \boxed{x = \frac{Q}{y-z}}$ (simplify)

EXAMPLE 6: **Solve for the temperature T in the gas law $PV = nRT$ mentioned in the Introduction.**

Solution: $PV = nRT$ (the original gas law)

$\Rightarrow \quad \frac{PV}{\boldsymbol{nR}} = \frac{nRT}{\boldsymbol{nR}}$ (divide each side by nR)

$\Rightarrow \quad \frac{PV}{nR} = T$ (simplify the right side)

$\Rightarrow \quad \boxed{T = \frac{PV}{nR}}$ (reverse the equation)

EXAMPLE 7: **Solve each formula for x:**

A. $\frac{x}{a} = b + c$

Now comes a new issue — we know we need to multiply each side of the equation by a (in order to isolate the x). Multiplying the left side by a is easy; it cross-cancels with

the a, leaving just x, the unknown. But how do we indicate that the right-hand side of the equation, the quantity $b + c$, <u>all of it</u>, must be multiplied by a? We put parentheses around the quantity $b + c$, that's how:

note the parentheses

$$\frac{x}{a}\left[\boldsymbol{a}\right] = (b+c)\ \left[\boldsymbol{a}\right]$$

Simplifying the left side of the equation gives x. Simplifying the right side gives simply $(b + c)a$. Thus,

$x = (b + c)a$,

or, by the commutative property for multiplication,

$\boldsymbol{x = a(b + c)}$, and we're done.

Note: Another way to write the final solution is to distribute and get $x = ab + ac$. But this is not necessary, since the goal of this chapter is to learn how to isolate variables, not simplify answers.

B. $\frac{x}{y-z} = w$

Again, we are trying to isolate the x, and again we will accomplish this by multiplying each side of the equation by the denominator, in this case $y - z$. Remembering the use of parentheses (or brackets), we get

$\frac{x}{y-z}\left[\boldsymbol{y-z}\right] = w\left[\boldsymbol{y-z}\right]$, and our final answer is

$\boldsymbol{x = w[y - z]}$, or $\boldsymbol{x = w(y - z)}$

C. $\frac{x}{c-e} = a+u$

Multiplying each side of the equation by $c - e$ produces

$$\frac{x}{c-e}\left[\boldsymbol{c-e}\right] = (a+u)\left[\boldsymbol{c-e}\right]$$

$\Rightarrow \quad x = (a + u)(c - e)$

Homework

1. Solve each formula for *x*:

a. $x + b = a$ b. $cx = d$ c. $x - y = z$

d. $\frac{x}{L} = T$ e. $c + x = m$ f. $xy = z$

g. $-R + x = w$ h. $\frac{x}{-a} = -b$ i. $x - m = n$

j. $x + a = b$ k. $-b + x = W$ l. $\frac{x}{y} = -z$

m. $a + x = w$ n. $x - ab = c$ o. $x + bc = d$

2. Solve each formula for *a*:

a. $a(b - c) = d$ b. $a(Q + R) = S$ c. $(b_1 + b_2)a = A$

d. $abc = d$ e. $mag = P$ f. $a(r_1 + r_2) = T$

g. $bca = M$ h. $a(x - y) = z$ i. $a(w - u) = n$

j. $axy = z$ k. $W = abc$ l. $Q = a(g - h)$

3. Solve each formula for *y*:

a. $\frac{y}{a} = b + c$ b. $\frac{y}{c} = d - e$ c. $\frac{y}{x} = x + a$

d. $\frac{y}{t} = b + c - d$ e. $\frac{y}{c} = x - w + z$ f. $\frac{y}{x - z} = A$

g. $\frac{y}{w + 7} = c$ h. $\frac{y}{a - b} = Q$ i. $\frac{y}{a - b + c} = w$

j. $\frac{y}{x + g - h} = k$ k. $\frac{y}{L + M - N} = P$ l. $\frac{y}{u + w} = x + z$

m. $\frac{y}{a - R} = Q + T$ n. $\frac{y}{m - n} = g - h$

o. $\frac{y}{a + b - c} = w + u$ p. $\frac{y}{x + w} = a + b + c$

q. $\dfrac{y}{u+w-x} = d-e+h$

❑ *TWO-STEP EXAMPLES*

Let's review a standard two-step equation. If we solve the equation $3x - 17 = 82$ for x, we remember the dilemma of deciding which to rid ourselves of first, the 3 or the 17. We agreed that reversing the Order of Operations was the secret. So, from the x's point of view, it was multiplied by 3 first, and then 17 was subtracted. Reversing this sequence means we first add 17 to each side, giving $3x = 99$; then we divide each side by 3 to get the value $x = 33$. Well, it's even easier with letters [Let's hope!].

EXAMPLE 8: **Solve for y: $ay - b = c$**

Solution: The Order of Operations tells us that first the a and the y were multiplied, and then b was subtracted. To isolate the y we need to reverse the Order of Operations: Add b to each side, and then divide each side by a:

	$ay - b = c$	(the original formula)
$\Rightarrow$	$ay - b + \boldsymbol{b} = c + \boldsymbol{b}$	(add b to each side)
$\Rightarrow$	$ay = c + b$	(simplify)
$\Rightarrow$	$\dfrac{ay}{\boldsymbol{a}} = \dfrac{c+b}{\boldsymbol{a}}$	(divide each side by a)
$\Rightarrow$	$\boxed{y = \dfrac{c+b}{a}}$	(simplify)

Note: It's common to use alphabetical order when writing variables, so the answer can also be written $y = \dfrac{b+c}{a}$.

EXAMPLE 9: **Solve for x:** $\frac{x}{a} + e = d$

Solution: The x has been divided by a, and then e has been added. Reversing these operations produces the following steps:

$\frac{x}{a} + e = d$ (the original formula)

$\Rightarrow \quad \frac{x}{a} + e - \boldsymbol{e} = d - \boldsymbol{e}$ (subtract e from each side)

$\Rightarrow \quad \frac{x}{a} = d - e$ (simplify)

Now recall from the previous section how we isolate the x. We need to multiply each side of the equation by a, and we need to put parentheses around the quantity $d - e$ when we do.

$\Rightarrow \quad \frac{x}{a}\boldsymbol{[a]} = (d-e)\boldsymbol{[a]}$ (multiply each side by a; notice the parentheses)

$\Rightarrow \quad \boxed{x = a(d-e)}$ (simplify)

Homework

4. Solve each formula for x:

a. $ax + b = c$ b. $cx - d = R$ c. $a + bx = n$

d. $m - nx = Q$ e. $\frac{x}{b} - e = h$ f. $\frac{x-b}{c} = a$

g. $ab + xy = cd$ h. $\frac{x+b}{d} = p$ i. $\frac{x}{a} + k = L$

j. $ax - d = c$ k. $cx + d = e$ l. $a - bx = n$

m. $t + nx = R$ n. $\frac{x}{b} + h = h$ o. $\frac{x+c}{c} = a$

p. $2a + xy = cd$ q. $\frac{x+7}{e} = \pi$ r. $\frac{x}{b} - j = N$

5. Solve each formula for y:

Example: Solve for y: $3x + 5y + 1 = 0$

Solution:

$5y + 1 = -3x$ (subtract $3x$ from each side)

$5y = -3x - 1$ (subtract 1 from each side)

$y = \frac{-3x-1}{5}$ (divide each side by 5)

a. $y + 2x = 10$

b. $y - 3x = -13$

c. $y + 4x - 12 = 0$

d. $3x + y = -7$

e. $-4x + y - 1 = 0$

f. $-3x + y = 7$

g. $-2x + y + 1 = 0$

h. $3x + 2y = 10$

i. $2x - 7y = 14$

j. $-8x - 2y + 7 = 0$

k. $-3x + 8y = 1$

l. $9x - 5y + 2 = 0$

❑ *MULTI-STEP EXAMPLES*

EXAMPLE 10: **Solve for n:** $\frac{Tn+a}{b} = c$

Solution: To solve this formula for n, we must remove the T, the a, and the b. At each step, determine the final operation, and perform the reverse operation.

Start with the original problem: $\frac{Tn+a}{b} = c$

The final operation is division by b. Remove the b by multiplying each side of the equation by b: $b\left(\frac{Tn+a}{b}\right) = b(c)$

Simplify each side: $Tn + a = bc$

Now the final operation is the addition of the a. Remove the a by subtracting it from both sides: $Tn = bc - a$

Now divide both sides by T: $\dfrac{Tn}{T} = \dfrac{bc-a}{T}$

Simplify each side, and we're done: $\boxed{n = \dfrac{bc-a}{T}}$

EXAMPLE 11: **Solve for x:** $\dfrac{nx-w}{y+z} = e-f$

Solution: This formula has the same structure as the one in the previous example, except that the denominator and the quantity on the right consist of two terms instead of one. The only thing to remember is to use parentheses wherever appropriate.

$\dfrac{nx-w}{y+z} = e-f$ (original formula)

$\Rightarrow \quad \dfrac{nx-w}{y+z}\left[y+z\right] = \left(e-f\right)\left[y+z\right]$ (multiply each side by $y+z$)

$\Rightarrow \quad nx-w = (e-f)(y+z)$ (simplify)

$\Rightarrow \quad nx = (e-f)(y+z)+w$ (add w to each side)

$\Rightarrow \quad \boxed{x = \dfrac{(e-f)(y+z)+w}{n}}$ (divide each side by n)

EXAMPLE 12: **Solve for R:** $\dfrac{aR-b}{n} + d = Q$

Solution: This might be the place to show you another approach to solving complicated formulas. The eventual steps we will take will be exactly as we've done in the previous examples. But some students like to see the Order of Operations explicitly numbered; this helps them see what operation to undo at each stage of the problem.

Pretend you were R, the unknown. Let's list exactly what's been done to you, using the Order of Operations:

1. you were multiplied by a
2. b was then subtracted
3. the whole thing was then divided by n
4. last, d was added on

To untangle this mess, we need to reverse the Order of Operations, and reverse the operation at each stage of the Order of Operations:

1. subtract d
2. multiply by n
3. add b
4. divide by a

The original formula: $\frac{aR-b}{n}+d = Q$

1. Subtract d from each side: $\frac{aR-b}{n}+d-d = Q-d$

   Simplify: $\frac{aR-b}{n} = Q-d$

2. Multiply each side by n: $\frac{aR-b}{n}\left[n\right] = (Q-d)\left[n\right]$

   Simplify: $aR-b = n(Q-d)$

3. Add b to each side: $aR-b+b = n(Q-d)+b$

   Simplify: $aR = n(Q-d)+b$

4. Divide each side by a: $\frac{aR}{a} = \frac{n(Q-d)+b}{a}$

   We made it! $\boxed{R = \frac{n(Q-d)+b}{a}}$

Homework

6. Solve each formula for x:

a. $\frac{x+a}{b} = c$ b. $\frac{3x-a}{d} = w$ c. $\frac{Tx+\pi}{\pi} = A$

d. $\frac{x+a}{c+d} = R$ e. $\frac{bx-c}{y} + L = M$ f. $\frac{nx+m}{a-b} = y-w$

g. $\frac{ax+u-w}{b} = c$ h. $\frac{xy-a+b}{T} = Z$ i. $\frac{ax-c+d}{L} = T$

j. $\frac{tx+a}{c} = b$ k. $\frac{ax+b}{y+z} = g-h$ l. $\frac{ax-b}{Q} + c = d$

m. $\frac{7x+u-w}{z} = a$ n. $\frac{ax}{b-c} = d$ o. $\frac{hx+p}{p-q} = a+b$

p. $\frac{x-a}{b} = d$ q. $\frac{\pi x+a}{d} = w$ r. $\frac{Tx-e}{e} = b$

s. $\frac{x-a}{c+d} = R$ t. $\frac{bx+c}{c} - L = M$ u. $\frac{nx-n}{a+b} = y+w$

v. $\frac{ax-u+w}{c} = d$ w. $\frac{xy+y+z}{S} = A$ x. $\frac{ax+b-L}{Q} = T$

y. $\frac{wx-a}{c} = b$ z. $\frac{ax-b}{y-z} = g+h$

❑ *SOLVING REAL FORMULAS*

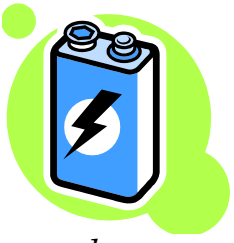
Homework

7. Solve for m in the kilometers/miles formula: $k = 1.61m$.

8. Solve for k in the kilometers/miles formula: $m = \frac{k}{1.61}$.

9. Solve for C in the temperature formula: $F = 1.8C + 32$.

10. Solve for F in the temperature formula $C = \frac{F - 32}{1.8}$.
11. Solve for L in the assets/liabilities/capital formula: $A = L + C$.
12. Solve for R in the profit/revenue/expense formula: $P = R - E$.
13. Solve for t in the distance/rate/time formula: $d = rt$.
14. Solve for m in the density/mass/volume formula: $d = \frac{m}{V}$.
15. Solve for S in the average formula: $A = \frac{S}{n}$.
16. Solve for i in Ohm's Law: $V = iR$.
17. Solve for R in the gas law: $PV = nRT$.
18. Solve for m in the potential energy formula: $P = mgh$.
19. Solve for m in Einstein's equation: $E = mc^2$.
20. Solve for l in the perimeter formula: $P = 2l + 2w$.
21. Solve for π in the circle area formula: $A = \pi r^2$.



❑ *Final Note*

Here's a parting comment regarding the solving of any formula:

The final answer can <u>never</u> contain
the variable you're solving for.

For example, a final solution such as

$$x = ab + cx$$

is outright insanity. Here's why: This solution says that to <u>determine</u> the value of <u>x</u>, we need to <u>know</u> the values of a, b, c, <u>and x</u> ! A fatal case of circular reasoning, indeed.



And now for something ironic: In computer languages the statement $x = x + 3$ is perfectly valid and tremendously useful (it merely increases the current value of x by 3.) But it is not an equation – and we're not trying to solve for anything – so it doesn't violate the rule above.

Review Problems

22. Solve for w in the perimeter formula: $P = 2l + 2w$.

23. Solve for w in the area formula: $A = lw$.

24. Solve for r in the circumference formula: $C = 2\pi r$.

25. Solve for d in the radius formula: $r = \frac{d}{2}$.

26. Solve for A in the formula: $\frac{A}{r^2} = \pi$.

27. Solve each formula for x:

a. $x - c = d$ b. $2x + b = R$ c. $abx = c$

d. $\frac{x}{u} = N$ e. $x(y + z) = a$ f. $\frac{x}{n} = c - d$

g. $\frac{x}{a+b} = m - n$ h. $\frac{x}{c-Q} = c + Q$ i. $\frac{x}{R} = a - b + c$

j. $x(b_1 + b_2) = A$ k. $\frac{x}{a} - e = m$ l. $\frac{x+a}{b} = y$

m. $9x - 7y + 13 = 0$

28. Your neighbor solves the formula $ax + bx = c$ for x, and comes up with a solution of $x = \frac{c - bx}{a}$. Your comments, please.

Solutions

1. a. $x = a - b$ b. $x = \frac{d}{c}$ c. $x = z + y$, or $x = y + z$

d. $x = TL$, or $x = LT$ e. $x = m - c$ f. $x = \frac{z}{y}$

g. $x = W + R$ h. $x = ab$ i. $x = n + m$

j. $x = b - a$ k. $x = W + b$ l. $x = -zy$, or $x = -yz$

m. $x = w - a$ n. $x = c + ab$ o. $x = d - bc$

2. a. $a = \frac{d}{b - c}$ b. $a = \frac{S}{Q + R}$ c. $a = \frac{A}{b_1 + b_2}$

d. $a = \frac{d}{bc}$ e. $a = \frac{P}{mg}$ f. $a = \frac{T}{r_1 + r_2}$

g. $a = \frac{M}{bc}$ h. $a = \frac{z}{x - y}$ i. $a = \frac{n}{w - u}$

j. $a = \frac{z}{xy}$ k. $a = \frac{W}{bc}$ l. $a = \frac{Q}{g - h}$

3. a. $y = a(b + c)$ b. $y = c(d - e)$ c. $y = x(x + a)$

d. $y = t(b + c - d)$ e. $y = c(x - w + z)$ f. $y = A(x - z)$

g. $y = c(w + 7)$ h. $y = Q(a - b)$ i. $y = w(a - b + c)$

j. $y = k(x + g - h)$ k. $y = P(L + M - N)$ l. $y = (x + z)(u + w)$

m. $y = (Q + T)(a - R)$ n. $y = (g - h)(m - n)$

o. $y = (w + u)(a + b - c)$ p. $y = (x + w)(a + b + c)$

q. $y = (d - e + h)(u + w - x)$

4. a. $x = \frac{c - b}{a}$ b. $x = \frac{R + d}{c}$ c. $x = \frac{n - a}{b}$

d. $x = \frac{Q - m}{-n}$ e. $x = (h + e)b$, or $x = b(h + e)$

f. $x = ac + b$ g. $x = \frac{cd - ab}{y}$ h. $x = pd - b$

i. $x = (L - k)a$, or $x = a(L - k)$ j. $x = \frac{c + d}{a}$

k. $x = \frac{e - d}{c}$ l. $x = \frac{n - a}{-b}$ m. $x = \frac{R - t}{n}$

n. $x = 0$ o. $x = ac - c$ p. $x = \frac{cd - 2a}{y}$

q. $x = \pi e - 7$ r. $x = b(N + j)$, or $x = (N + j)b$

5. a. $y = 10 - 2x$ b. $y = -13 + 3x$ c. $y = 12 - 4x$

d. $y = -7 - 3x$ e. $y = 1 + 4x$ f. $y = 7 + 3x$

g. $y = 2x - 1$

h. $3x + 2y = 10 \Rightarrow 2y = 10 - 3x \Rightarrow y = \frac{10 - 3x}{2}$

i. $y = \frac{14 - 2x}{-7}$ j. $y = \frac{8x - 7}{-2}$ k. $y = \frac{1 + 3x}{8}$

l. $y = \frac{-2 - 9x}{-5}$

6. a. $x = bc - a$ b. $x = \frac{dw + a}{3}$ c. $x = \frac{A\pi - \pi}{T}$

d. $x = R(c + d) - a$ e. $x = \frac{y(M - L) + c}{b}$

f. $x = \frac{(y - w)(a - b) - m}{n}$ g. $x = \frac{bc + w - u}{a}$ h. $x = \frac{ZT - b + a}{y}$

i. $x = \frac{LT - d + c}{a}$ j. $x = \frac{bc - a}{t}$

k. $x = \frac{(g - h)(y + z) - b}{a}$ l. $x = \frac{(d - c)Q + b}{a}$ or $x = \frac{Q(d - c) + b}{a}$

m. $x = \frac{az + w - u}{7}$ n. $x = \frac{d(b - c)}{a}$

o. $x = \frac{(a + b)(p - q) - p}{h}$ p. $x = db + a$ q. $x = \frac{wd - a}{\pi}$

r. $x = \frac{be + e}{T}$ s. $x = R(c + d) + a$

t. $x = \frac{(M + L)c - c}{b}$ u. $x = \frac{(y + w)(a + b) + n}{n}$

v. $x = \dfrac{cd - w + u}{a}$

w. $x = \dfrac{AS - z - y}{y}$

x. $x = \dfrac{TQ + L - b}{a}$

y. $x = \dfrac{bc + a}{w}$

z. $x = \dfrac{(g + h)(y - z) + b}{a}$

7. $m = \dfrac{k}{1.61}$

8. $k = 1.61m$

9. $C = \dfrac{F - 32}{1.8}$

10. $F = 1.8C + 32$

11. $L = A - C$

12. $R = P + E$

13. $t = \dfrac{d}{r}$

14. $m = dV$

15. $S = An$

16. $i = \dfrac{V}{R}$

17. $R = \dfrac{PV}{nT}$

18. $m = \dfrac{P}{gh}$

19. $m = \dfrac{E}{c^2}$

20. $l = \dfrac{P - 2w}{2}$

21. $\pi = \dfrac{A}{r^2}$

22. $w = \dfrac{P - 2l}{2}$

23. $w = \dfrac{A}{l}$

24. $r = \dfrac{C}{2\pi}$

25. $d = 2r$

26. $A = \pi r^2$

27. a. $x = d + c$

b. $x = \dfrac{R - b}{2}$

c. $x = \dfrac{c}{ab}$

d. $x = Nu$

e. $x = \dfrac{a}{y + z}$

f. $x = n(c - d)$

g. $x = (m - n)(a + b)$

h. $x = (c + Q)(c - Q)$

i. $x = R(a - b + c)$

j. $x = \dfrac{A}{b_1 + b_2}$

k. $x = a(m + e)$

l. $x = by - a$

m. $x = \dfrac{7y - 13}{9}$

28. When solving a formula for x, the only letter that can't possibly be in the final answer is x itself. For one thing, solving a formula for x means to <u>isolate</u> it — how isolated is it when there's an x on each side of the equals sign?

Second, the bogus solution is circular reasoning. According to your neighbor's solution, to find the value of x we would need the values of a, b, and c, <u>and x!</u> That is, to <u>find</u> the value of x, we would <u>need</u> the value of x. That's crazy.

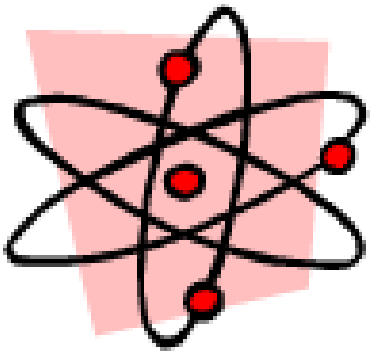
❑ *TO ∞ AND BEYOND*

Solve for *x*: $\dfrac{\dfrac{\dfrac{ax+b}{c}-d+e}{f-g}+gh-k}{w+z}+y = n$

"That is what learning is.
You suddenly understand something
you've understood all your life,
but in a new way."

– *Doris Lessing*

USING FORMULAS



In 1905, while working in a patent office in Bern, Switzerland, Albert Einstein published his statement describing the equivalence of mass and energy:

The energy contained in an object is equal to the mass of the object times the square of the speed of light.

That's one way to say it. But sometimes English is not the best way to say something. In Algebra, we would write this statement as

$$E = mc^2$$

[where c represents the *speed of light*, about 186,000 miles per second]

Can you see the advantage of Algebra over English?

❑ *IMPORTANT TERMS*

We begin with some basic terminology and notation:

Sum The *sum* of two quantities is the result of **adding** them. For example, the sum of the constants 20 and 4 is 24, and the sum of the variables x and y is written $x + y$.

Difference The *difference* of two quantities is the result of **subtracting** them. For instance, the difference of 20 and 4 is 16, and the difference of x and y is written $x - y$.

Product The *product* of two quantities is the result of **multiplying** them. We can say that 80 is the product of 20 and 4. The product of x and y can be written in any of the following ways:

$x \times y$ $x \cdot y$ $x(y)$ $(x)y$ $(x)(y)$ xy ⟵ the best way

In each product of x with y, both the x and the y are called ***factors***, while the answer is called the ***product***. And you might know that a computer would expect to see $x * y$ if you wanted the computer to find the product of x and y.

Quotient The *quotient* of two quantities is the result of **dividing** them. For example, the quotient of 20 and 4 is 5. The quotient of x and y can be written in a variety of ways:

$y\overline{)x}$	This is the classic "long division" sign from elementary school, but will also be used in Algebra.
$x \div y$	This is the standard symbol displayed on most calculators, and is the notation when dividing fractions.
x / y	This is how division is expressed in spreadsheets and computer languages.
$\frac{x}{y}$	This is the standard fraction way of showing division — and for your algebra course — it's the best way.

Square The *square* of a number results from multiplying that number by itself. For instance, the square of 7 is 49. The square of x is written x^2, and means $x \cdot x$, or xx. It is read "x squared," or "x to the second power," or simply "x to the second."

Parentheses When you must clarify that a certain operation must be done before another operation, use parentheses. For example, if the sum of 7 and 8 is to be multiplied by 3, we can write

$$3(7 + 8) = 3(15) = 45$$

Homework

1. Calculate (evaluate) each expression:

a. $34 + 66$	b. $32 - 20$	c. 20×3	d. $12 \cdot 4$
e. $5(7)$	f. $(3)9$	g. $(17)(16)$	h. $25\overline{)600}$
i. $38 \div 2$	j. $3 \div 12$	k. $18/2$	l. $\frac{34}{17}$
m. $\frac{8}{10}$	n. $3(4 + 5)$	o. $6(10 - 3)$	p. 12^2

2. Computers use the asterisk, *****, to signify multiplication. Calculate 2 * 3 * 4.

3. a. What is the sum of 19 and 6?
   b. What is the sum of a and b?
   c. What is the difference of 50 and 20?
   d. What is the difference of Q and R?
   e. What is the product of 9 and 10?
   f. What is the product of u and w?
   g. What is the quotient of 55 and 11?
   h. What is the quotient of m and n?
   i. What is the square of 7?
   j. What is the square of x?
   k. What is 15 squared?
   l. What is 11 to the second power?

❑ *CONVERTING MILES AND KILOMETERS*

In almost every country, large distances are measured not in miles, but in the metric system unit called the **kilometer** (accent on either the first or second syllable). One meter is a little longer than a yard, and a kilometer is 1000 meters (sometimes spelled metres). What we need is

a pair of formulas we can use to convert between miles and kilometers. Here they are (they're actually just good approximations), where "mi" represents *miles* and "km" represents *kilometers*:

$$\text{km} = 1.6 \times \text{mi}$$

$$\text{mi} = \frac{\text{km}}{1.6}$$

Suggestion: Ask your teacher whether you need to memorize the formulas in this chapter.

EXAMPLE 1:

A. **Convert 7 miles to kilometers.**

The hardest part of this problem is determining which of the two formulas to use. Because of how they are written, the first formula tells us how to find kilometers if we know miles, while the second formula tells us how to find miles if we know kilometers. Since the first formula starts with km, and since km is what the question is asking for, we use the first formula.

$\text{km} = 1.6 \times \text{mi} = 1.6 \times 7 =$ **11.2**

Therefore, 7 miles = **11.2 kilometers**

B. **Convert 13.9 kilometers to miles, and round your answer to the hundredths place.**

We're trying to find miles in this problem; since the second formula starts with mi, it's the appropriate formula to use.

$\text{mi} = \frac{\text{km}}{1.6} = \frac{13.9}{1.6} = 8.7$ (rounded off)

Thus, 13.9 kilometers = **8.7 miles** (approximately)

Homework

4. Each of the following distances is given in miles. Convert to kilometers, and round to the tenths place:

   a. 5 mi b. 29 mi c. 12.5 mi d. 0.23 mi e. 1.07 mi

5. Each of the following distances is given in kilometers. Convert to miles, and round to the tenths place:

   a. 12 km b. 400 km c. 17.3 km d. 0.88 km e. 0.09 km

❑ *KINETIC ENERGY*

EXAMPLE 2: **The energy an object due to its motion is called *kinetic energy*, and is given by the formula**

$$K = \frac{1}{2}mv^2$$

where *m* is the mass and *v* is the velocity of the object. Find the kinetic energy of a car with a mass of 1250 kg (kilograms) and a velocity of 30 m/s (meters per second).



Solution: The formula for kinetic energy is

$$K = \frac{1}{2}mv^2$$

Placing the given values of m and v into the formula gives:

If *m* is measured in kg and *v* is measured in m/s, then kinetic energy is measured in ***joules***, named after James Prescott **Joule.**

$$K = \frac{1}{2}(1250)(30^2)$$

$$\Rightarrow \quad K = \frac{1}{2}(1250)(900)$$

$$\Rightarrow \quad K = (625)(900) \qquad \left(\frac{1}{2}(1250) \text{ is } 1250 \div 2\right)$$

$$\Rightarrow \quad K = 562{,}500$$

Therefore, the kinetic energy of the car is

562,500 joules

Homework

6. Find the kinetic energy (in joules, J) given the mass (in kg) and the velocity (in m/s):

a. $m = 14;\ v = 5$

b. $m = 2;\ v = 10$

c. $m = 5;\ v = 1$

d. $m = 30;\ v = 12$

❑ *CONVERTING TEMPERATURES*

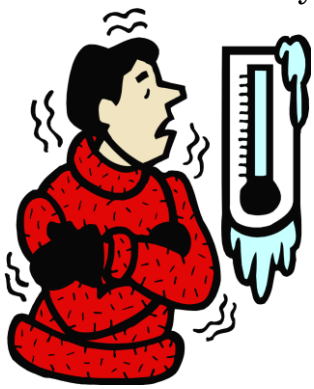


Consider a job announcement trying to lure U.S. workers to work at the new office in Victoria, British Columbia (Canada). People relocating usually want to know the new area's climate. In your research, you found the following data for average daily highs and lows for each of the four seasons in Victoria.

	Average High	Average Low
Fall	14°C	4°C

Winter	10°C	–6°C
Spring	22°C	16°C
Summer	30°C	18°C

It appears that Victoria has a pretty chilly climate — putting data like this in an announcement isn't likely to produce many job applicants. But look closely; these temperatures are given in degrees *Celsius* (in the past, called centigrade) — not in Fahrenheit, the scale with which most of us are familiar. So it's clear we must translate from Celsius to Fahrenheit before we compose the announcement.



On the ***Fahrenheit*** scale, water freezes at **32°F** and boils at **212°F**. These numbers are a bit awkward, so the ***Celsius*** scale was created to make things easier. On this scale, the freezing point of water is assigned the number **0°C** and the boiling point is declared to be **100°C**.

What we need now is a conversion formula that allows us to translate from Celsius to Fahrenheit, so that we can convert the given table to a table Americans will understand. For all we know, the climate could be even chillier than the table suggests. Here's the formula we need; it's derived in the chapter *Temperature Formulas*:

$$F = 1.8C + 32$$

This formula tells us that to find the Fahrenheit temperature, we need to perform two steps:

1) multiply the Celsius temperature by 1.8
2) add 32 to that result

EXAMPLE 3: **The average high temperature in Victoria during the Fall is 14°C. Convert this temperature to Fahrenheit.**

Solution: We state the relevant formula, plug in the given values, and find the value of the unknown.

$F = 1.8C + 32$	(the Celsius-to-Fahrenheit formula)
$F = 1.8(\mathbf{14}) + 32$	(plug in the 14°C that is given)
$F = 25.2 + 32$	(multiply 1.8 by 14)
$F = 57.2$	(add 25.2 and 32)

Thus, the equivalent temperature is $57.2°\text{F}$

So, we have a formula that converts a temperature in Celsius degrees to one in Fahrenheit degrees. Now we need to reverse the process — we need a formula that will convert Fahrenheit degrees into Celsius degrees, and here it is:

$$C = \frac{F - 32}{1.8}$$

This formula tells us that to find the Celsius temperature, we need to perform two steps:

1) subtract 32 from the Fahrenheit temperature
2) divide that result by 1.8

EXAMPLE 4: **Convert a room temperature of 72°F to Celsius.**

Solution: Using the Fahrenheit-to-Celsius formula, we write

$$C = \frac{F - 32}{1.8}$$ (the Fahrenheit-to-Celsius formula)

$$C = \frac{\mathbf{72} - 32}{1.8}$$ (plug in the 72°F that is given)

$$C = \frac{40}{1.8}$$ (do the top first)

$C = 22.2222\ldots$ (and then divide 40 by 1.8)

Therefore, when rounded to the nearest tenth, room temperature is about

22.2°C

Homework

7. Use the two formulas to perform the temperature conversions:

   a. Convert 0°C to °F.
   b. Convert 100°C to °F.
   c. Convert 32°F to °C.
   d. Convert 212°F to °C.

   How could you have predicted the answers to these questions without using the formulas?

8. Convert the given temperature to the other scale, and round to the tenths place:

   a. 100°F b. 123°F c. 35°F d. 451°F
   e. 14°C f. 200°C g. 24°C h. 98°C
   i. 50.7°F j. 123.9°C k. 43.5°F l. 1000°C

Trivia Question: One of these problems is part of the title of a classic science-fiction novel. Do you know the title and author?

9. Convert body temperature, 98.6°F, to °C. Give a Celsius temperature which would represent a serious fever.

Review Problems

10. Calculate each expression:

a. $82 + 28$ b. $100 - 17$ c. $(55)(10)$ d. $\frac{88}{44}$

e. $7(7 + 3)$ f. $10(12 - 2)$ g. 9^2 h. $1^2 + 1^2$

11. a. What is the sum of 90 and 10?

b. What is the sum of x and y?

c. What is the difference of 30 and 20?

d. What is the difference of A and B?

e. What is the product of 30 and 15?

f. What is the product of g and h?

g. What is the quotient of 100 and 50?

h. What is the quotient of c and x?

i. What is the square of 20?

j. What is the square of the quantity $a + b$?

12. a. Convert 23.9 miles to kilometers.

b. Convert 1000 kilometers to miles.

13. A student with a mass of 60 kg, who hasn't been studying his algebra, has been flung across the campus with a velocity of 7 m/s. Find the kinetic energy of this poor sap.

14. a. Convert 123.7°F to °C.

    b. Convert 0.5°C to °F.

15. Air pressure at sea level is 14.7 pounds per square inch (psi). When underwater, the water pressure increases by 14.7 psi for each additional 33 ft of depth. The formula for calculating the total pressure, P, at a depth, d, is given by



$$P = 0.445d + 14.7 \qquad [d \text{ is feet; } P \text{ is psi}]$$

    a. Confirm that at a depth of 0 ft (at the surface of the water), the pressure is 14.7 psi.

    b. Compute the pressure at a depth of 100 ft.

Solutions

1.

a.	100	b.	12	c.	60	d.	48
e.	35	f.	27	g.	272	h.	24
i.	19	j.	$\frac{1}{4}$, or 0.25	k.	9	l.	2
m.	$\frac{4}{5}$, or 0.8	n.	27	o.	42	p.	144

2. 24

3. a. $19 + 6 = 25$; the sum of 19 and 6 could be written 6 + 19, since adding can be done in either order, but writing 19 + 6 is consistent with the order of the numbers in the phrase "sum of 19 and 6."

b. $a + b$ (or $b + a$ if you insist)

c. $50 - 20 = 30$; turning the 50 and the 20 around will <u>not</u> produce the same answer. As we'll see later, $20 - 50 = -30$.

d. $Q - R$; just write the letters in the order in which they appear in the phrase.

e. $9 \cdot 10 = 90$; again, I have to admit that the product of 9 and 10 could also be written as $10 \cdot 9$, but don't do it!

f. The product of u and w, in that order, is uw, which can also be written $u(w)$ or $u \cdot w$

g. $\frac{55}{11}$, or $55 \div 11$, both of which = 5

h. The quotient of m and n can be written in four different ways, but the best way is $\frac{m}{n}$.

i. $7^2 = 7 \times 7 = 49$

j. x^2

k. 225

l. 121

4. a. 8 km b. 46.4 km c. 20 km
d. 0.4 km e. 1.7 km

5. a. 7.5 mi b. 250 mi c. 10.8 mi
d. 0.6 mi e. 0.1 mi

6. a. 175 J b. 100 J c. 2.5 J d. 2160 J

7. a. 32°F b. 212°F c. 0°C d. 100°C

8. a. 37.8°C b. 50.6°C c. 1.7°C d. 232.8°C
e. 57.2°F f. 392°F g. 75.2°F h. 208.4°F
i. 10.4°C j. 255°F k. 6.4°C l. 1832°F

9. 37°C; a serious fever might be 40°C.

10. a. 110 b. 83 c. 550 d. 2
e. 70 f. 100 g. 81 h. 2

11. a. 100 b. $x + y$ c. 10 d. $A - B$ e. 450
f. gh g. 2 h. $\frac{c}{x}$ i. 400 j. $(a + b)^2$

12. a. 38.24 km b. 625 mi

13. 1470 J

14. a. 50.9°C b. 32.9°F

15. a. $P = 0.445d + 14.7$
$P = 0.445(\mathbf{0}) + 14.7$
$P = 0 + 14.7$
$P = 14.7$ ✓ It's true; at a depth of 0, the pressure is 14.7 psi.

b. $P = 59.2$ psi

❑ *To ∞ and Beyond*

A. What happens to the *kinetic energy* of an object if the mass is doubled?

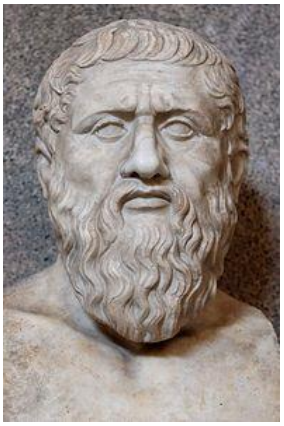
B. What happens to the kinetic energy of an object if the velocity is tripled?

C. What happens to the kinetic energy of an object if the mass is tripled and the velocity is quadrupled?

D. Our temperature formulas could be used to convert 300° below zero on the Celsius scale (written $-300°C$) to Fahrenheit. Do some research to determine why, in actuality, this problem makes no sense.

E. Use the formula for water pressure and depth to calculate the depth of a diver if the pressure is 459.7 psi.

"The educated differ from the uneducated as much as the living from the dead."



Αριστοτλε (Aristotle)

ADDING AND SUBTRACTING FRACTIONS – PART I

A prior chapter, *Algebraic Fractions, an Introduction*, showed us that to add or subtract fractions, we must ensure that the denominators are the same. So, even though the fractions in this chapter are a little more challenging than we've seen before, the same rules and techniques apply.



❑ *MORE ADDING AND SUBTRACTING*

EXAMPLE 1: **Add:** $\dfrac{4u-21}{u+7} + \dfrac{u^2}{u+7}$

Solution: We are adding fractions with the same denominator ($u + 7$), so all we have to do is add the numerators. The common denominator becomes the denominator of the sum:

$$\frac{4u-21+u^2}{u+7}$$

We've added the two fractions so that it's now a single fraction, but all fractional answers need to be reduced — we'll rearrange the terms in the numerator so it's easier to factor:

$$\frac{u^2+4u-21}{u+7}$$

Factor the numerator and divide out the common factor:

$$\frac{(u+7)(u-3)}{u+7} = \frac{1\cancel{(u+7)}(u-3)}{\cancel{u+7}_1} = \boxed{u-3}$$

EXAMPLE 2: **Subtract:** $\dfrac{\mathbf{12x-10}}{\mathbf{x^2-2x-8}} - \dfrac{\mathbf{11x-12}}{\mathbf{x^2-2x-8}}$

Solution: Remember that both addition and subtraction of fractions require a *common denominator*. The fractions in this example already have the same denominator, so there's nothing to worry about here.

The tricky part is subtracting the numerators. We must subtract the second numerator (all of it!) from the first numerator — this is where parentheses come to the rescue:

$$\frac{12x-10}{x^2-2x-8} - \frac{11x-12}{x^2-2x-8}$$ (the original problem)

$$= \frac{(12x-10)-(11x-12)}{x^2-2x-8}$$ (one numerator minus all of the other)

$$= \frac{12x-10-11x+12}{x^2-2x-8}$$ (remove the parentheses, carefully)

$$= \frac{x+2}{x^2-2x-8}$$ (combine like terms)

$$= \frac{x+2}{(x+2)(x-4)}$$ (factor the denominator)

$$= \frac{\cancel{x+2}^{1}}{1\cancel{(x+2)}(x-4)}$$ (divide out the common factor)

$$= \boxed{\frac{1}{x-4}}$$ (note that the numerator is 1)

EXAMPLE 3: **Subtract:** $\dfrac{10}{a^6} - \dfrac{7}{a^2}$

Solution: The denominators are different, so we can't subtract the fractions yet. To make the denominators the same, the second denominator (a^2) needs to be built up to match the first denominator (a^6). We ask, what should we multiply a^2 by to get a^6? Since a^2 is two factors of a, and a^6 is six factors of a, we would need four more factors of a to make them match. So we'll multiply the second fraction, top and bottom, by a^4.

$$\frac{10}{a^6} - \frac{7}{a^2}$$

$$= \frac{10}{a^6} - \frac{7 \times a^4}{a^2 \times a^4}$$

$$= \frac{10}{a^6} - \frac{7a^4}{a^6}$$

same denominator

$$= \boxed{\frac{10 - 7a^4}{a^6}}$$

EXAMPLE 4: **Add:** $\dfrac{2}{x^2y^3} + \dfrac{7}{y^7z}$

Solution: This problem also requires a common denominator. To make the denominators the same, we notice that the first denominator needs four more factors of y (so that there will be seven of them) and a factor of z; the second denominator needs an x^2 factor in it. Here's how we do it:

$$\frac{2}{x^2y^3} + \frac{7}{y^7z}$$ (the original problem)

$$= \frac{2 \cdot y^4 z}{x^2 y^3 \cdot y^4 z} + \frac{7 \cdot x^2}{y^7 z \cdot x^2}$$ (build to an LCD)

$$= \underbrace{\frac{2y^4 z}{x^2 y^7 z} + \frac{7x^2}{x^2 y^7 z}}_{\text{same denominator}}$$ (simplify tops and bottoms)

Now the denominators are the same, so we add the numerators and put that result over the common denominator:

$$\boxed{\frac{2y^4 z + 7x^2}{x^2 y^7 z}}$$

EXAMPLE 5: **Subtract:** $\frac{6k+4}{3} - \frac{2k+3}{2}$

Solution: The least common denominator (LCD) is 6, so we'll multiply the top and bottom of the first fraction by 2 and the top and bottom of the second fraction by 3:

$$\frac{6k+4}{3} - \frac{2k+3}{2}$$ (the original problem)

$$= \frac{\mathbf{2}(6k+4)}{\mathbf{2}(3)} - \frac{\mathbf{3}(2k+3)}{\mathbf{3}(2)}$$ (create the LCD of 6)

$$= \frac{12k+8}{6} - \frac{6k+9}{6}$$ (simplify tops and bottoms)

Notice the Parentheses!

$$= \frac{(12k+8)-(6k+9)}{6}$$ (combine into a single fraction)

$$= \frac{12k+8-6k-9}{6}$$ (distribute)

$$= \boxed{\frac{6k-1}{6}}$$ (combine like terms)

Homework

1. Add or subtract:

a. $\frac{d^2}{d-9} - \frac{d+72}{d-9}$

b. $\frac{3p+1}{p^2-6p-27} - \frac{2p-2}{p^2-6p-27}$

c. $\frac{n^2}{n-6} - \frac{7n-6}{n-6}$

d. $\frac{16x+63}{x+7} + \frac{x^2}{x+7}$

e. $\frac{-w-9}{w^2+8w-48} - \frac{-2w-5}{w^2+8w-48}$

f. $\frac{-6a-11}{a^2-10a-24} - \frac{-7a+1}{a^2-10a-24}$

2. Add or subtract:

a. $\frac{8}{5w^6} - \frac{3}{2w^4}$

b. $\frac{3}{b^6} + \frac{9}{5b^3}$

c. $\frac{2}{k^2} + \frac{7}{4k^5}$

d. $\frac{1}{9w^2} - \frac{4}{5w^4}$

e. $\frac{2}{5r^2} + \frac{5}{4r^4}$

f. $\frac{1}{z^3} - \frac{7}{6z^6}$

3. Add or subtract:

a. $\frac{2}{c^2m^2} + \frac{3}{m^2w^6}$

b. $\frac{4}{a^5c^5} - \frac{2}{c^6v^2}$

c. $\frac{5}{a^3b^2} + \frac{5}{b^5m^5}$

d. $\frac{2}{b^3h^2} - \frac{6}{h^3x^4}$

e. $\frac{5}{a^3c^2} + \frac{5}{c^4s^6}$

f. $\frac{6}{k^3t^4} - \frac{4}{t^5v^6}$

4. Add or subtract:

a. $\frac{2z-9}{8} + \frac{2z-9}{5}$

b. $\frac{8b+6}{6} - \frac{6b-8}{2}$

c. $\frac{-9m-9}{9} - \frac{-5m-9}{7}$

d. $\frac{4s-9}{3} + \frac{5s-3}{3}$

e. $\frac{-7u+9}{5} - \frac{4u-1}{6}$

f. $\frac{-2k-5}{4} + \frac{9k+1}{5}$

Solutions

1. a. $d + 8$ b. $\frac{1}{p-9}$ c. $n - 1$

d. $x + 9$ e. $\frac{1}{w+12}$ f. $\frac{1}{a+2}$

2. a. $\frac{16-15w^2}{10w^6}$ b. $\frac{15+9b^3}{5b^6}$ c. $\frac{8k^3+7}{4k^5}$

d. $\frac{5w^2-36}{45w^4}$ e. $\frac{8r^2+25}{20r^4}$ f. $\frac{6z^3-7}{6z^6}$

3. a. $\frac{2w^6+3c^2}{c^2m^2w^6}$ b. $\frac{4cv^2-2a^5}{a^5c^6v^2}$ c. $\frac{5b^3m^5+5a^3}{a^3b^5m^5}$

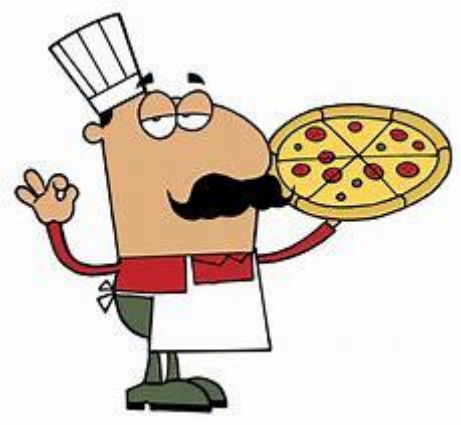
d. $\frac{2hx^4-6b^3}{b^3h^3x^4}$ e. $\frac{5c^2s^6+5a^3}{a^3c^4s^6}$ f. $\frac{6tv^6-4k^3}{k^3t^5v^6}$

4. a. $\frac{26z-117}{40}$ b. $\frac{-5b+15}{3}$ c. $\frac{-2m+2}{7}$

d. $3s - 4$ e. $\frac{-62u+59}{30}$ f. $\frac{26k-21}{20}$

ADDING AND SUBTRACTING FRACTIONS – PART II

We continue our analysis of algebraic fractions. Now the fractions will have more stuff in their numerators and denominators, but the basic rule still applies: To **add** or **subtract** fractions, we must have a common denominator.



❑ *MORE ADDING AND SUBTRACTING*

EXAMPLE 1: **Subtract:** $\dfrac{7}{x+3} - \dfrac{3}{x-2}$

Solution: The two denominators have no factor in common. (Yes, they both have an x, but this is a term, not a factor.) The LCD is the product of the two denominators: $(x + 3)(x - 2)$. So we'll multiply the top and bottom of the first fraction by $(x - 2)$, and multiply the top and bottom of the second fraction by $(x + 3)$:

$$\frac{7}{x+3} - \frac{3}{x-2} \qquad \text{(the original problem)}$$

$$= \frac{7}{x+3}\underbrace{\left[\frac{x-2}{x-2}\right]}_{=1} - \frac{3}{x-2}\underbrace{\left[\frac{x+3}{x+3}\right]}_{=1} \qquad \text{(build up to the LCD)}$$

$$= \frac{7(x-2)}{(x+3)(x-2)} - \frac{3(x+3)}{(x-2)(x+3)}$$ (the bottoms are the same)

Same denominator!

$$= \frac{7(x-2)-3(x+3)}{(x-2)(x+3)}$$ (combine the two fractions)

$$= \frac{7x-14-3x-9}{x^2+x-6}$$ (distribute)

$$= \boxed{\frac{4x-23}{x^2+x-6}}$$ (combine like terms)

EXAMPLE 2: **Add:** $\dfrac{\mathbf{5}}{\boldsymbol{m^2+5m+6}} + \dfrac{\mathbf{3}}{\boldsymbol{m+2}}$

Solution: This is a tricky one. To add these fractions, we have to multiply tops and bottoms by factors that will result in the LCD. But to know what factors to multiply by, we have to know what factors are already in the denominators. So, we factor the denominators:

$$\frac{5}{m^2+5m+6} + \frac{3}{m+2} = \frac{5}{(m+2)(m+3)} + \frac{3}{m+2}$$

Now we can easily see the factors in the denominators. To make the denominators the same, we need to multiply the top and bottom of the second fraction by $m + 3$. Then we can add the fractions:

$$\frac{5}{(m+2)(m+3)} + \frac{3}{m+2}\left[\frac{m+3}{m+3}\right] = \frac{5}{(m+2)(m+3)} + \frac{3(m+3)}{(m+2)(m+3)}$$

$$= \frac{5+3(m+3)}{(m+2)(m+3)} = \frac{5+3m+9}{m^2+5m+6} = \boxed{\frac{3m+14}{m^2+5m+6}}$$

EXAMPLE 3: **Subtract:** $\dfrac{8}{u^2+3u-4} - \dfrac{7-5u}{u^2-3u-28}$

Solution: First factor all the denominators:

$$\frac{8}{(u-1)(u+4)} - \frac{7-5u}{(u+4)(u-7)}$$

Now build up the fractions so that the LCD is created:

$$= \frac{8}{(u-1)(u+4)}\left[\frac{u-7}{u-7}\right] - \frac{7-5u}{(u+4)(u-7)}\left[\frac{u-1}{u-1}\right]$$

$$= \frac{8(u-7)}{(u-1)(u+4)(u-7)} - \frac{(7-5u)(u-1)}{(u-4)(u-7)(u-1)}$$

Notice that the denominators are the same.

$$= \frac{8(u-7) - (7-5u)(u-1)}{(u-1)(u+4)(u-7)}$$ (subtract the numerators)

Notice the parentheses.

$$= \frac{8u-56-(7u-7-5u^2+5u)}{(u-1)(u+4)(u-7)}$$ (distribute)

$$= \frac{8u-56-(-5u^2+12u-7)}{(u-1)(u+4)(u-7)}$$ (combine like terms)

$$= \frac{8u-56+5u^2-12u+7}{(u-1)(u+4)(u-7)}$$ (distribute the minus sign)

$$= \boxed{\frac{5u^2-4u-49}{(u-1)(u+4)(u-7)}}$$ (combine like terms)

You may wonder why we left the denominator in factored form. It's just a matter of opinion, but many teachers figure that the problem's hard enough without multiplying out those three binomials.

Homework

1. Perform the indicated operation:

a. $\frac{6}{d+1} + \frac{-4}{6d+5}$

b. $\frac{-6}{5h-2} - \frac{5}{4h-1}$

c. $\frac{-6}{q+1} + \frac{3}{5q+4}$

d. $\frac{3}{2t+5} - \frac{-6}{2t+1}$

2. Perform the indicated operation:

a. $\frac{3}{z-3} + \frac{1}{z^2+12z+32}$

b. $\frac{-3}{u^2+u-72} - \frac{3}{u+9}$

c. $\frac{-2}{w^2-19w+90} - \frac{2}{w-10}$

d. $\frac{4}{h+7} + \frac{-5}{h^2+11h+28}$

3. Perform the indicated operation:

a. $\frac{7}{b^2+5b-6} + \frac{3b-1}{b^2-4b-60}$

b. $\frac{-6}{a^2-16} - \frac{-5a+3}{a^2-5a+4}$

c. $\frac{-1}{s^2-s-42} - \frac{6s-7}{s^2-4s-21}$

d. $\frac{9}{v^2-9} + \frac{2v+1}{v^2+13v+30}$

Solutions

1. a. $\frac{32d+26}{(d+1)(6d+5)}$ or $\frac{32d+26}{6d^2+11d+5}$

b. $\frac{-49h+16}{(5h-2)(4h-1)}$

c. $\frac{-27q-21}{(q+1)(5q+4)}$

d. $\frac{18t+33}{(2t+5)(2t+1)}$

2. a. $\frac{3z^2+37z+93}{(z+4)(z+8)(z-3)}$

b. $\frac{-3u+21}{(u+9)(u-8)}$

c. $\frac{-2w+16}{(w-10)(w-9)}$

d. $\frac{4h+11}{(h+7)(h+4)}$

3. a. $\frac{3b^2+3b-69}{(b-1)(b+6)(b-10)}$

b. $\frac{5a^2+11a-6}{(a+4)(a-4)(a-1)}$

c. $\frac{-6s^2-30s+39}{(s+6)(s-7)(s+3)}$

d. $\frac{2v^2+4v+87}{(v-3)(v+3)(v+10)}$

"The illiterate of the 21st century will not be those who cannot read and write, but those who cannot learn, unlearn, and relearn."

– Alvin Toffler

ALGEBRAIC FRACTIONS – AN INTRODUCTION



We're about to embark on a study of algebraic fractions. If you've understood arithmetic fractions before, you'll find that algebraic fractions follow the same rules. If fractions are still a mystery, let's make this the semester you master them.

❑ *FRACTIONS, THE BASICS*

First, no matter what a fraction is used for or how complicated it looks, a fraction ultimately represents DIVISION. For example, $\frac{6}{3}$ is a fraction, but it is also the division problem $6 \div 3$, which is why we write

$$\frac{6}{3} = 2$$

The fraction $\frac{1}{4}$ is a division problem (it equals 0.25) even if we never actually carry out the division.

The top of a fraction is called the **numerator**; the bottom is called the **denominator**.

The **reciprocal** of a fraction is obtained by swapping the numerator and denominator; it's created by "inverting" the fraction. As examples, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, and the reciprocal of x is $\frac{1}{x}$. The only number that does not possess a reciprocal is 0 — for if 0 had a reciprocal, it would have to be $\frac{1}{0}$, which is undefined. Last, an interesting property

of reciprocals is that the product of any number and its reciprocal is always 1. For example, $\frac{2}{3}\times\frac{3}{2} = \frac{6}{6} = 1$.

Homework

1. Convert each fraction to a whole number (if possible):

   a. $\frac{10}{2}$ b. $\frac{99}{9}$ c. $\frac{18}{1}$ d. $\frac{34}{17}$ e. $\frac{0}{75}$ f. $\frac{3}{0}$

2. Convert each fraction to a decimal (if possible):

   a. $\frac{1}{8}$ b. $\frac{3}{16}$ c. $\frac{1}{32}$ d. $\frac{43}{10}$ e. $\frac{9}{5}$ f. $\frac{14}{0}$

3. When converting the fraction $\frac{\text{numerator}}{\text{denominator}}$ to a long-division problem, which of the following is the proper setup?

   $\text{numerator}\overline{)\,\text{denominator}}$ $\text{denominator}\overline{)\,\text{numerator}}$

4. Find the **reciprocal**:

   a. $\frac{3}{2}$ b. $-\frac{4}{5}$ c. $-\frac{8}{7}$ d. 7 e. -14

   f. $-\frac{1}{7}$ g. a h. $\frac{1}{x}$ i. $\frac{x}{y}$ j. $-\frac{y}{x}$

   k. $-T$ l. $-\frac{1}{w}$ m. 1 n. -1 o. 0

5. a. What is the reciprocal of $-\frac{7}{19}$?

   b. What is the reciprocal of $-\frac{19}{7}$?

   c. Prove that $-\frac{17}{9}$ and $-\frac{9}{17}$ are reciprocals of each other by calculating their product.

❑ *Adding and Subtracting Fractions With the Same Denominator*

$$\frac{2}{9}+\frac{5}{9} = \frac{2+5}{9} = \frac{7}{9} \qquad \Rightarrow \qquad \frac{a}{b}+\frac{c}{b} = \frac{a+c}{b}$$

To **add** fractions with the same (common) denominator

i) the numerator of the answer is the sum of the two numerators

ii) the denominator of the answer is the common denominator

$$\frac{5}{11}-\frac{3}{11} = \frac{5-3}{11} = \frac{2}{11}$$

$$\frac{2}{13}-\frac{8}{13} = \frac{-6}{13} = -\frac{6}{13} \qquad \Rightarrow \qquad \frac{x}{y}-\frac{z}{y} = \frac{x-z}{y}$$

To **subtract** fractions with the same (common) denominator

i) the numerator of the answer is the difference of the two numerators

ii) the denominator of the answer is the common denominator

Homework

6. Add or subtract the fractions:

a. $\frac{7}{10}+\frac{2}{10}$ b. $\frac{43}{101}+\frac{44}{101}$ c. $\frac{7}{8}-\frac{3}{8}$ d. $\frac{1}{5}-\frac{3}{5}$

e. $\frac{x}{y}+\frac{z}{y}$ f. $\frac{m}{u}-\frac{q}{u}$ g. $\frac{c}{a}+\frac{d}{a}$ h. $\frac{t}{w}-\frac{z}{w}$

i. $\frac{b}{c}+\frac{7}{c}$ j. $\frac{9}{Q}-\frac{7}{Q}$ k. $\frac{5}{R}-\frac{10}{R}$ l. $\frac{1}{3a}+\frac{7}{3a}$

m. $\frac{a}{bc}-\frac{d}{bc}$ n. $\frac{a}{x^2}+\frac{b}{x^2}$ o. $\frac{3}{10x}+\frac{4}{10x}$ p. $\frac{u}{a^3}-\frac{b}{a^3}$

❑ ADDING AND SUBTRACTING FRACTIONS WITH DIFFERENT DENOMINATORS

$$\begin{aligned} & \frac{2}{3}+\frac{5}{7} \\ = \quad & \frac{2\bullet 7}{3\bullet 7}+\frac{5\bullet 3}{7\bullet 3} \\ = \quad & \frac{14}{21}+\frac{15}{21} \\ = \quad & \frac{29}{21} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} & \frac{a}{b}+\frac{c}{d} \\ = \quad & \frac{a\bullet d}{b\bullet d}+\frac{c\bullet b}{c\bullet b} \\ = \quad & \frac{ad}{bd}+\frac{bc}{bd} \\ = \quad & \frac{ad+bc}{bd} \end{aligned}$$

$$\begin{aligned} & 3+\frac{4}{5} \\ = \quad & \frac{3}{1}+\frac{4}{5} \\ = \quad & \frac{3\bullet 5}{1\bullet 5}+\frac{4}{5} \\ = \quad & \frac{15}{5}+\frac{4}{5} \\ = \quad & \frac{19}{5} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} & x+\frac{y}{z} \\ = \quad & \frac{x}{1}+\frac{y}{z} \\ = \quad & \frac{x\bullet z}{1\bullet z}+\frac{y}{z} \\ = \quad & \frac{xz}{z}+\frac{y}{z} \\ = \quad & \frac{xz+y}{z} \end{aligned}$$

To **add** or **subtract** fractions with <u>different</u> denominators, we need each fraction written with the <u>same</u> denominator. This is accomplished by multiplying one or both fractions by a fraction equal to 1. A fraction equals 1 when the top and bottom are the same (and also assuming that neither the top nor the bottom is zero). Examples of fractions that are equal to 1 are $\frac{93}{93}$, $\frac{5\pi}{5\pi}$, and $\frac{-ab}{-ab}$.

In the first example above, we needed to change each denominator into 21. We did this by multiplying the top and bottom of the first fraction by 7, and the top and bottom of the second fraction by 3.

Similarly, to achieve a common denominator in the second example, we multiplied the top and bottom of the first fraction by ***d***, and the top and bottom of the second fraction by ***b***. This converted both fractions into fractions with the same denominator, ***bd***. Then they were ready to be added together.

Homework

7. Add or subtract the fractions:

a. $\frac{1}{2}+\frac{1}{3}$ b. $\frac{2}{5}-\frac{1}{10}$ c. $\frac{1}{4}-\frac{5}{6}$ d. $\frac{1}{3}-\frac{9}{2}$

e. $\frac{a}{b}+\frac{w}{x}$ f. $\frac{c}{d}-\frac{g}{h}$ g. $\frac{m}{n}+\frac{m}{q}$ h. $\frac{a}{b}-\frac{a}{c}$

i. $\frac{6}{a}+\frac{3}{b}$ j. $\frac{a}{R}-\frac{3}{T}$ k. $a+\frac{b}{c}$ l. $w-\frac{x}{y}$

m. $\frac{k}{j}+n$ n. $\frac{w}{x}-z$ o. $\frac{x^2}{a}+\frac{y^2}{a}$ p. $\frac{w}{x}+\frac{y}{z}$

❑ *MULTIPLYING AND DIVIDING FRACTIONS*

$$\frac{7}{10}\cdot\frac{11}{15}=\frac{77}{150} \quad \Rightarrow \quad \frac{m}{n}\cdot\frac{p}{q}=\frac{mp}{nq}$$

To **multiply** fractions

i) the numerator of the answer is the product of the numerators

ii) the denominator of the answer is the product of the denominators

$$\frac{2}{7}\div\frac{5}{9}=\frac{2}{7}\times\frac{9}{5}=\frac{18}{35} \quad \Rightarrow \quad \frac{c}{d}\div\frac{u}{w}=\frac{c}{d}\times\frac{w}{u}=\frac{cw}{du}$$

To **divide** fractions, multiply the first fraction (the *dividend*) by the reciprocal of the second fraction (the *divisor*).

Shortcut: When multiplying fractions, before actually multiplying the tops and bottoms, sometimes dividing out common factors can simplify reducing the final answer.

$$\frac{5}{6}\times\frac{13}{5}=\frac{\cancel{5}^{1}}{6}\times\frac{13}{\cancel{5}_{1}}=\frac{13}{6} \quad \Rightarrow \quad \frac{x}{y}\times\frac{z}{x}=\frac{\cancel{x}^{1}}{y}\times\frac{z}{\cancel{x}_{1}}=\frac{z}{y}$$

Homework

8. Perform the indicated operation:

a. $\frac{2}{3}\cdot\frac{5}{9}$ b. $\frac{1}{2}\times\frac{5}{7}$ c. $\left(\frac{8}{9}\right)\left(\frac{9}{10}\right)$ d. $\frac{4}{5}\cdot\frac{5}{4}$

e. $\frac{w}{x}\cdot\frac{w}{z}$ f. $\frac{a}{b}\times\frac{c}{b}$ g. $\frac{x}{y}\cdot\frac{c}{d}$ h. $\left[\frac{a}{b}\right]\left[\frac{b}{a}\right]$

i. $a\times\frac{b}{c}$ j. $\frac{m}{n}\times Q$ k. $z\div\frac{w}{z}$ l. $\frac{u}{w}\div a$

m. $\frac{x}{y}\div\frac{x}{y}$ n. $\frac{a}{b}\div\frac{b}{a}$ o. $\frac{a}{b}\div\frac{a}{c}$ p. $\frac{g}{h}\cdot\frac{g}{h}$

q. $3\cdot\frac{x}{y}$ r. $a\left(\frac{b}{c}\right)$ s. $\left(\frac{w}{7}\right)A$ t. $\frac{w}{7}+A$

9. Perform the indicated operation:

a. $\frac{K}{L} \div \frac{K}{M}$ b. $\frac{n}{b} + \frac{n}{b}$ c. $\frac{x}{a} - \frac{x}{a}$ d. $\frac{a}{b} \cdot \frac{b}{c}$

e. $\frac{G}{H} \div \frac{G}{H}$ f. $\frac{1}{a} + \frac{1}{b}$ g. $\frac{2}{c} - \frac{3}{d}$ h. $\frac{a}{w} - \frac{a}{z}$

i. $\frac{R}{T} \times \frac{T}{R}$ j. $6 + \frac{x}{y}$ k. $\frac{u}{m} - n$ l. $\frac{x}{y} \div \frac{x}{z}$

m. $\left(\frac{b}{c}\right)\left(\frac{b}{c}\right)$ n. $\frac{b}{c} \times \frac{c}{b}$ o. $\frac{m}{n} \div \frac{p}{q}$ p. $\frac{p}{q} + \frac{m}{n}$

q. $a\left(\frac{b}{c}\right)$ r. $\frac{w}{n} \div 5$ s. $\frac{p}{m} + E$ t. $\frac{x}{y} + \frac{a}{b} \cdot \frac{b}{a}$

Review Problems

10. a. What is the reciprocal of $-\frac{2}{9}$? b. What is the reciprocal of 0?

11. Perform the indicated operation:

a. $\frac{a}{b} + \frac{c}{b}$ b. $\frac{x}{y} + \frac{y}{x}$ c. $a - \frac{w}{u}$ d. $\frac{m}{n} - A$

e. $\frac{x}{y} \cdot \frac{w}{z}$ f. $\frac{x}{z} \times \frac{z}{y}$ g. $\frac{a}{b} \div \frac{c}{d}$ h. $c \div \frac{d}{e}$

i. $\frac{g}{h} \div \pi$ j. $\frac{m}{\pi} \cdot \frac{\pi}{m}$ k. $h + \frac{k}{h}$ l. $\frac{a}{b}\left(\frac{b}{c}\right)$

m. $\left(\frac{a}{t}\right)\left(\frac{t}{a}\right)$ n. $\frac{a}{n^2} + \frac{b}{n^2}$ o. $\frac{8}{ab} + \frac{6}{ab}$ p. $\frac{a}{x^3} - \frac{b}{x^3}$

q. $\frac{w}{4c}+\frac{w}{4c}$ r. $\frac{a}{mn}-\frac{a}{mn}$ s. $\frac{A}{B}\cdot\frac{C}{D}$ t. $\frac{C}{D}\div\frac{B}{A}$

u. $\frac{s}{r}\div s$ v. $Q\div\frac{Q}{R}$ w. $\frac{6}{L}\div\frac{6}{L}$ x. $\frac{M}{Q}\div\frac{Q}{M}$

y. $\frac{ab}{xy}+\frac{c}{xy}$ z. $\frac{abc}{d}-\frac{def}{d}$

Solutions

1. a. 5 b. 11 c. 18 d. 2 e. 0
f. Undefined

2. a. 0.125 b. 0.1875 c. 0.03125 d. 4.3 e. 1.8
f. Undefined

3. The second one

4. a. $\frac{2}{3}$ b. $-\frac{5}{4}$ c. $-\frac{7}{8}$ d. $\frac{1}{7}$ e. $-\frac{1}{14}$
f. -7 g. $\frac{1}{a}$ h. x i. $\frac{y}{x}$ j. $-\frac{x}{y}$
k. $-\frac{1}{T}$ l. $-w$ m. 1 n. -1 o. Undefined

5. a. $-\frac{19}{7}$ b. $-\frac{7}{19}$ c. $\left(-\frac{17}{9}\right)\left(-\frac{9}{17}\right)=\left(-\frac{{}^{1}\cancel{17}}{\cancel{9}_{1}}\right)\left(-\frac{\cancel{9}^{1}}{\cancel{17}_{1}}\right)=1$ ✓

6. a. $\frac{9}{10}$ b. $\frac{87}{101}$ c. $\frac{1}{2}$ d. $-\frac{2}{5}$ e. $\frac{x+z}{y}$
f. $\frac{m-q}{u}$ g. $\frac{c+d}{a}$ h. $\frac{t-z}{w}$ i. $\frac{b+7}{c}$ j. $\frac{2}{Q}$
k. $-\frac{5}{R}$ l. $\frac{8}{3a}$ m. $\frac{a-d}{bc}$ n. $\frac{a+b}{x^2}$ o. $\frac{7}{10x}$
p. $\frac{u-b}{a^3}$

7. a. $\frac{5}{6}$ b. $\frac{3}{10}$ c. $-\frac{7}{12}$ d. $-\frac{25}{6}$

e. $\frac{ax+bw}{bx}$ f. $\frac{ch-dg}{dh}$ g. $\frac{mq+mn}{nq}$ h. $\frac{ac-ab}{bc}$

i. $\frac{6b+3a}{ab}$ j. $\frac{aT-3R}{RT}$ k. $\frac{ac+b}{c}$ l. $\frac{wy-x}{y}$

m. $\frac{k+jn}{j}$ n. $\frac{w-xz}{x}$ o. $\frac{x^2+y^2}{a}$ p. $\frac{wz+xy}{xz}$

8. a. $\frac{10}{27}$ b. $\frac{5}{14}$ c. $\frac{4}{5}$ d. 1

e. $\frac{w^2}{xz}$ f. $\frac{ac}{b^2}$ g. $\frac{cx}{dy}$ h. 1

i. $\frac{ab}{c}$ j. $\frac{mQ}{n}$ k. $\frac{z^2}{w}$ l. $\frac{u}{aw}$

m. 1 n. $\frac{a^2}{b^2}$ o. $\frac{c}{b}$ p. $\frac{g^2}{h^2}$

q. $\frac{3x}{y}$ r. $\frac{ab}{c}$ s. $\frac{Aw}{7}$ t. $\frac{w+7A}{7}$

9. a. $\frac{M}{L}$ b. $\frac{2n}{b}$ c. 0 d. $\frac{a}{c}$

e. 1 f. $\frac{a+b}{ab}$ g. $\frac{2d-3c}{cd}$ h. $\frac{az-aw}{wz}$

i. 1 j. $\frac{x+6y}{y}$ k. $\frac{u-mn}{m}$ l. $\frac{z}{y}$

m. $\frac{b^2}{c^2}$ n. 1 o. $\frac{mq}{np}$ p. $\frac{np+mq}{qn}$

q. $\frac{ab}{c}$ r. $\frac{w}{5n}$ s. $\frac{p+Em}{m}$ t. $\frac{x+y}{y}$

10. a. $-\frac{9}{2}$ b. 0 does not have a reciprocal.

11. a. $\frac{a+c}{b}$ b. $\frac{x^2+y^2}{xy}$ c. $\frac{au-w}{u}$ d. $\frac{m-An}{n}$

e. $\frac{wx}{yz}$ f. $\frac{x}{y}$ g. $\frac{ad}{bc}$ h. $\frac{ce}{d}$

i. $\frac{g}{\pi h}$ j. 1 k. $\frac{h^2+k}{h}$ l. $\frac{a}{c}$

m. 1 n. $\frac{a+b}{n^2}$ o. $\frac{14}{ab}$ p. $\frac{a-b}{x^3}$

q. $\frac{w}{2c}$ r. 0 s. $\frac{AC}{BD}$ t. $\frac{AC}{BD}$

u. $\frac{1}{r}$ v. R w. 1 x. $\frac{M^2}{Q^2}$

y. $\frac{ab+c}{xy}$ z. $\frac{abc-def}{d}$

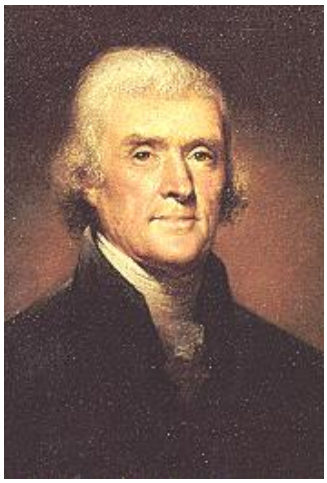
❑ *To ∞ and Beyond*

A. $\frac{1}{a}+\frac{2}{b}+\frac{3}{c}$ =

B. $\frac{a}{b}\left(\frac{c}{d}+\frac{e}{f}\right)$ =

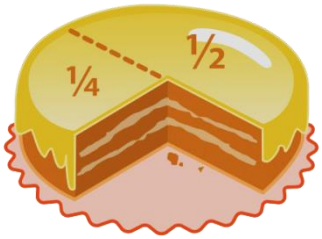
"On matters of style, swim with the current; on matters of principle, stand like a rock."

Thomas Jefferson



SIGNED FRACTIONS, POWERS, ROOTS, AND EQUATIONS

In this chapter we tie up some loose ends. First, we split a single fraction into two fractions, followed by performing our standard math operations on positive and negative fractions, including powers and square roots. Then we finish the chapter by solving equations and number problems that contain fractions.



❑ *SPLITTING UP A FRACTION*

In some courses you will see the need to "split up" a fraction in a couple of ways, one using multiplication and the other using addition.

EXAMPLE 1:

A. **Express $\frac{7x}{2}$ as the product of two quantities:**

Solution: The easiest way to see this process is to just do it and then check that it's right. Here's the claim:

$$\frac{7x}{2} = \frac{7}{2}x$$

and here's the reason:

$$\frac{7}{2}x = \frac{7}{2}\cdot\frac{x}{1} = \frac{7\cdot x}{2\cdot 1} = \frac{7x}{2} \quad ✓$$

B. **Express $\frac{7x+9}{5}$ as the sum of two fractions.**

Solution: What two fractions have a sum of $\frac{7x+9}{5}$? The denominator 5 tells us that we could use two fractions, each with a denominator of 5. Since the numerator is $7x + 9$, we can make one of the numerators $7x$ and the other one 9. Thus,

$$\frac{7x+9}{5} = \mathbf{\frac{7x}{5} + \frac{9}{5}}$$ (check by adding the two fractions)

C. **Combine the ideas of parts A and B to split up $\frac{8x-5}{7}$.**

Solution:

$$\frac{8x-5}{7} = \frac{8x}{7} - \frac{5}{7} = \mathbf{\frac{8}{7}x - \frac{5}{7}}$$

D. **Express $\frac{8x+16}{16}$ as the sum of two quantities.**

Solution:

$$\frac{8x+16}{16} = \frac{8x}{16} + \frac{16}{16} = \frac{x}{2} + 1 = \mathbf{\frac{1}{2}x + 1}$$

Homework

1. Express each fraction as the product of two quantities, in simplified form, using part A of Example 1 as a guide:

a. $\frac{9x}{4}$ b. $\frac{17x}{2}$ c. $\frac{3y}{16}$ d. $\frac{-33a}{17}$ e. $\frac{5n}{-23}$

f. $\frac{3x}{6}$ g. $\frac{-6x}{-2}$ h. $\frac{22m}{33}$ i. $\frac{-9x}{-15}$ j. $\frac{-39z}{52}$

2. Express each fraction as the sum or difference of two simplified quantities, using parts C and D of Example 1 as a guide:

a. $\frac{3x+8}{4}$ b. $\frac{-9x+18}{6}$ c. $\frac{y-1}{8}$ d. $\frac{3n+15}{-5}$

e. $\frac{-3w-24}{2}$ f. $\frac{45x-75}{15}$ g. $\frac{33x+44}{55}$ h. $\frac{-31x-17}{-20}$

❑ *SIGNED FRACTIONS*

Adding and Subtracting

$$-\frac{2}{3}-\frac{1}{2} = -\frac{4}{6}-\frac{3}{6} = -\mathbf{\frac{7}{6}}$$

$$\frac{4}{5}-\left(-\frac{2}{3}\right) = \frac{4}{5}+\frac{2}{3} = \frac{12}{15}+\frac{10}{15} = \mathbf{\frac{22}{15}}$$

$$\frac{2}{9}-7 = \frac{2}{9}-\frac{63}{9} = -\mathbf{\frac{61}{9}}$$

Multiplying and Dividing

$$\left(\frac{2}{3}\right)\left(-\frac{5}{7}\right) = -\mathbf{\frac{10}{21}}$$

$$-\frac{4}{7}\div -2 = -\frac{4}{7}\times -\frac{1}{2} = \frac{4}{14} = \mathbf{\frac{2}{7}}$$

$$\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{5}\right) = \mathbf{\frac{1}{120}}$$

$$\frac{-8}{\frac{1}{3}} = -\frac{8}{1}\times\frac{3}{1} = -\mathbf{24}$$

$$\frac{-\frac{9}{4}}{-2} = -\frac{9}{4}\div -2 = -\frac{9}{4}\times -\frac{1}{2} = \mathbf{\frac{9}{8}}$$

Powers

An **exponent** still means what it always has, so these next examples should be clear.

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3}\cdot\frac{2}{3} = \mathbf{\frac{4}{9}}$$

$$\left(-\frac{1}{4}\right)^3 = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right) = \mathbf{-\frac{1}{64}}$$

$$\left(-\frac{9}{4}\right)^1 = \mathbf{-\frac{9}{4}}$$

$$\left(\frac{123}{456}\right)^0 = \mathbf{1}$$

$$\left(\frac{1}{2}\right)^8 = \underbrace{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}}_{\text{8 factors of 1/2}} = \mathbf{\frac{1}{256}}$$

Square Roots

As for the **square root** sign, we still ask the question: What number (that's not negative) times itself gives the number in the radical sign?

$\sqrt{\frac{9}{25}} = \mathbf{\frac{3}{5}}$ This is true because $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$.

$\sqrt{\frac{1}{144}} = \mathbf{\frac{1}{12}}$ This is because $\frac{1}{12}\times\frac{1}{12} = \frac{1}{144}$.

$\sqrt{-\frac{4}{49}}$ does <u>not</u> exist, because $-\frac{4}{49}$ is a negative number, and square roots of negative numbers lie outside the real numbers.

$\sqrt{\frac{-4}{-49}}$ does exist, because the radicand (the fraction) is actually a positive number: $\sqrt{\frac{-4}{-49}} = \sqrt{\frac{4}{49}} = \mathbf{\frac{2}{7}}$.

Homework

Perform the indicated operation:

3. a. $-\frac{1}{2}-\frac{4}{5}$ b. $-\frac{1}{3}-\left(-\frac{1}{3}\right)$ c. $\frac{2}{3}-\left(-\frac{5}{6}\right)$

   d. $-\frac{4}{5}+\frac{2}{3}$ e. $9-\frac{4}{5}$ f. $-1-\frac{2}{3}$

   g. $\frac{8}{3}-5$ h. $-\frac{2}{3}-(-1)$ i. $-\frac{1}{4}-\frac{2}{7}$

4. a. $\left(-\frac{1}{2}\right)\left(-\frac{5}{6}\right)$ b. $\left(-\frac{2}{3}\right)\left(\frac{3}{2}\right)$ c. $-\frac{5}{6}\cdot-\frac{6}{5}$

   d. $-\frac{2}{3}\div-\frac{3}{2}$ e. $\frac{1}{2}\div-9$ f. $7\div-\frac{3}{4}$

   g. $\dfrac{-\frac{2}{3}}{-\frac{1}{9}}$ h. $\dfrac{\frac{4}{5}}{-8}$ i. $\dfrac{-\frac{4}{5}}{\frac{5}{8}}$

5. a. $\left(-\frac{2}{3}\right)^2$ b. $\left(-\frac{1}{2}\right)^3$ c. $\left(-\frac{1}{3}\right)^4$

   d. $\left(-\frac{9}{10}\right)^0$ e. $\left(-\frac{14}{19}\right)^1$ f. $\left[\frac{1}{2}-\frac{9}{10}\right]^0$

   g. $\left(-\frac{1}{2}\right)^5$ h. $\left(-\frac{2}{3}\right)^6$ i. $\left(\frac{99}{-99}\right)^{99}$

6. a. $\sqrt{\frac{81}{100}}$ b. $\sqrt{\frac{36}{64}}$ c. $\sqrt{\frac{1}{4}}$

d. $\sqrt{\frac{1}{9}}$ e. $\sqrt{\frac{121}{144}}$ f. $\sqrt{-\frac{25}{81}}$

g. $\sqrt{\frac{-256}{-289}}$ h. $\sqrt{-\frac{14}{17}-\left(-\frac{14}{17}\right)}$ i. $\sqrt{(abc)^0}$

❑ *EQUATIONS AND WORD PROBLEMS*

EXAMPLE 2: **Solve for x:** $-\frac{2}{3}x-\frac{1}{8}=\frac{5}{6}$

Solution: To isolate the x, we will first add $\frac{1}{8}$ to each side of the equation:

$$-\frac{2}{3}x-\frac{1}{8}+\frac{1}{8}=\frac{5}{6}+\frac{1}{8}$$

Now we simplify each side of the equation:

Left side: $-\frac{2}{3}x-\frac{1}{8}+\frac{1}{8}=-\frac{2}{3}x$

Right side: $\frac{5}{6}+\frac{1}{8}=\frac{5}{6}\left[\frac{4}{4}\right]+\frac{1}{8}\left[\frac{3}{3}\right]=\frac{20}{24}+\frac{3}{24}=\frac{23}{24}$

The LCD is 24

So now the equation reads:

$$-\frac{2}{3}x=\frac{23}{24}$$

The last step is to divide each side of the equation by $-\frac{2}{3}$ (in order to isolate the x):

$$\frac{-\frac{2}{3}x}{-\frac{2}{3}}=\frac{\frac{23}{24}}{-\frac{2}{3}}$$

$$\Rightarrow \quad \frac{-\frac{2}{3}x}{-\frac{2}{3}} = \frac{23}{24}\left(-\frac{3}{2}\right) \qquad \text{(multiply by the reciprocal)}$$

$$\Rightarrow \quad x = \frac{23}{3 \cdot 8} \cdot -\frac{3}{2} \qquad \text{(factor and cross-cancel)}$$

$$\Rightarrow \quad \boxed{x = -\frac{23}{16}}$$

EXAMPLE 3: **I'm thinking of a number. If $\frac{1}{2}$ is added to 3 times the number, the result is $\frac{3}{4}$. Find the number.**

Solution: You might recall this type of word problem. The only new issue is what to do with those fractions. Suppose we let n represent the unknown number. The equation which results from the wording of the problem is

$$3n + \frac{1}{2} = \frac{3}{4}$$

We could subtract $\frac{1}{2}$ from each side of the equation and then divide by 3, or we could try a different technique — one that is really useful in more advanced problems — and here it is:

Determine the LCD of all the fractions in the problem. Noting that the denominator of the first term $3n$ is 1, we calculate the LCD of 1, 2 and 4 to be **4**.

Now multiply each side of the equation by the LCD, **4**:

$$4\left[3n + \frac{1}{2}\right] = 4\left[\frac{3}{4}\right]$$

Distribute:

$$4\left[3n\right] + 4\left[\frac{1}{2}\right] = 4\left[\frac{3}{4}\right]$$

And simplify:

$$12n + 2 = 3$$

Solving for n gives the result $n = \frac{1}{12}$.

Therefore, the number I was thinking of is

$$\boxed{\frac{1}{12}}$$

Homework

7. Solve each equation: (You should solve some of them using the method of Example 2, and some using Example 3.)

a. $\frac{1}{2}x + 2 = -5$ b. $\frac{2}{3}x - \frac{1}{2} = \frac{1}{2}$ c. $\frac{2}{3}n + \frac{1}{2} = \frac{1}{3}$

d. $\frac{3}{4}t - \frac{1}{4} = \frac{1}{4}$ e. $-\frac{2}{3}n + \frac{1}{2} = -\frac{1}{2}$ f. $-\frac{1}{9}z - \frac{2}{3} = \frac{1}{3}$

g. $\frac{1}{2}x + \frac{3}{4} = -\frac{6}{8}$ h. $\frac{2}{5}a + \frac{1}{2} = -\frac{1}{10}$ i. $-\frac{2}{3}y - \frac{1}{4} = 12$

8. I'm thinking of a number. If $\frac{2}{3}$ is added to 4 times the number, the result is $\frac{9}{10}$. Find the number.

9. I'm thinking of a number. If $\frac{1}{2}$ is subtracted from 3 times the number, the result is –1. Find the number.

10. I'm thinking of a number. If $\frac{1}{7}$ is added to twice the number, the result is $-\frac{9}{10}$. Find the number.

11. I'm thinking of a number. If $\frac{2}{5}$ is subtracted from 5 times the number, the result is $-\frac{1}{2}$. Find the number.

12. If 5 times a number is increased by $\frac{2}{3}$, the result is –1. What is the number?

13. If –3 times a number is decreased by $\frac{3}{4}$, the result is $\frac{1}{2}$. What is the number?

14. If 4 times a number is increased by $\frac{1}{5}$, the result is $-\frac{2}{3}$. What is the number?

15. If –6 times a number is decreased by $\frac{2}{7}$, the result is 5. What is the number?

Review Problems

16. Express $\frac{-18y}{20}$ as a product.

17. Express $\frac{6x-11}{-2}$ as the sum of two quantities.

18. Evaluate each expression:

a. $-\frac{1}{2}+\frac{4}{5}$ b. $-\frac{1}{3}-\left(-\frac{1}{3}\right)$ c. $12-\frac{9}{13}$ d. $-\frac{2}{3}-\frac{1}{2}$

e. $\left(-\frac{1}{2}\right)\left(-\frac{1}{5}\right)$ f. $\frac{-\frac{2}{3}}{-\frac{1}{4}}$ g. $\left(-\frac{19}{40}\right)^0$ h. $\left(-\frac{1}{3}\right)^2$

i. $\left(-\frac{3}{4}\right)^3$ j. $\sqrt{\frac{9}{144}}$ k. $-\sqrt{\frac{25}{64}}$ l. $\sqrt{-\frac{9}{16}}$

19. Solve each equation:

a. $\frac{2}{3}x+\frac{1}{2}=\frac{1}{3}$ b. $-\frac{4}{5}y+\frac{1}{2}=-\frac{2}{3}$ c. $\frac{1}{4}z-\frac{3}{4}=-3$

20. I'm thinking of a number. If $\frac{1}{3}$ is added to 7 times the number, the result is $-\frac{1}{2}$. Find the number.

21. If 8 times a number is decreased by $\frac{2}{5}$, the result is –9. What is the number?

Solutions

1. a. $\frac{9}{4}x$ b. $\frac{17}{2}x$ c. $\frac{3}{16}y$ d. $-\frac{33}{17}a$
e. $-\frac{5}{23}n$ f. $\frac{1}{2}x$ g. $3x$ h. $\frac{2}{3}m$
i. $\frac{3}{5}x$ j. $-\frac{3}{4}z$

2. a. $\frac{3}{4}x+2$ b. $-\frac{3}{2}x+3$ c. $\frac{1}{8}y-\frac{1}{8}$ d. $-\frac{3}{5}n-3$
e. $-\frac{3}{2}w-12$ f. $3x-5$ g. $\frac{3}{5}x+\frac{4}{5}$ h. $\frac{31}{20}x+\frac{17}{20}$

3. a. $-\frac{13}{10}$ b. 0 c. $\frac{3}{2}$ d. $-\frac{2}{15}$ e. $\frac{41}{5}$
f. $-\frac{5}{3}$ g. $-\frac{7}{3}$ h. $\frac{1}{3}$ i. $-\frac{15}{28}$

4. a. $\frac{5}{12}$ b. -1 c. 1 d. $\frac{4}{9}$ e. $-\frac{1}{18}$
f. $-\frac{28}{3}$ g. 6 h. $-\frac{1}{10}$ i. $-\frac{32}{25}$

5. a. $\frac{4}{9}$ b. $-\frac{1}{8}$ c. $\frac{1}{81}$ d. 1 e. $-\frac{14}{19}$
f. 1 g. $-\frac{1}{32}$ h. $\frac{64}{729}$ i. -1

6. a. $\frac{9}{10}$ b. $\frac{3}{4}$ c. $\frac{1}{2}$ d. $\frac{1}{3}$ e. $\frac{11}{12}$
f. Does not exist g. $\frac{16}{17}$ h. 0 i. 1

7. a. -14 b. $\frac{3}{2}$ c. $-\frac{1}{4}$ d. $\frac{2}{3}$ e. $\frac{3}{2}$
f. -9 g. -3 h. $-\frac{3}{2}$ i. $-\frac{147}{8}$

8. $4n+\frac{2}{3} = \frac{9}{10}$

$\Rightarrow \quad 4n = \frac{9}{10}-\frac{2}{3} = \frac{27}{30}-\frac{20}{30} = \frac{7}{30}$

$\Rightarrow \quad 4n = \frac{7}{30} \Rightarrow n = \frac{\frac{7}{30}}{4} = \frac{7}{30}\div 4 = \frac{7}{30}\cdot\frac{1}{4} = \frac{7}{120}$

Alternate (and easier) Method: Multiply each side of the equation by the LCD, 30.

9. $3n-\frac{1}{2} = -1 \Rightarrow n = -\frac{1}{6}$ **10**. $-\frac{73}{140}$ **11**. $-\frac{1}{50}$

12. $-\frac{1}{3}$ **13**. $-\frac{5}{12}$ **14**. $-\frac{13}{60}$ **15**. $-\frac{37}{42}$

16. $-\frac{9}{10}y$ **17**. $-3x+\frac{11}{2}$

18. a. $\frac{3}{10}$ b. 0 c. $\frac{147}{13}$ d. $-\frac{7}{6}$ e. $\frac{1}{10}$

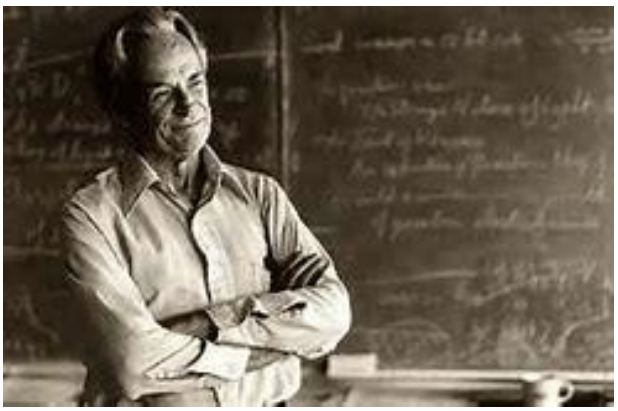
f. $\frac{8}{3}$ g. 1 h. $\frac{1}{9}$ i. $-\frac{27}{64}$ j. $\frac{1}{4}$

k. $-\frac{5}{8}$ l. Does not exist

19. a. $-\frac{1}{4}$ b. $\frac{35}{24}$ c. –9

20. $7n+\frac{1}{3}=-\frac{1}{2}$; $n=-\frac{5}{42}$ **21**. $8n-\frac{2}{5}=-9$; $n=-\frac{43}{40}$

"You can know the name of a bird in all the languages of the world, but when you're finished, you'll know absolutely nothing whatever about the bird... So let's look at the bird and see what it's doing – that's what counts. I learned very early the difference between knowing the name of something and knowing something."



Richard Feynman

MULTIPLYING AND DIVIDING FRACTIONS

EXAMPLE 1: $\dfrac{\boldsymbol{a^4k^5}}{\boldsymbol{u^2b^2}} \cdot \dfrac{\boldsymbol{a^3k^4}}{\boldsymbol{u^2b}}$

$= \dfrac{a^4k^5a^3k^4}{u^2b^2u^2b}$ (multiply tops and multiply bottoms)

$= \dfrac{a^4a^3k^5k^4}{u^2u^2b^2b}$ (rearrange the factors, top and bottom)

$= \boxed{\dfrac{a^7k^9}{u^4b^3}}$ (simplify by adding exponents)

EXAMPLE 2: $\dfrac{\boldsymbol{k^4u^6}}{\boldsymbol{ws^2}} \cdot \dfrac{\boldsymbol{ws^4}}{\boldsymbol{k^6u}}$

$= \dfrac{k^4u^6ws^4}{ws^2k^6u}$ (multiply tops and multiply bottoms)

$= \boxed{\dfrac{s^2u^5}{k^2}}$ (reduce the fraction)

Alternate solution, with factoring and cross-canceling:

$\dfrac{\boldsymbol{k^4u^6}}{\boldsymbol{ws^2}} \cdot \dfrac{\boldsymbol{ws^4}}{\boldsymbol{k^6u}}$

$= \dfrac{k^4u^5u}{ws^2} \cdot \dfrac{ws^2s^2}{k^4k^2u}$

$$= \frac{k^4 u^5 \cancel{u}}{\cancel{w} s^2} \cdot \frac{\cancel{w} s^2 s^2}{k^4 k^2 \cancel{u}}$$

$$= \boxed{\frac{u^5 s^2}{k^2}}$$

EXAMPLE 3: $\dfrac{\boldsymbol{x^2 y}}{\boldsymbol{z^2}} \div \dfrac{\boldsymbol{y^4}}{\boldsymbol{x^6 z^5}}$

$$= \frac{x^2 y}{z^2} \cdot \frac{x^6 z^5}{y^4}$$ (multiply by the reciprocal)

$$= \boxed{\frac{x^8 z^3}{y^3}}$$ (cross-cancel z's and y's)

EXAMPLE 4: **Multiply:** $\dfrac{\boldsymbol{x^2 - 9}}{\boldsymbol{x^2 - 6x + 9}} \cdot \dfrac{\boldsymbol{x + 2}}{\boldsymbol{x^2 + 5x + 6}}$

Solution:

$$\frac{x^2 - 9}{x^2 - 6x + 9} \cdot \frac{x + 2}{x^2 + 5x + 6}$$ (the given problem)

$$= \frac{(x+3)(x-3)}{(x-3)(x-3)} \cdot \frac{x+2}{(x+2)(x+3)}$$ (factor everything)

$$= \frac{\cancel{(x+3)}\cancel{(x-3)}}{(x-3)\cancel{(x-3)}} \cdot \frac{\cancel{x+2}}{\cancel{(x+2)}\cancel{(x+3)}}$$ (cancel)

$$= \boxed{\frac{1}{x-3}}$$ ($x - 3$ is on the bottom)

EXAMPLE 5: **Divide:** $\dfrac{x+10}{x^2+6x-7} \div \dfrac{x-12}{x^2-13x+12}$

Solution:

$$\frac{x+10}{x^2+6x-7} \div \frac{x-12}{x^2-13x+12}$$ (the given problem)

$$= \frac{x+10}{x^2+6x-7} \times \frac{x^2-13x+12}{x-12}$$ (invert and multiply)

$$= \frac{x+10}{(x+7)(x-1)} \times \frac{(x-1)(x-12)}{x-12}$$ (factor everything)

$$= \frac{x+10}{(x+7)\cancel{(x-1)}} \times \frac{\cancel{(x-1)}\,\cancel{(x-12)}}{\cancel{x-12}}$$ (cancel)

$$= \boxed{\frac{x+10}{x+7}}$$

Homework

1. Multiply or divide:

a. $\dfrac{x^2k}{t^4u^4} \div \dfrac{t^4u^5}{x^4k^3}$ b. $\dfrac{r^6a^6}{v^5u} \div \dfrac{r^4a^4}{vu^6}$ c. $\dfrac{w^2r^5}{k^6x} \div \dfrac{kx^6}{w^2r^5}$

d. $\dfrac{r^5z^5}{b^2n^4} \cdot \dfrac{b^6n}{r^6z^4}$ e. $\dfrac{u^4w}{v^3m^2} \div \dfrac{uw^6}{v^6m}$ f. $\dfrac{ku^2}{v^6w^3} \cdot \dfrac{v^6w^5}{k^4u^3}$

2. Multiply or divide:

a. $\dfrac{c-2}{c^2} \cdot \dfrac{c^2+4c}{c+4}$ b. $\dfrac{m}{m^2-2m-35} \cdot \dfrac{m^2-m-42}{m+6}$

c. $\dfrac{c-2}{c^2} \div \dfrac{c-3}{c^2-4c}$ d. $\dfrac{v}{v^2+4v-21} \div \dfrac{v+10}{v^2-2v-63}$

e. $\frac{q+6}{q^2+6q} \div \frac{q}{q^2+6q}$

f. $\frac{v-1}{v^2-v-6} \cdot \frac{v^2-2v-3}{v+1}$

3. Multiply or divide:

a. $\frac{m-6}{m^2-21m+108} \div \frac{m+5}{m^2-4m-45}$

b. $\frac{c+7}{c^2-18c+80} \cdot \frac{c^2-6c-40}{c+4}$

c. $\frac{z-1}{z^2-15z+44} \cdot \frac{z^2+2z-24}{z+6}$

d. $\frac{z+4}{z^2+8z+7} \div \frac{z-4}{z^2-3z-4}$

e. $\frac{m+8}{m^2+22m+120} \div \frac{m+3}{m^2+18m+80}$

f. $\frac{u-8}{u^2-6u-7} \div \frac{u-10}{u^2-9u-10}$

Solutions

1. a. $\frac{x^6k^4}{t^8u^9}$ b. $\frac{r^2a^2u^5}{v^4}$ c. $\frac{w^4r^{10}}{k^7x^7}$

d. $\frac{zb^4}{rn^3}$ e. $\frac{u^3v^3}{w^5m}$ f. $\frac{w^2}{k^3u}$

2. a. $\frac{c-2}{c}$ b. $\frac{m}{m+5}$ c. $\frac{c^2-6c+8}{c^2-3c}$

d. $\frac{v^2-9v}{v^2+7v-30}$ e. $\frac{q+6}{q}$ f. $\frac{v-1}{v+2}$

3. a. $\frac{m-6}{m-12}$ b. $\frac{c+7}{c-8}$ c. $\frac{z-1}{z-11}$

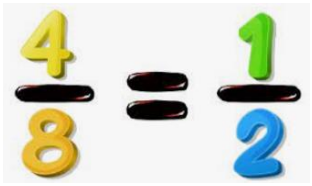
d. $\frac{z+4}{z+7}$ e. $\frac{m^2+16m+64}{m^2+15m+36}$ f. $\frac{u-8}{u-7}$

“Education is the power to think clearly, the power to act well in the world’s work, and the power to appreciate life.”

— *Brigham Young*

REDUCING QUADRATIC FRACTIONS

❑ *INTRODUCTION*



Now that we're proficient at factoring, we can knock some ugly fractions down to size. Let's review a few reducing problems to prepare for this chapter.

First Fraction: Let's reduce the fraction $\frac{39}{52}$. By prime factoring the top and bottom, we get $\frac{39}{52} = \frac{3\cdot 13}{2\cdot 2\cdot 13} = \frac{3\cdot \cancel{13}^{1}}{2\cdot 2\cdot \cancel{13}_{1}} = \mathbf{\frac{3}{4}}$. Once the top and bottom have been factored (i.e., written as a product), we can divide out (or cancel) the common factor of 13.

Second Fraction: Applying the same logic to the fraction $\frac{ax}{a^2+7a}$, we can reduce it by writing $\frac{ax}{a^2+7a} = \frac{ax}{a(a+7)} = \frac{\cancel{a}x}{\cancel{a}(a+7)} = \mathbf{\frac{x}{a+7}}$.

And a Third Fraction: $\frac{LT-QT}{AL-AQ} = \frac{\overbrace{T(L-Q)}^{\text{top factored}}}{\underbrace{A(L-Q)}_{\text{bottom factored}}} = \frac{T(\cancel{L-Q})^{1}}{A(\cancel{L-Q})_{1}} = \mathbf{\frac{T}{A}}$

And don't forget that when something is factored, it consists of a single term (where *multiplication* is the *final operation*). Thus, $a(b-c)$ is factored, but $nx + ny$ is <u>not</u> factored. Most importantly, notice that in each fraction we have canceled *factors* (things multiplied), <u>not</u> *terms* (things added).

Homework

1. Verify that the fraction $\frac{ab+ac}{ad}$ reduces to $\frac{b+c}{d}$ by letting $a = 2$, $b = 3$, $c = 4$, and $d = 6$ in both fractions.

2. Prove that the fraction $\frac{x+z}{yz}$ does <u>not</u> reduce to $\frac{x}{y}$ by choosing numerical values for x, y, and z. Also, prove to yourself that $\frac{x+z}{yz}$ does <u>not</u> reduce to $\frac{x+1}{y}$, either.

3. First, consider the fraction $\frac{x^2+2}{x^2+3}$. By letting $x = 1$, prove that the fraction does <u>not</u> reduce to $\frac{2}{3}$.

   Second, look at the fraction $\frac{2x^2}{3x^2}$. Note that this fraction <u>is</u> reducible and that it reduces to $\frac{2}{3}$.

   Third, explain why the first fraction is <u>not</u> reducible while the second fraction <u>is</u> reducible.

❑ *REDUCING QUADRATIC FRACTIONS*

Whether we're reducing $\frac{39}{52}$ (like the one at the top of the chapter) or something like what you're about to see, the process is the same:

1) Factor the top and factor the bottom.
2) Divide out any common factors.

See *Factoring Quadratics – The Real Deal* for a reminder on factoring quadratics.

EXAMPLES: **Reduce each fraction to lowest terms:**

A. $\dfrac{x^2+5x+6}{x^2-4} = \dfrac{(x+3)(x+2)}{(x+2)(x-2)} = \dfrac{(x+3)\cancel{(x+2)}}{\cancel{(x+2)}(x-2)} = \boldsymbol{\dfrac{x+3}{x-2}}$

B. $\dfrac{6n^2+5n-21}{3n^2+22n+35} = \dfrac{(3n+7)(2n-3)}{(3n+7)(n+5)} = \dfrac{\cancel{(3n+7)}(2n-3)}{\cancel{(3n+7)}(n+5)} = \boldsymbol{\dfrac{2n-3}{n+5}}$

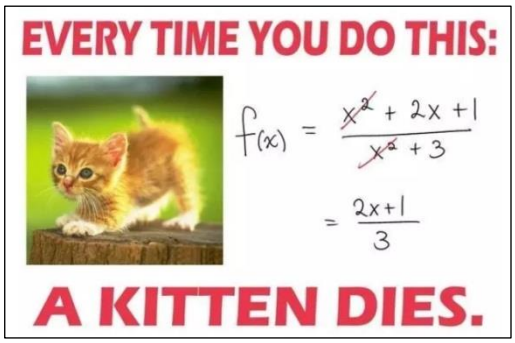
C. $\dfrac{x+5}{2x^2+7x-15} = \dfrac{x+5}{(2x-3)(x+5)} = \dfrac{\cancel{x+5}}{(2x-3)\cancel{(x+5)}} = \boldsymbol{\dfrac{1}{2x-3}}$

D. $\dfrac{3w^2-2w-5}{3w-5} = \dfrac{(3w-5)(w+1)}{3w-5} = \dfrac{\cancel{(3w-5)}(w+1)}{\cancel{3w-5}} = \boldsymbol{w+1}$

E. $\dfrac{a^2-25}{a^2+5a+6} = \dfrac{(a+5)(a-5)}{(a+3)(a+2)}$,

which contains no common factors.

Therefore, this fraction is **not reducible**.



Homework

4. Reduce each fraction to lowest terms:

a. $\dfrac{n^2-9}{n^2+6n+9}$

b. $\dfrac{4a^2+4a+1}{2a^2+5a+2}$

c. $\dfrac{c+7}{c^2-49}$

d. $\dfrac{4w^2-9}{2w-3}$

e. $\dfrac{6x^2+11x-7}{2x^2+17x-9}$

f. $\dfrac{h^2+3h+2}{h^2-3h+2}$

g. $\dfrac{3k^2-17k+1}{3k^2-17k+1}$

h. $\dfrac{100-w^2}{w^2+10w}$

i. $\dfrac{10x^2+11x-6}{5x^2-12x+4}$

j. $\dfrac{x^2+7x+10}{x^2+3x+2}$

k. $\dfrac{n^2-9}{n^2+4n+3}$

l. $\dfrac{y-7}{y^2-14y+49}$

m. $\dfrac{w^2-81}{w+9}$

n. $\dfrac{m^2+10m+25}{m^2-25}$

o. $\dfrac{x^2+2x+1}{x^2-4}$

p. $\dfrac{6a^2+13a-5}{9a^2+12a-5}$

q. $\dfrac{k^2-6k+7}{7-6k+k^2}$

r. $\dfrac{16u^2+34u-15}{2u^2+3u-5}$

Review Problems

5. Reduce: $\dfrac{6a^2-5a-21}{3a^2-4a-7}$

6. Reduce: $\dfrac{x^2-9}{x^2-4}$

7. Reduce: $\dfrac{10a^2+29a-21}{5a^2-38a+21}$

8. Reduce: $\dfrac{x-3}{x^2-9}$

9. Reduce: $\dfrac{n^2+14n+49}{(n+7)^2}$

Solutions

1. Note that this does not constitute a complete proof.

2. Be sure you don't let the y or the z be zero. How come?

3. Letting $x = 1$, $\frac{x^2+2}{x^2+3} = \frac{1^2+2}{1^2+3} = \frac{1+2}{1+3} = \frac{3}{4}$, which is not $\frac{2}{3}$.

The first fraction has two terms top and bottom; you cannot cancel terms. The second fraction has one term top and bottom (the final operation is multiplication); you are allowed to cancel factors.

4. a. $\frac{n-3}{n+3}$ b. $\frac{2a+1}{a+2}$ c. $\frac{1}{c-7}$ d. $2w + 3$

e. $\frac{3x+7}{x+9}$ f. Not reducible g. 1 h. $\frac{10-w}{w}$

i. $\frac{2x+3}{x-2}$ j. $\frac{x+5}{x+1}$ k. $\frac{n-3}{n+1}$ l. $\frac{1}{y-7}$

m. $w - 9$ n. $\frac{m+5}{m-5}$ o. Not Reducible p. $\frac{2a+5}{3a+5}$

q. 1 r. $\frac{8u-3}{u-1}$

5. $\frac{2a+3}{a+1}$ **6.** Not reducible **7.** $\frac{2a+7}{a-7}$

8. $\frac{1}{x+3}$ **9.** 1

❑ *TO ∞ AND BEYOND*

A. Consider the fraction

$$\frac{3}{x-7}$$

Let's choose a number for x. We see that if $x = 4$, for instance, then the value of the fraction is

$$\frac{3}{4-7} = \frac{3}{-3} = -1$$

And x can also take on the value 0, since $\frac{3}{\mathbf{0}-7} = \frac{3}{-7} = -\frac{3}{7}$. In fact, x can be any number at all . . . with one exception. What is that number?

B. Consider the fraction

$$\frac{x^2-4}{x^2-11x+30}$$

Find the two values of x which must never be used in this fraction.

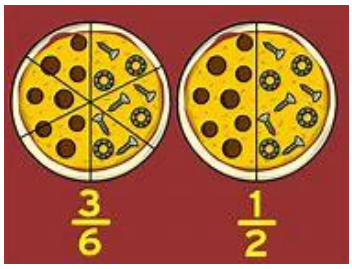
"*Education is what survives when what has been learned has been forgotten.*"

– **B. F. Skinner** (1904 - 1990)



USING THE GCF TO REDUCE FRACTIONS

❑ *REDUCING ALGEBRAIC FRACTIONS*



Most of us learned to reduce an arithmetic fraction by dividing the top and the bottom of the fraction by the same (non-zero) number. For example,

$$\frac{30}{75} = \frac{30 \div \mathbf{15}}{75 \div \mathbf{15}} = \boxed{\frac{2}{5}}$$

When it comes to algebra, though, we need to look at reducing a fraction a little differently, since it's kind of hard to divide letters by letters.

So now we re-examine the reducing of arithmetic fractions so that we get a method more appropriate to algebraic fractions. Watch as we reduce the same fraction by *factoring*:

$$\frac{30}{75} = \frac{2 \cdot 3 \cdot 5}{3 \cdot 5 \cdot 5} = \frac{2 \cdot \cancel{3} \cdot \cancel{5}}{\cancel{3} \cdot \cancel{5} \cdot 5} = \boxed{\frac{2}{5}}$$

which, of course, is the same answer as before. Here's what we did: First we factored the top and the bottom into prime factors — the very building blocks of the whole numbers. Then we "divided out" any factor that was common to both the top and the bottom (since any number divided by itself is 1). Whatever factors remain constitute the final answer.

EXAMPLE 1: **Reduce each fraction:**

A. $\frac{18}{54} = \frac{2\cdot 3\cdot 3}{2\cdot 3\cdot 3\cdot 3} = \frac{\cancel{2}\cdot\cancel{3}\cdot\cancel{3}}{\cancel{2}\cdot\cancel{3}\cdot\cancel{3}\cdot 3} = \mathbf{\frac{1}{3}}$

Notice that we don't leave a zero in the numerator just because every factor in the numerator canceled out. Remember that canceling is really dividing, so each time we cancel a pair of factors, we're really dividing a number by itself, which is always 1.

B. $\frac{ab}{a} = \frac{\cancel{a}b}{\cancel{a}} = \boldsymbol{b}$

Because a and b are being multiplied (they're one term), we can divide top and bottom by a.

C. $\frac{bc}{cw} = \frac{b\cancel{c}}{\cancel{c}w} = \mathbf{\frac{b}{w}}$

The top and bottom each consist of one term, so we can divide top and bottom by c.

EXAMPLE 2: **Reduce:** $\boldsymbol{\frac{a+b}{a+c}}$

Solution: Can we divide top and bottom by a? I don't think so. But maybe you think so; give me a moment to explain. Since dividing is designed to remove multiplication, and since the top and bottom have addition in them (they're each two terms), we cannot cancel the a's. Therefore, this fraction is **not reducible**.

Here's a more concrete way to verify this fact. If you think you should be able to cancel the a's in this fraction, let's do it and see what happens when we use numbers. Suppose $a = 6$, $b = 10$, and $c = 12$. Using the Order of Operations, we would get

$$\frac{a+b}{a+c} = \frac{6+10}{6+12} = \frac{16}{18} = \frac{8}{9}$$ (which is absolutely correct)

On the other hand, canceling the a's completely would produce the following outcome:

$$\frac{a+b}{a+c} = \frac{b}{c} = \frac{10}{12} = \frac{5}{6}$$ (not the correct answer)

And if you think that it would be legal to divide out the a's and leave 1's in their place, the following would be the consequence:

$$\frac{a+b}{a+c} = \frac{1+b}{1+c} = \frac{1+10}{1+12} = \frac{11}{13}$$ (still not the correct answer)

That's three different answers, depending upon our theory of reducing fractions. Which one's right? Well, since the Order of Operations has been with us for a long time now, I prefer to stick with what I know, and conclude that the only valid answer is $\frac{8}{9}$. I therefore think we should reject all this goofy canceling; none of it worked.



The only time we can "cancel" in a fraction is when both the top and the bottom consist of <u>one</u> term (that is, the final operation is multiplication on the top and on the bottom).

Since reducing a fraction requires one term on the top and one term on the bottom, we can now utilize our knowledge of factoring to reduce fractions which may have more than one term on the top or bottom.

<u>EXAMPLE 3:</u> **Reduce each fraction:**

A. $\frac{2x+8}{2a-10} = \frac{2(x+4)}{2(a-5)} = \frac{\cancel{2}(x+4)}{\cancel{2}(a-5)} = \boldsymbol{\frac{x+4}{a-5}}$

B. $\frac{ax+bx}{xy-xz} = \frac{x(a+b)}{x(y-z)} = \frac{\cancel{x}(a+b)}{\cancel{x}(y-z)} = \boldsymbol{\frac{a+b}{y-z}}$

C. $\frac{bn+an}{aq+bq} = \frac{n(b+a)}{q(a+b)} = \frac{n(a+b)}{q(a+b)} = \frac{n\cancel{(a+b)}}{q\cancel{(a+b)}} = \boldsymbol{\frac{n}{q}}$

Using the GCF to Reduce Fractions

D. $\dfrac{ux-uw}{ax+aw} = \dfrac{u(x-w)}{a(x+w)}$

There's no common factor to cancel. So the original fraction is **not reducible**.

EXAMPLE 4: **Reduce each fraction:**

A. $\dfrac{x^2+x}{x} = \dfrac{x(x+1)}{x} = \dfrac{\cancel{x}(x+1)}{\cancel{x}} = \boldsymbol{x+1}$

B. $\dfrac{n^2-n}{n-1} = \dfrac{n(n-1)}{n-1} = \dfrac{n\cancel{(n-1)}}{\cancel{n-1}} = \boldsymbol{n}$

C. $\dfrac{a}{a^2-3a} = \dfrac{a}{a(a-3)} = \dfrac{\cancel{a}}{\cancel{a}(a-3)} = \dfrac{\mathbf{1}}{\boldsymbol{a-3}}$

D. $\dfrac{u-4}{u^2-4u} = \dfrac{1(u-4)}{u(u-4)} = \dfrac{1\cancel{(u-4)}}{u\cancel{(u-4)}} = \dfrac{\mathbf{1}}{\boldsymbol{u}}$

We now have a technique for reducing fractions to lowest terms, but it may still be a little hard to believe, for instance, that $\dfrac{n^2-n}{n-1} = n$, as in part B of the previous example. Perhaps you'll feel a little more confident in this answer if we substitute a number for n and verify the equality ourselves. So let's choose n = 10; then

$\dfrac{n^2-n}{n-1} = \dfrac{\mathbf{10}^2-\mathbf{10}}{\mathbf{10}-1} = \dfrac{100-10}{10-1} = \dfrac{90}{9} = 10$, which equals n. ☺

THE STEPS NEEDED TO REDUCE A FRACTION:

1. Be sure the top is factored.
2. Be sure the bottom is factored.
3. Divide top and bottom by any *common* factors.

Homework

1. Reduce each fraction by factoring into primes:

a. $\frac{20}{22}$ b. $\frac{17}{51}$ c. $\frac{84}{174}$ d. $\frac{22}{33}$ e. $\frac{20}{41}$

f. $\frac{30}{50}$ g. $\frac{34}{51}$ h. $\frac{13}{39}$ i. $\frac{26}{65}$ j. $\frac{32}{128}$

2. Reduce each fraction:

a. $\frac{3x-12}{3n+21}$ b. $\frac{ax+bx}{ay+by}$ c. $\frac{tx+tz}{ty-tz}$

d. $\frac{xy+yz}{ay-by}$ e. $\frac{aR+aT}{bR+bT}$ f. $\frac{mx+my}{ax-ay}$

g. $\frac{ax+bx}{ax-cx}$ h. $\frac{ax-bx}{ay-by}$ i. $\frac{a}{am+an}$

j. $\frac{PT-QT}{T}$ k. $\frac{gn+hn}{gn-hn}$ l. $\frac{ab-ac}{cx-xy}$

3. Reduce each fraction:

a. $\frac{x^2+3x}{x}$ b. $\frac{n^2-n}{n-1}$ c. $\frac{z}{z+z^2}$ d. $\frac{Q-3}{QR-3R}$

e. $\frac{y^2-y}{y}$ f. $\frac{a^2+a}{a+1}$ g. $\frac{t-1}{t^2-t}$ h. $\frac{n}{n^2+9n}$

i. $\frac{a-3}{a^2-3a}$ j. $\frac{c+2}{c^2+c}$ k. $\frac{z^2-3z}{z-3}$ l. $\frac{Q^2-10Q}{Q}$

m. $\frac{c}{ac-c^2}$ n. $\frac{rx+x^2}{r+1}$ o. $\frac{ax+bx^2}{xy-xz^2}$ p. $\frac{an+b}{n}$

Review Problems

4. Reduce each fraction:

a. $\frac{5x+20}{10x-45}$ b. $\frac{ax-bx}{cx+dx}$ c. $\frac{am+an}{a}$ d. $\frac{w}{w^2-3w}$

e. $\frac{t^2+t}{t+1}$ f. $\frac{R^2-4R}{R}$ g. $\frac{2x+8}{3x-12}$ h. $\frac{ax+ay}{bx+by}$

5. Reduce each fraction:

a. $\frac{5x+10}{10x+45}$ b. $\frac{a^2-ab}{ac+ad}$ c. $\frac{cm+cn}{c}$ d. $\frac{u}{u^2+7u}$

e. $\frac{t-1}{t^2-t}$ f. $\frac{R^2+99R}{R}$ g. $\frac{2x+8}{6x-12}$ h. $\frac{ax+ay}{bx-by}$

Solutions

1. a. $\frac{20}{22} = \frac{\cancel{2}\cdot 2\cdot 5}{\cancel{2}\cdot 11} = \frac{10}{11}$ b. $\frac{1}{3}$ c. $\frac{14}{29}$ d. $\frac{2}{3}$ e. $\frac{20}{41}$

f. $\frac{3}{5}$ g. $\frac{2}{3}$ h. $\frac{1}{3}$ i. $\frac{2}{5}$ j. $\frac{1}{4}$

2. a. $\frac{x-4}{n+7}$ b. $\frac{x}{y}$ c. $\frac{x+z}{y-z}$ d. $\frac{x+z}{a-b}$

e. $\frac{a}{b}$ f. Not reducible g. $\frac{a+b}{a-c}$ h. $\frac{x}{y}$

i. $\frac{1}{m+n}$ j. $P-Q$ k. $\frac{g+h}{g-h}$ l. Not reducible

3. a. $x + 3$ b. n c. $\frac{1}{1+z}$ d. $\frac{1}{R}$

e. $y - 1$ f. a g. $\frac{1}{t}$ h. $\frac{1}{n+9}$

i. $\frac{1}{a}$ j. Not reducible k. z l. $Q - 10$

m. $\frac{1}{a-c}$ n. Not reducible o. $\frac{a+bx}{y-z^2}$ p. Not reducible

4. a. $\frac{x+4}{2x-9}$ b. $\frac{a-b}{c+d}$ c. $m + n$ d. $\frac{1}{w-3}$

e. t f. $R - 4$ g. Not reducible h. $\frac{a}{b}$

5. a. $\frac{x+2}{2x+9}$ b. $\frac{a-b}{c+d}$ c. $m + n$ d. $\frac{1}{u+7}$

e. $\frac{1}{t}$ f. $R + 99$ g. $\frac{x+4}{3x-6}$ h. Not reducible



"I'd plead the fifth, but we haven't learned fractions yet."

“Education is not received.

It is achieved.”

– Anonymous

DOMAIN

Have you ever gotten an ERROR message on your calculator? You may have tried to divide by zero — or perhaps you attempted to compute the square root of a negative number. In either case, you tried to use a number outside the *domain* of the *function* you were trying to calculate.



❑ *TWO WAYS TO DESCRIBE THE DOMAIN*

The ***domain*** of a function is the set of **all inputs** to the function.

There are two ways we specify the domain of a function.

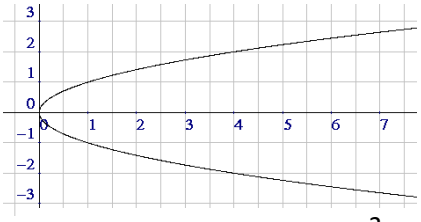
Sometimes, the domain is **explicitly** given (very common in Calculus). For instance,

> Let h be the function defined on the set of numbers $[0, 3]$ by the formula $h(x) = x^2$.

The domain of this function is the set **[0, 3]**. Why? Because the definition of the function h says so. Thus, while $h(0) = 0$ and $h(2.5) = 6.25$ and $h(3) = 9$, both $h(-1)$ and $h(4)$ are undefined. It's not that -1 and 4 can't be squared — they simply are not in the explicitly given domain, $[0, 3]$.

Other times, the domain is <u>not</u> explicitly given, so we agree to use the ***natural domain***. In this scenario, the function's domain is <u>every real number that's legal to use in the formula</u>. For example, the function $g(x) = x^2$ would have a domain of $\mathbb{R}$, because any real number can be squared. But in the formula $y = \frac{3}{x-7}$, x can be any real number <u>except</u> 7 (otherwise, we're dividing by 0). Therefore, the domain is $\mathbb{R} - \{7\}$. [All real numbers except 7.]

The concept of domain also applies to formulas that are not functions. Consider the formula $x = y^2$. We know this is not a function because an input of $x = 25$, for instance, produces two outputs, $y = \pm 5$. Nevertheless, we can ask what inputs are allowed. Since x is the square of y, it should be clear that x must be greater than or equal to zero. That is, the domain is **[0, ∞)**.



The non-function $x = y^2$ has domain [0, ∞)

Homework

1. Consider the function $f(x) = \frac{6x-6}{\sqrt{2x-8}}$. Calculate each functional value: a. $f(12)$ b. $f(36)$ c. $f(4.5)$ d. $f(4)$ e. $f(3)$
   From these results we see that 12, 36, and 4.5 are in the _____ of the function, while 4 and 3 are not.

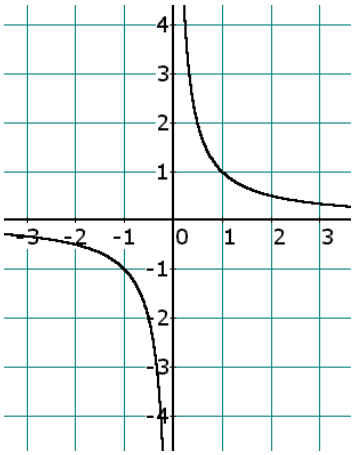
2. What is the domain of the function defined by $y = x^3$, $x \in [3, 5]$?

3. Consider the function given by the formula $g(x) = x^3 - x^2$. What is the domain of g?

❑ *PRELIMINARY EXAMPLES*

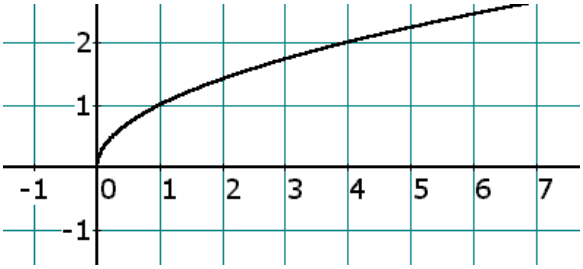
We have a pretty good notion of what a function is. There's a set of *inputs*, which are in some way associated with a set of *outputs*. Although this association is usually given by a formula (e.g., $y = x^2$), don't forget about the four quarters of the football game and the graphs in a previous chapter. Most importantly, remember that this association between the inputs and the outputs is such that *each element in the set of inputs is associated with exactly one output*. This chapter focuses on what inputs are allowed in a function.

First Example: Consider the function $y = \frac{1}{x}$. What inputs are allowed in this function? That is, what can x legally be? Well, x had better not be 0, since division by 0 is undefined. But if x is any other number, no problems will occur. So x can be any real number except 0.



> The set "All real numbers except 0" can be written $\mathbb{R} - \{0\}$.

Second Example: Now consider the function $y = \sqrt{x}$. We ask the same question: What x's are allowed in this formula? In other words, what kinds of real numbers are we allowed to take the square root of? The answer is that we can take the square root of any number that is zero or greater — which is equivalent to saying that we cannot take the square root of a negative number. Thus, x can be any number greater than or equal to 0, which we also write as $\boldsymbol{x \geq 0}$, or, if you're familiar with interval notation: **[0, ∞)**.



Homework

4. Let $y = 2x + 10$. Which of the following are legal values of x?

   a. 30 b. $-\pi$ c. $\sqrt{7}$ d. 0

5. Let $y = \frac{3}{x-2}$. Which of the following are legal values of x?

   a. 3 b. 2 c. π d. 0

6. Let $y = \sqrt{x}$. Which of the following are legal values of x?

   a. 7 b. 0 c. –9 d. 99

7. Let $y = \frac{10}{x^2 - 144}$. Which of the following are legal values of x?

   a. 12 b. 20 c. 0 d. –12

8. Let $y = \frac{x}{x^2 + 9}$. Which of the following are legal values of x?

   a. 3 b. 0 c. –3 d. 2

❑ *THE DOMAIN OF A FUNCTION*

The **domain** of a function is the set of all legal inputs.

We know that the ***domain*** of a function is the set of legal inputs to a function. Why is the domain a vital idea to consider? Total havoc can result when a number outside the domain is introduced into the function (like in a spreadsheet or a programming language).

For example, if our computer application tries to let $x = 0$ in the function $y = \frac{1}{x}$, we're sunk — the computer will stop execution of the

program and give an error message (or worse, freeze up!). And if we allow $x = -4$ in the function $y = \sqrt{x}$, then we're really up a creek, since $\sqrt{-4}$ is not a real number, and we're assuming that real numbers are all that you have at our disposal in your course.

To review, the **domain** of a typical function of x found in an algebra or trig course is the set of all real numbers that are legal for x to be in the formula. For example, the domain of the function $y = x^2$ is $\mathbb{R}$, all the real numbers (since any real number can be squared without any serious problem). But in the formula $y = \frac{3}{x-7}$, x can be any real number except 7 (why?), and therefore the domain is $\mathbb{R} - \{7\}$, all the real numbers with 7 removed.

Homework

9. Consider the function $y = \sqrt{2x-8}$.
Calculate the functional value for each x:

a. 12 b. 36 c. 4.5 d. 4 e. 3

From these results, we see that the values 12, 36, 4.5, and 4 are in the **function's domai**n, while 3 is not.

10. Let h be the function defined by $y = \frac{1}{x^2-9}$. Calculate each functional value:

x	−4	−3	−2	0	1	3	4	5
y								

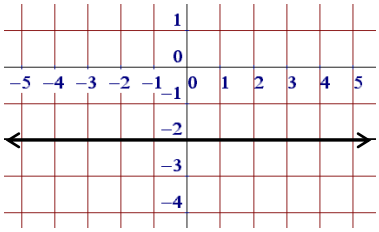
From this table of inputs and outputs, guess what the function's domain is.

11. Consider the function given by the formula $y = x^3 - x^2$. What is the domain of g?

12. What is the domain of the function $y = \frac{2}{x+3}$?

❑ *FINDING THE DOMAIN*

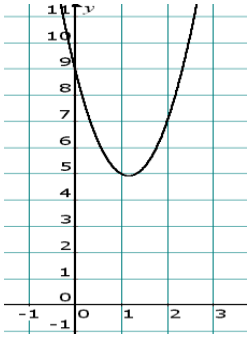
I. $y = -2$



In this equation, the x isn't even mentioned. There can, therefore, be no restrictions on x. That is, x can be any real number. Thus, the domain is

$\mathbb{R}$

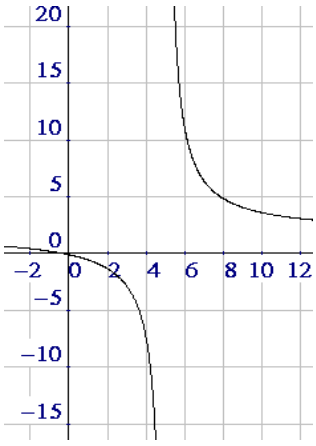
II. $y = 3x^2 - 7x + 9$



We ask ourselves: What are the legal x's? Well, x can be anything, since the operations in the formula could not possibly be a cause for concern. The domain is therefore

$\mathbb{R}$

III. $y = \dfrac{7x+2}{4x-20}$



Now we've got something interesting to look at. Question: What can go wrong in a division problem? Answer: The possibility of dividing by zero. We must ensure that x is <u>never</u> allowed to be a number that would make the denominator zero. So we find out what value(s) of x <u>would</u> make the bottom zero, and then <u>don't allow those x's to be in the domain</u>! Setting the bottom to zero gives

$4x - 20 = \mathbf{0}$ ⟵ This is what we <u>don't</u> want to happen.

$\Rightarrow \quad 4x = 20$

$\Rightarrow \quad x = 5$

Thus, if $x = 5$, the denominator is zero, which is absolutely forbidden! So, the domain of this function is the set of all real numbers except 5:

$$\mathbb{R} - \{5\}$$

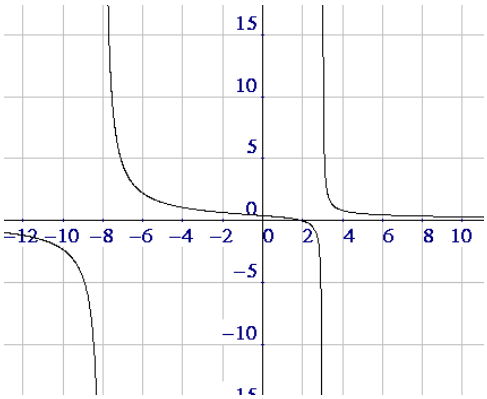
IV. $y = \dfrac{5x - 10}{x^2 + 5x - 24}$

As in the previous example, we must make certain that the denominator is never zero; let's see what values of x make it zero, and then exclude such values from our domain:

$$\begin{aligned} & x^2 + 5x - 24 = 0 \\ \Rightarrow \quad & (x + 8)(x - 3) = 0 \\ \Rightarrow \quad & x + 8 = 0 \text{ or } x - 3 = 0 \\ \Rightarrow \quad & x = -8 \text{ or } x = 3 \end{aligned}$$

We conclude that the domain is all real numbers except –8 and 3:

$$\mathbb{R} - \{-8, 3\}$$



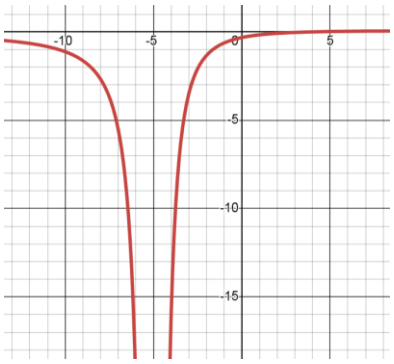
The Quadratic Formula could have been used to solve the quadratic equation in this problem.

V. $y = \dfrac{2x - 8}{x^2 + 10x + 25}$

First, we note that it's perfectly O.K. for the numerator of a fraction to be 0, so this numerator has no role in determining the function's domain. We focus on determining what values of x would make the bottom zero, and then exclude those values from the domain:

$$x^2 + 10x + 25 = 0$$

$$\Rightarrow \quad (x+5)(x+5) = 0$$
$$\Rightarrow \quad x+5 = 0 \text{ or } x+5 = 0$$
$$\Rightarrow \quad x = -5 \text{ or } x = -5$$



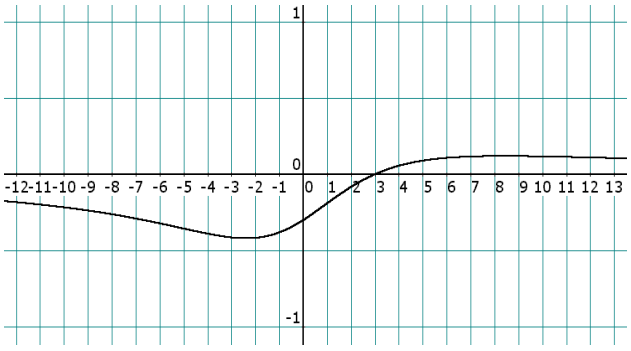
We see that the only value of x that is not allowed in the domain is $x = -5$. Hence, the domain of the function is all real numbers except -5:

$$\boxed{\mathbb{R} - \{-5\}}$$

VI. $y = \dfrac{2x-6}{x^2+20}$

Let's see what makes the denominator zero (so we can exclude it from the domain.)

$$x^2 + 20 = 0$$
$$\Rightarrow \quad x^2 = -20$$
$$\Rightarrow \quad x = \pm\sqrt{-20}, \text{ which are not real numbers.}$$



What do we conclude here? Well, we tried to figure out what values of x would make the denominator zero, so that we could exclude such an x from being in the domain. But there aren't any values of x that make the denominator zero. So there is nothing to exclude from the domain. Therefore, every real number is allowed in the function. The domain is

$$\boxed{\mathbb{R}}$$

Homework

Find the domain of each function:

13. $y = \pi$

14. $y = \frac{x^2 - 9}{9x - 7}$

15. $y = \frac{2x + 1}{x^2 - 100}$

16. $y = \frac{x^2 - 25}{x^2 + 49}$

17. $y = \sqrt{2} + \sqrt{3}$

18. $y = \sqrt{3}$

19. $y = \frac{2x - 3}{2x^2 + 3x}$

20. $y = \frac{3}{2x - x^2}$

21. $y = x^3 - 8$

22. $y = \frac{9x - 7}{2x + 9}$

23. $y = \frac{5x + 25}{x^2 - 144}$

24. $y = \frac{x}{x^2 + 3}$

25. $y = \sqrt{2}\,x + 9$

26. $y = \frac{x - 2}{x^2 - 81}$

27. $y = \frac{3}{2x^2 - 5x - 3}$

28. $y = 9$

29. $y = \frac{1}{x^2 + 10x + 25}$

30. $y = \frac{2x - 7}{8x^2 - 10x - 3}$

31. $y = \frac{x + 10}{x^2 + 100}$

32. $y = \frac{x^2 + 5x + 1}{4x^2 - x - 3}$

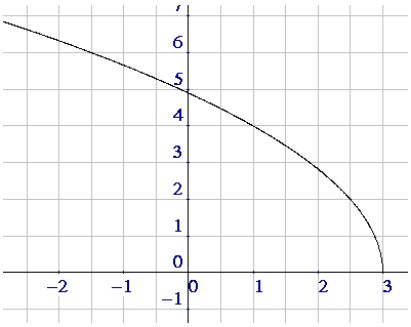
33. $y = \frac{3x + 7}{10x^2 - 20x}$

34. $y = \frac{x^2 - 14}{14x^2 - 42x}$

❑ THE DOMAIN OF SQUARE ROOT FUNCTIONS

EXAMPLE 1: **Find the domain of $y = \sqrt{24 - 8x}$.**

Solution: In this formula, we ask ourselves if anything could possibly go wrong using certain values of x. Well, the square root of a negative number doesn't exist in the real numbers, so certainly something could go wrong — for instance, if $x = 4$, then we'd have $y = \sqrt{-8}$.



Now we need a statement that describes the legal x's. How about this:

The square root of a quantity is defined (as a real number) only when that quantity is greater than or equal to zero.

So, in this example, the radicand, $24 - 8x$, must be greater than or equal to 0; i.e., **≥ 0**

	$24 - 8x \geq 0$	(**the radicand must be 0 or positive**)
$\Rightarrow$	$-8x \geq -24$	(subtract 24 from each side)
$\Rightarrow$	$x \leq 3$	(divide by –8, and reverse the inequality)

Therefore, the domain is the set of all real numbers which are less than or equal to 3:

$$\boxed{x \leq 3}$$

or, **$(-\infty, 3]$** in interval notation

Homework

Find the domain of each function:

35. $y = \sqrt{3x+12}$

36. $y = \sqrt{8-2x}$

37. $y = \sqrt{7x-15}$

38. $y = \sqrt{-4x+1}$

39. $y = \sqrt{17-17x}$

40. $y = \sqrt{4x+12}$

41. $y = \sqrt{4x+1}$

42. $y = \sqrt{10-10x}$

43. $y = \sqrt{20x}$

44. $y = \sqrt{20-x}$

45. $y = \sqrt{-3x-15}$

46. $y = \sqrt{-30-7x}$

Review Problems

Find the domain of each function:

47. $y = \sqrt{2}$

48. $y = \dfrac{2x-6}{x^2+4x+3}$

49. $f(x) = \pi$

50. $y = \pi x^3 - 17x + 1$

51. $y = \dfrac{x+2}{x-3}$

52. $y = \dfrac{3}{x^2-x-20}$

53. $y = \dfrac{20}{x^2 - x}$

54. $y = \dfrac{\pi}{13x^2 + 26x}$

55. $y = \sqrt{18 - 9x}$

56. $y = \dfrac{4 - x}{x^2 - 400}$

57. $y = \dfrac{13x - 13}{9x^2 - 42x + 49}$

58. $y = \dfrac{17x - 17}{25x^2 + 20x + 4}$

59. $y = \dfrac{x^2 - 5x + 6}{50}$

60. $y = \dfrac{\sqrt{2}}{3}$

61. $y = \sqrt{12x - 16}$

62. $y = \sqrt{10 - 5x}$

63. $y = \dfrac{x^2 - 49}{2x + 17}$

64. $y = \dfrac{\sqrt{3}}{9x^2 - 49}$

Solutions

1. a. 33/2 b. 105/4 c. 21 d. Undefined
e. Undefined Domain

2. [3, 5]

3. $\mathbb{R}$

4. All four numbers are legal inputs, since all four numbers could be multiplied by 2 and then have 10 added on with no problem.

5. 3 is a legal input, since $\frac{3}{3-2} = \frac{3}{1} = 3$; no muss, no fuss.
2 is <u>not</u> legal, since $\frac{3}{2-2} = \frac{3}{0} = \textit{Undefined}$.
π is legal because nothing can go wrong.
0 is legal because $\frac{3}{-2}$ is a perfectly fine number.

6. $\sqrt{7}$ is a perfectly fine real number, so 7 is a legal input.
$\sqrt{0}$ is also a real number (= 0), so 0 is legal.
$\sqrt{-9}$ is <u>not</u> a real number, so this number doesn't exist in our current world; hence, –9 is <u>not</u> a legal input.
$\sqrt{99}$ is some real number (9 point something), so 99 is legal.

7. When $x = 12$, we get $12^2 - 144 = 144 - 144 = 0$ in the denominator. Zero in the denominator? I don't think so! Thus, 12 is <u>not</u> a legal input. 20 is O.K. to use for x, just as 0 is fine for x. So 20 and 0 are legal inputs. But when $x = -12$, we're in the same boat as when x was 12. This is because when $x = -12$ we get $144 - 144 = 0$ in the denominator, which is certainly not allowed. Therefore, –12 is <u>not</u> legal.

8. All four values of x are legal, since none of them makes the denominator zero.

9. a. 4 b. 8 c. 1 d. 0 e. Undefined

10. 1/7; Undefined; –1/5; –1/9; –1/8; Undefined; 1/7; 1/16
When $x = 3$ or –3, the function is undefined due to dividing by zero. It seems that no other inputs would produce a zero in the denominator. So, our guess would be that the domain of h is all real numbers <u>except</u> 3 and –3. This domain can be written $\mathbb{R} - \{3, -3\}$, or $\mathbb{R} - \{\pm 3\}$

11. $\mathbb{R}$ **12**. $\mathbb{R} - \{-3\}$ **13**. $\mathbb{R}$ **14**. $\mathbb{R} - \left\{\frac{7}{9}\right\}$ **15**. $\mathbb{R} - \{\pm 10\}$

16. $\mathbb{R}$, and here's why: No matter what x is, x^2 is at least 0 (since x^2 is never negative). Now add 49 to something that is at least 0, and you now have a number which is at least 49; so the denominator $x^2 + 49$ can <u>never</u> be 0. Since dividing by 0 is the only critical issue in this function, the domain is all real numbers.

17. $\mathbb{R}$ **18**. $\mathbb{R}$ **19**. $\mathbb{R} - \left\{0, -\frac{3}{2}\right\}$ **20**. $\mathbb{R} - \{0, 2\}$

21. $\mathbb{R}$ **22**. $\mathbb{R} - \left\{-\frac{9}{2}\right\}$ **23**. $\mathbb{R} - \{\pm 12\}$ **24**. $\mathbb{R}$

25. $\mathbb{R}$ **26**. $\mathbb{R} - \{\pm 9\}$ **27**. $\mathbb{R} - \left\{3, -\frac{1}{2}\right\}$ **28**. $\mathbb{R}$

29. $\mathbb{R} - \{-5\}$ **30**. $\mathbb{R} - \left\{-\frac{1}{4}, \frac{3}{2}\right\}$ **31**. $\mathbb{R}$ **32**. $\mathbb{R} - \left\{1, -\frac{3}{4}\right\}$

33. $\mathbb{R} - \{0, 2\}$ **34**. $\mathbb{R} - \{0, 3\}$ **35**. $x \geq -4$ **36**. $x \leq 4$

37. $x \geq \frac{15}{7}$ **38**. $x \leq \frac{1}{4}$ **39**. $x \leq 1$ **40**. $x \geq -3$

41. $x \geq -\frac{1}{4}$ **42**. $x \leq 1$ **43**. $x \geq 0$ **44**. $x \leq 20$

45. $x \leq -5$ **46**. $x \leq -\frac{30}{7}$ **47**. $\mathbb{R}$ **48**. $\mathbb{R} - \{-1, -3\}$

49. $\mathbb{R}$ **50**. $\mathbb{R}$ **51**. $\mathbb{R} - \{3\}$ **52**. $\mathbb{R} - \{5, -4\}$

53. $\mathbb{R} - \{0, 1\}$ **54**. $\mathbb{R} - \{0, -2\}$ **55**. $x \leq 2$ **56**. $\mathbb{R} - \{\pm 20\}$

57. $\mathbb{R} - \left\{\frac{7}{3}\right\}$ **58**. $\mathbb{R} - \left\{-\frac{2}{5}\right\}$ **59**. $\mathbb{R}$ **60**. $\mathbb{R}$

61. $x \geq \frac{4}{3}$ **62**. $x \leq 2$ **63**. $\mathbb{R} - \left\{-\frac{17}{2}\right\}$ **64**. $\mathbb{R} - \left\{\pm \frac{7}{3}\right\}$

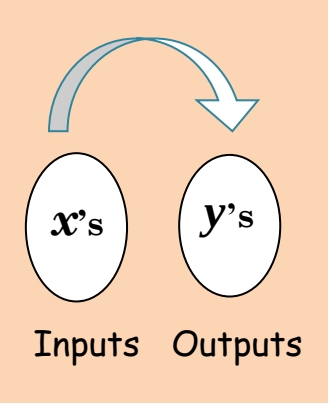
NOTE: All of the domains in the solutions involving inequalities could also be written in Interval Notation. Examples:

$x \geq 4$ can be written $[4, \infty)$.

$x < -3$ can be written $(-\infty, -3)$.

FUNCTIONS – FORMULAS AND GRAPHS

An earlier chapter gave us the basic notion of a **function**: an entity where each input produces exactly one output. In this chapter we look at functions once again, this time using the more useful versions of functions: Formulas and Graphs. As the graphic to the right indicates, when a function is written as a formula with *x*'s and *y*'s, we usually assume that the *x*'s are the inputs and the *y*'s are the outputs.



❑ *FORMULAS*

EXAMPLE 1:

A. $y = 2x + 7$ is a function.

Given an input x, there is exactly one output y. For example, if $x = 4$, then $y = 15$ and nothing else. [You may recall that this would be called a *linear* function, and its graph is a straight line.]

B. $y = x^4$ is a function.

Given an input x, there is exactly one output y. For instance, $x = 3$ produces $y = 81$. If $x = 0$, then $y = 0$. And if $x = -3$, then the only output is 81. Remember that even

though 81 is the output for two different inputs, the fact remains that $x = 3$ has only 81 for its output, and $x = -3$ also has only 81 for its output. So, for any value of x, the formula produces exactly one value of y. This is why it's a function.

C. $y = -3x^2 + 7x - 19$ is a function. [In fact, it's a *quadratic* function.]

D. $y = |x|$ is a function, called an *absolute value* function.

If $x = 7$, then $y = 7$, exactly one output for the input of 7. If $x = -7$, then $y = 7$, exactly one output for the input of -7.

E. $x = |y|$ is not a function.

Why isn't it a function? We need to conjure up some input (an x) which has more than one output (y). Let's choose $x = 4$. This gives us the equation $4 = |y|$, and there are two solutions to this equation; namely, y could be 4, or y could be -4. We have more than one output for a single input. It follows that this formula does not represent a function.

F. $3x - 7y = 19$ is a function.

Here we could solve for y,

$$y = \frac{3}{7}x - \frac{19}{7},$$

and see that given an x, there's only one y for it, so this is a (linear) function.

G. $x^2 + y^2 = 4$ is not a function.

Let $x = 0$. Then $y^2 = 4 \Rightarrow y = \pm 2$. That is, an input of 0 results in two outputs, 2 and -2. Thus, this formula (which you might know is a circle) is not a function.

H. $y = \pm\sqrt{x}$ is <u>not</u> a function.

If we choose 81 as the input, we get two outputs: ± 9, violating the fundamental notion of a function.

I. $y = 5$ <u>is</u> a function.

Given any input (the x), there's only one output, namely 5. Therefore, this horizontal line is a function.

J. $x = 2$ is <u>not</u> a function.

In fact, this example probably holds the world's record for being a non-function. After all, given an input (the only choice being 2), there are an <u>infinite</u> number of outputs (y can be <u>anything</u>). We thus see that a vertical line is not a function.

x	y
2	5
2	0
2	–3

Homework

1. Consider the formula $y = x^3$. Recall that x is an input and y is an output. When a value of x has been assigned, there's only one y value. What does this mean?

2. Consider $y^2 = x$. When $x = 25$, y has two values, 5 and –5. What does this mean?

3. Consider $x = |y + 10|$.

   a. When $x = 20$, y has two values. Verify this fact.

   b. What does this mean?

4. Determine whether the formula is a function:

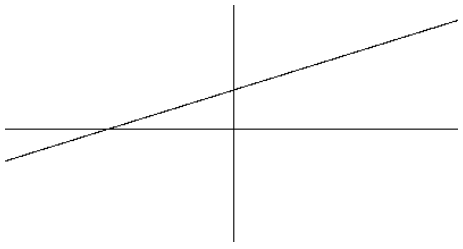
a. $y = -7x + 9$	b. $x = 2y + 5$	c. $y = x^5$
d. $y^2 = x$	e. $x^2 + y^2 = 14$	f. $y = \lvert 7x + 2 \rvert$
g. $x^2 - y^2 = 9$	h. $x = \pi$	i. $y = \sqrt{2}$
j. $y = \pm\sqrt{x-1}$	k. $y = \frac{x+1}{x-3}$	l. $y = \sqrt{2x^2 + x + 1}$
m. $x = \lvert y + 5 \rvert$	n. $x^2 + y^2 = 1$	o. $y = \sqrt{1-x}$
p. $y = x^3 - x^2 + x$	q. $x - 5 = 0$	r. $y + 2\pi = 0$
s. $3x - 7y = 8$	t. $y = \lvert 10 - x \rvert$	u. $x = \lvert 1 - y \rvert$

❑ *GRAPHS*

Now it's time to determine whether or not a given graph is a function. We will assume that the standard x- and y-axes are used, and as before, we agree that x is the input and y is the output. Recalling that a function must produce exactly one output for each legal input, we look at the following three graphs.

EXAMPLE 2:

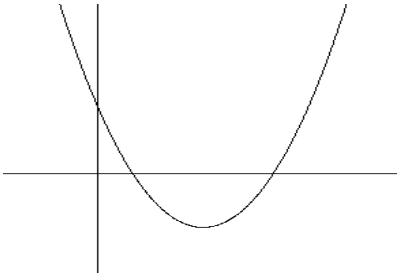
A.



Choose any x-value on the x-axis — this is the input. Now go straight up or down until you get to the graph to find the y-value — this is the output. How many outputs did you get? You should have gotten exactly one output. Whatever

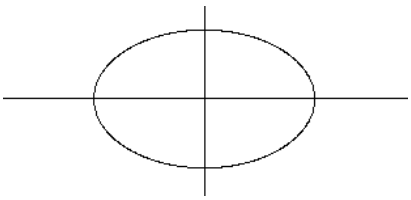
legal x you use, there's exactly one y for it. The graph <u>is</u> a function.

B.



Pick any input on the x-axis. Go up or down until you hit the graph — the y-value is the output. Every legal value of x produces exactly one output. This graph <u>is</u> also a function.

C.



Choose a legal input (say, $x = 0$). Now go find the graph. This time we run into the graph <u>twice</u>, once going up and once going down. Our x has two different y's. That is, we have a legal input with more than one output. This graph is definitely <u>not</u> a function.

The following chart indicates other examples of graphs and their status as functions:

FUNCTIONS	NON-FUNCTIONS
single point	vertical line
horizontal line	circle
top-half of a circle	ellipse (oval)
non-vertical line	right-half of a circle
parabola opening up	parabola opening to the left

Homework

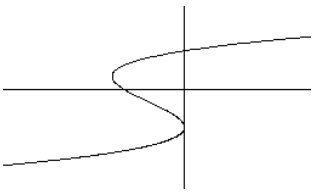
5. T/F: No line is a function.

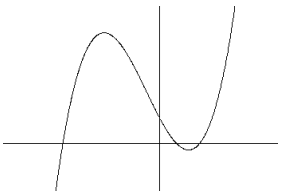
6. Which of the following are functions?

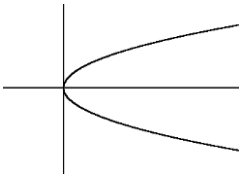
   a. horizontal line b. bottom half of a circle c. left half of a circle

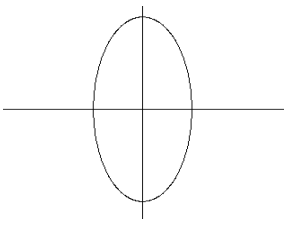
7. Explain why a graph consisting of a single point is a function.

8. Which of the following are functions?

a. 

b. 

c. 

d. 

Review Problems

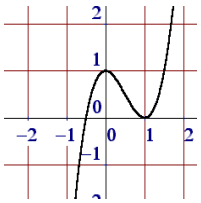
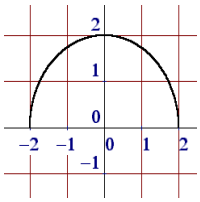
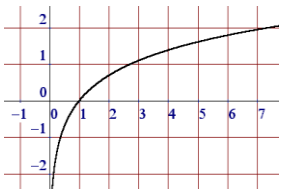
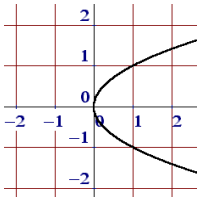
9. Consider the formula $x^2 + y^2 = 1$. Explain why it's not a function.

10. Consider the formula $y = \pm\sqrt{x}$. If $x = 9$, what is y? Is this a function?

11. Is it a function?

a. $x = \lvert y + 1 \rvert$	b. $y = x^3$	c. $y = \pm\sqrt{x-3}$
d. $x = 5$	e. $y = -\pi$	f. $x^2 = y^2$
g. $7x - 9y = 10$	h. $y = \lvert x + 5 \rvert$	i. $y = 7x^2 - \pi x^{10}$
j. $x^2 + y^2 = 49$	k. $x = 0$	l. $y = 0$

12. Which are functions?

13. True/False:

a. $x = y^4$ is a function.

b. $y = \lvert x - 2 \rvert$ is a function.

c. $y = \pm\sqrt{x-\pi}$ is a function.

d. $x = 9$ is a function.

e. $y = -\pi$ is a function.

f. $x^2 + y^2 = 2$ is a function.

g. Every line is a function.

h. Every semicircle is a function.

i. $x = \lvert y - 1 \rvert$ is a function.

Solutions

1. It means that the formula represents a function, since each input has a unique output.

2. It means that the formula does <u>not</u> represent a function, since there's an input with more than one output.

3. a. If $x = 20$, then $20 = |y+10|$. This implies that

$y + 10 = 20$ or $y + 10 = -20$, giving two solutions: $y = 10, -30$.

b. One input (20) produced two outputs (10 and –30). Therefore, the formula does <u>not</u> represent a function.

4. a. Yes, given an input (x), there's only one output (y).
b. Yes, solve for y and it's just like part a.
c. Yes, one input produces exactly one output.
d. No, an x-value of 4 produces two y-values, namely 2 and –2.
e. No, an x-value of 0 produces two y-values.
f. Yes
g. No, if $x = 10$, then y has two values.
h. No, for $x = \pi$, there are an infinite number of y's.
i. Yes, given any input, the output must be $\sqrt{2}$, so there's a unique output for each input.
j. No, if $x = 9$, then y has two values, 2 and –2.
k. Yes, put in any legal value of x, and only one y-value will result.
l. Yes, it may be complicated but only one y-value will appear for a given x-value.
m. No, if $x = 2$, then y can be either –3 or –7.
n. No, let $x = 0$ and see what you get.
o. Yes, as long as an appropriate x is chosen, the positive square root produces exactly one answer.
p. Yes q. No r. Yes s. Yes t. Yes u. No

5. F (actually, most lines are functions)

6. a. and b.

7. When you choose the only legal x, you're already at the y-value, and there's only one.

8. b. only

9. If $x = 0$, $y = \pm 1$; an input has produced two outputs, so it's not a function. Or, the graph is a circle, which is certainly not a function.

10. $y = \pm 3$; it is not a function, since a single input of 9 produced two outputs.

11. a. No b. Yes c. No d. No e. Yes f. No
g. Yes h. Yes i. Yes j. No k. No l. Yes

12. All but the 4th

13. a. F b. T c. F d. F e. T f. F
g. F h. F i. F

“Ninety-nine percent of the failures come from people who have the habit of making excuses.”

– George Washington

LIMITS

The title of this book is *From Logic to Limits*, and the first chapter is entitled **Logic**. So, obviously, this last chapter must be called **Limits**.

EXAMPLE 1:

Infinity may not be a number, but it's a pretty interesting concept. Although a variable x cannot actually be infinity, we can nevertheless ask ourselves: What happens to a function as x approaches infinity as a limit?



Consider the function $y = \frac{1}{x}$.

If $x = 10$, then $y = \frac{1}{10}$.

If $x = 500$, then $y = \frac{1}{500}$.

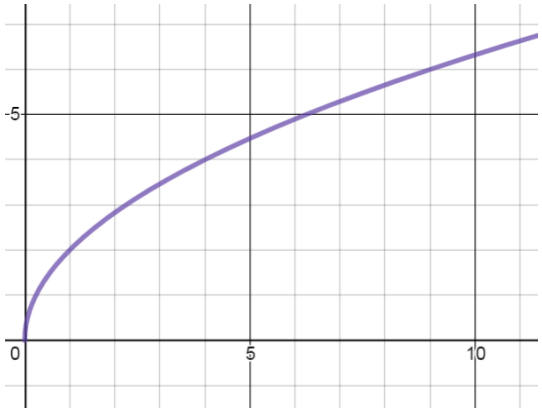
And if we let x be really big, say $x = 1{,}000{,}000$, then $y = \frac{1}{1{,}000{,}000}$.

It seems that as x grows larger and larger, the y-values are getting closer and closer to zero. In fact, we can say that x is approaching infinity, and this is causing y to approach zero. In other words,

As x approaches infinity, y approaches zero:

As $x \to \infty$, $y \to 0$

EXAMPLE 2: **Analyze the following limits for the graph:**



You might recognize this as being the graph of the function $y = \sqrt{x}$.

As $x \to \infty$, $y \to \infty$: Imagine x growing larger and larger. For example, if x = 100, then y = 10; if x = 10,000, then y = 100. It's clear that, as x grows larger and larger, so does y (though not nearly at the same speed as the x). On the graph, as we move farther and farther to the right, the graph keeps getting higher and higher. In fact, we can rise as high as we desire by choosing x appropriately large. This is why we say: As $x \to \infty$, $y \to \infty$.

As $x \to 0$, $y \to 0$: Note that, since $\sqrt{x}$ is defined only for $x \geq 0$, we'll make sure we use positive values of x that are getting smaller and smaller (approaching 0).

The table indicates that $\sqrt{x}$ approaches **0** also.

x	$\sqrt{x}$
1	1
0.5	0.707
0.25	0.5
0.01	0.1
0.001	0.032
0.0002	0.004

0 0

Note: It makes no difference at all what happens when $x = 0$. In a limit, we care only about what's happening as x gets closer and closer to the number it's approaching. In fact, we don't care whether or not x is even allowed to be 0.

As $x \to 9$, $y \to 3$: As x gets closer and closer to 9, y gets closer and closer to 3:

x	8	8.3	8.5	8.8	8.9	8.99	→ 9
$\sqrt{x}$	2.828	2.88	2.915	2.966	2.983	2.998	→ 3

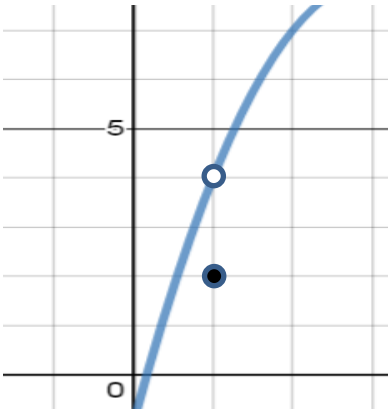
Final Note This last limit shows that as x approaches 9, y approaches 3. It is also the case that when $x = 9$, $y = 3$. Yet, this fact has NOTHING to do with the limit. Even if y were a hundred trillion when $x = 3$, the limit statement would still be correct.

"The educated differ from the uneducated
as much as the living from the dead."

□□□□□□□□□□(Aristotle)

Bottom line: Functional values are one thing, **a**nd limits are another. The following example might clarify this critical distinction. It also gives us limits which do not involve infinity.

EXAMPLE 3: **Let f be defined by the graph**



a. $f(1) =$ _____

b. As $x \to 1$, $y \to$ _____

Solution:

a. Looking at the solid dot part of the graph, we conclude that $f(1) = \mathbf{2}$.

b. As x approaches 1, the graph is heading toward a y-value of 4. The open circle at the point (1, 4) means that x never reaches 1 on the curve. Still, as x approaches 1, y approaches a value of **4**. That is, as $\boldsymbol{x \to 1, y \to 4}$

To summarize, at $x = 1$, the function value is 2. But as $x \to 1$, $y \to 4$. So we note again that calculating limits has nothing at all to do with what might happen if x actually becomes the value it's approaching. In fact, in the graph of this example, if we were to remove the solid dot, the answer to part a. would change to "Undefined," but the limit would remain exactly as it is.

❑ *DEFINITION AND NOTATION*

You should now be familiar with the following notation:
??????????????

$\lim_{x \to a^+} f(x) = L$	As x approaches a from the right, $f(x)$ is approaching L.
$\lim_{x \to a^-} f(x) = L$	As x approaches a from the left, $f(x)$ is approaching L.

Now for a new idea: a two-sided limit. We write

$$\lim_{x \to a} f(x) = L$$

when we want to indicate the following:

As x approaches a from either direction,

$f(x)$ is approaching L.

We can thus write:

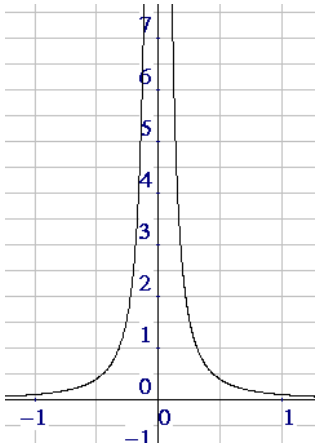
IF $\lim_{x \to a^+} f(x) = L$ AND $\lim_{x \to a^-} f(x) = L$

THEN $\lim_{x \to a} f(x) = L$.

In words, we say that if both of the one-sided limits exist and are equal, then the two-sided limit exists and its value is the same as the two one-sided limits.

❑ *LIMITS FROM GRAPHS*

EXAMPLE 1: **Calculate all the limits as $x \to 0$:**



Solution: We see from the right side of the graph that as $x \to 0^+$, $f(x) \to \infty$. That is,

$$\lim_{x \to 0^+} f(x) = \infty$$

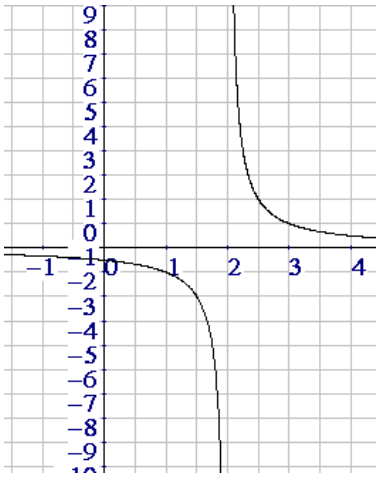
Also, as $x \to 0^-$, $f(x) \to \infty$, so we also have the fact that

$$\lim_{x \to 0^-} f(x) = \infty$$

By the definition of the two-sided limit, we conclude that

$$\boxed{\lim_{x \to 0} f(x) = \infty}$$

EXAMPLE 2: **Calculate all the limits as $x \to 2$:**



Solution: The part of the curve in Quadrant I tells us that

$$\lim_{x \to 2^+} f(x) = \infty,$$

whereas the part of the curve in Quadrant IV indicates that

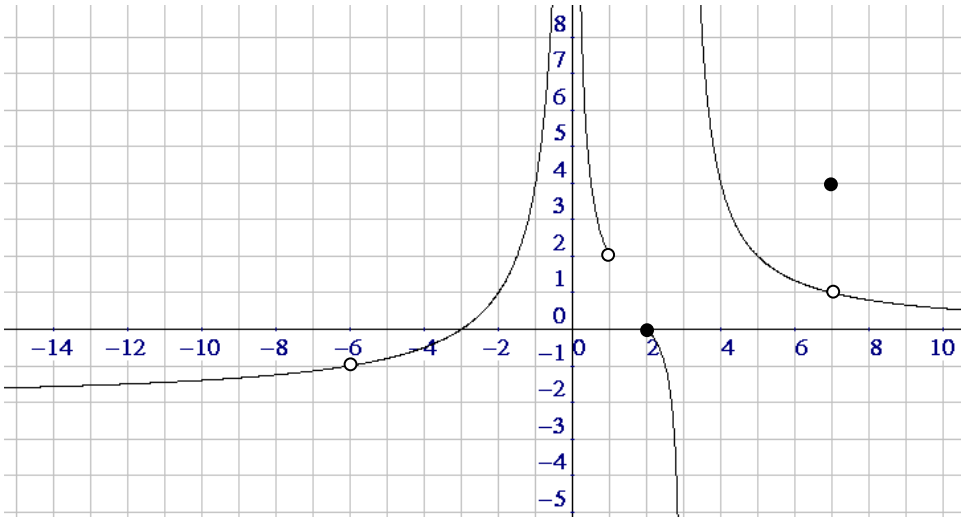
$$\lim_{x \to 2^-} f(x) = -\infty.$$

Thus, each one-sided limit exists, but since they are not equal, the two-sided limit

$$\boxed{\lim_{x \to 2} f(x) \text{ does not exist}}$$

Homework

Let *f* be defined by the following graph:



1. a. Explain why *f* is a function.
   b. Find the doman of *f*.
   c. Find the range of *f*.

2. a. $\lim_{x\to\infty} f(x)$ b. $\lim_{x\to-\infty} f(x)$

3. a. $f(-6)$ b. $\lim_{x\to-6^-} f(x)$ c. $\lim_{x\to-6^+} f(x)$ d. $\lim_{x\to-6} f(x)$

4. a. $f(0)$ b. $\lim_{x\to0^-} f(x)$ c. $\lim_{x\to0^+} f(x)$ d. $\lim_{x\to0} f(x)$

5. a. $f(1)$ b. $\lim_{x\to1^-} f(x)$ c. $\lim_{x\to1^+} f(x)$ d. $\lim_{x\to1} f(x)$

6. a. $f(2)$ b. $\lim_{x\to2^-} f(x)$ c. $\lim_{x\to2^+} f(x)$ d. $\lim_{x\to2} f(x)$

7. a. $f(3)$ b. $\lim_{x \to 3^-} f(x)$ c. $\lim_{x \to 3^+} f(x)$ d. $\lim_{x \to 3} f(x)$

8. a. $f(7)$ b. $\lim_{x \to 7^-} f(x)$ c. $\lim_{x \to 7^+} f(x)$ d. $\lim_{x \to 7} f(x)$

❑ *LIMITS FROM FORMULAS*

EXAMPLE 3: **Find the value of each of the following limits:**

A. $\lim_{x \to 7} 12 = \mathbf{12}$, since 12 is certainly infinitely close to 12 as x approaches 7 (in fact, it doesn't matter what x approaches).

B. $\lim_{x \to 2} 3x = \mathbf{6}$, because as x gets closer and closer to 2 (from either side), $3x$ gets closer and closer to 6.

C. $\lim_{x \to 0} 7x + 9 = \mathbf{9}$

D. $\lim_{x \to 3} x^2 - x + 5 = \mathbf{11}$

E. $\lim_{x \to 3} \frac{1}{x^2} = \mathbf{\frac{1}{9}}$

F. $\lim_{x \to 16} \sqrt{x} = \mathbf{4}$

G. $\lim_{x \to 0} \sqrt{x}$ *does not exist*

Here's why: Recall that "$x \to 0$" means that x approaches 0 from <u>both</u> sides — and if both the left and right-hand limits exist and are equal, then we can say that the limit exists. But in this case, the left-hand limit does not exist (it's

outside the domain of the square root function), and therefore the two-sided limit does not exist.

H. $\lim_{x \to 0} 2x^3 - x^2 + 8x - 9 = \mathbf{-9}$

EXAMPLE 4: **Find the value of each of the following limits:**

A. $\lim_{x \to 0} \dfrac{x^2 + x}{x}$

As x shrinks toward 0, the top approaches 0, and so does the bottom, leaving us with the fraction $\frac{0}{0}$, which is meaningless. But watch what some algebra can do:

$$\frac{x^2 + x}{x} = \frac{x(x+1)}{x} = \frac{\cancel{x}(x+1)}{\cancel{x}} = x + 1$$

Note: We can cancel out the x if we know for sure that it can never be zero. But this is clearly the case, since x is approaching zero, and therefore can never be zero.

So we can change the original problem to $\lim_{x \to 0} x + 1$, which is simply 1. Thus,

$$\boxed{\lim_{x \to 0} \frac{x^2 + x}{x} = 1}$$

B. $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

As x approaches 3, both the top and bottom of the fraction approach 0, resulting in the same meaningless fraction as the preceding example. But,

$$\frac{x^2-9}{x-3} = \frac{(x+3)(x-3)}{x-3} = \frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}} = x+3,$$

Therefore, the original problem is equal to $\lim_{x\to 3} x+3$, which has a value of 6. Thus,

$$\boxed{\lim_{x\to 3} \frac{x^2-9}{x-3} = 6}$$

C. $\lim_{x\to -4} \frac{x+4}{x^2-16}$

The top and bottom of the fraction are each approaching 0 as x approaches –4. However,

$$\frac{x+4}{x^2-16} = \frac{x+4}{(x+4)(x-4)} = \frac{\cancel{x+4}}{\cancel{(x+4)}(x-4)} = \frac{1}{x-4}, \text{ so}$$

$$\lim_{x\to -4} \frac{x+4}{x^2-16} = \lim_{x\to -4} \frac{1}{x-4} = \boxed{-\frac{1}{8}}$$

D. $\lim_{x\to 25} \frac{x-25}{\sqrt{x}-5}$

The top and bottom are each approaching 0 — this is undesirable — and in this problem, we have two ways to remedy the situation:

The 1st method for changing the fraction involves factoring:

$$\frac{x-25}{\sqrt{x}-5} = \frac{(\sqrt{x}+5)(\sqrt{x}-5)}{\sqrt{x}-5} = \frac{(\sqrt{x}+5)\cancel{(\sqrt{x}-5)}}{\cancel{\sqrt{x}-5}} = \sqrt{x}+5$$

The 2nd method for changing the fraction involves rationalizing the denominator:

$$\frac{x-25}{\sqrt{x}-5} = \frac{x-25}{\sqrt{x}-5}\left[\frac{\sqrt{x}+5}{\sqrt{x}+5}\right] = \frac{(x-25)(\sqrt{x}+5)}{x-25} = \sqrt{x}+5$$

Either way, $\lim_{x\to 25} \frac{x-25}{\sqrt{x}-5} = \lim_{x\to 25} \sqrt{x}+5 = 5+5 =$ $\boxed{10}$

E. $\lim_{x\to 7} \frac{x^2-49}{x+5}$

As x approaches 7, the top approaches 0, but the bottom approaches 12, giving us the fraction $\frac{0}{12}$, which is 0. There's no real problem here — no fancy algebra to perform. Thus,

$\lim_{x\to 7} \frac{x^2-49}{x+5} =$ $\boxed{0}$

Homework

9. $\lim_{x\to 99} 7$

10. $\lim_{x\to 3} 2x+10$

11. $\lim_{x\to -1} -x^2-x+7$

12. $\lim_{x\to 7} \frac{2}{x}$

13. $\lim_{x\to 0^+} \sqrt{x}$

14. $\lim_{x\to 0} \sqrt{x}$

15. $\lim_{x\to 24} \sqrt{x+1}$

16. $\lim_{x\to 0} \frac{x^2-x}{x}$

17. $\lim_{x\to 5} \frac{x^2-25}{x-5}$

18. $\lim_{x\to -3} \frac{x+3}{x^2-9}$

19. $\lim_{x\to 4} \frac{x+4}{x^2+16}$

20. $\lim_{x\to 100} \frac{x-100}{\sqrt{x}-10}$

Solutions

1. a. Given any x in the domain of the function, there exists <u>exactly one</u> y-value for it. In other words, it satisfies the Vertical Line Test: no vertical line cuts the graph in more than one place.

b. Domain = $(-\infty, -6) \sim (-6, 0) \sim (0, 1) \sim [2, 3) \sim (3, \infty)$

c. Range = $\oplus$, or in interval notation, $(-\infty, \infty)$.

2. a. 0 b. –2

3. a. Does not exist (DNE) b. –1 c. –1 d. –1

4. a. DNE b. ∞ c. ∞ d. ∞

5. a. DNE b. 2 c. DNE d. DNE

6. a. 0 b. DNE c. 0 d. DNE

7. a. DNE b. –∞ c. ∞ d. DNE

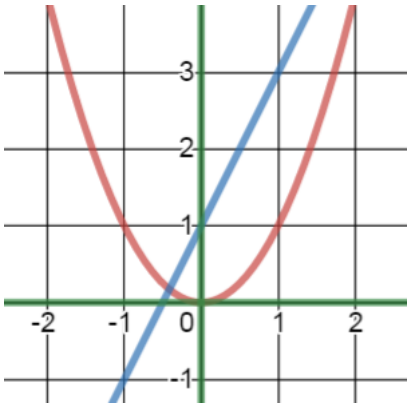
8. a. 4 b. 1 c. 1 d. 1

9. 7 **10**. 16 **11**. 7 **12**. $\frac{2}{7}$ **13**. 0 **14**. DNE

15. 5 **16**. –1 **17**. 10 **18**. $-\frac{1}{6}$ **19**. $\frac{1}{4}$ **20**. 20

FUNCTION NOTATION AND COMPOSITION

The notation "*y = whatever*" is a fine way to write a function, but it has a major flaw. Suppose we are talking about two functions in the same problem — for example, the line $y = 2x + 1$ and the parabola $y = x^2$. I now ask you what the y-value is when $x = 1$. Is the answer 3 (from the line), or is the answer 1 (from the parabola)? It depends on which formula you chose.



If $x = 1$, what does y equal? It depends on which graph you're looking at.

❑ *FUNCTION NOTATION*

Not knowing which function we're referring to is unacceptable — we'd never be able to communicate this way. This is why we use a new notation for functions: We give each function a different name, in order to distinguish one from the other. We can name our line and parabola functions something like

$$L(x) = 2x + 1 \quad \text{and} \quad P(x) = x^2$$

[The symbol $L(x)$ is read "L **of** x" or "L **at** x".]

The L and the P are the names of the functions, and the x's in parentheses just make it clear which variable is the input. So, for example, in the function L, the input is x, and the output is $L(x)$ (that is, $2x + 1$).

Now, with this new notation, if I specify an input of $x = 1$, it's fair to ask you what $L(1)$ is, and you'll know for sure that it's 3, because I gave you both the input (the 1) and the function L. And by the same token, if I asked what $g(1)$ is, you'd be confident that it's 1. Get it??

The following are some of the math functions you will find in programming languages and spreadsheets:

abs
sqrt
sin
log
exp
mod
round

EXAMPLE 1: **Given the three functions**

$$f(x) = 3x - 10$$
$$g(x) = x^2 + 1$$
$$h(x) = \frac{1}{x}$$

calculate each of the functional values:

A. $f(5) = 3(\mathbf{5}) - 10 = 15 - 10 = \mathbf{5}$

B. $g(5) = \mathbf{5}^2 + 1 = 25 + 1 = \mathbf{26}$

C. $h(5) = \frac{\mathbf{1}}{\mathbf{5}}$

D. $f(10) = 3(\mathbf{10}) - 10 = 30 - 10 = \mathbf{20}$

E. $g(\sqrt{7}) = (\sqrt{\mathbf{7}})^2 + 1 = 7 + 1 = \mathbf{8}$

F. $h(0) = \frac{1}{\mathbf{0}}$, which is **undefined**

Homework

1. Let $f(x) = x^2 + 3x$
   $g(x) = 5 - x$
   $h(x) = \sqrt[3]{x+1}$

Calculate:

a. $f(7)$	b. $g(7)$	c. $f(0)$	d. $g(0)$
e. $f(10)$	f. $g(10)$	g. $f(-5)$	h. $g(-5)$
i. $h(0)$	j. $h(-28)$	k. $h(-1)$	l. $f(-1)$

❑ *COMPUTER LANGUAGE FUNCTIONS*

Let's look at how a computer language handles functions by looking at three functions: two that are built into the language, `abs` and `sqrt`, and one that we'll define ourselves.

You may have guessed that **`abs`** stands for absolute value. For example, the `abs` function produces the outputs for the given inputs:

`abs`(7) = 7 `abs`(–12) = 12 `abs`(0) = 0

As for the square root function, **`sqrt`**, a computer would calculate like this:

`sqrt`(81) = 9 `sqrt`(0) = 0 `sqrt`(–4) = *Error*

Our third example will be a *user-defined* function; this is a function that does <u>not</u> come pre-built into the language, but one that we create ourselves.

Notice that the square root of a negative number does not exist as a real number — hence the *Error* in the `sqrt` calculations above. So how can a programmer absolutely guarantee that her program will never accidentally try to take the square root of a negative number? Here's one way: Take the number's absolute value first, then feed that result (which can't be negative) into the square root function:

`sqrt`(`abs`(–9)) would produce an output of **3**.

Do you see why? Starting with the input, –9, <u>first</u> its absolute value is calculated, which results in the number 9. Second, the positive square

root of the 9 is taken, with a final output of 3. Get it? What if we start with a positive number? No problem — for example,

`sqrt(abs(100))` would produce an output of **10**.

The absolute value of 100 is still 100, whose positive square root is 10. Whether the input is positive, zero, or negative, our new function will always be able to take a square root, with no Error message to worry about. And this new function will even work perfectly if the input is 0.

Just for the heck of it, let's reverse the order of the functions which comprise our new user-defined function, and see how it handles an input of –25:

`abs(sqrt(–25)) = abs(`***Error!***`)`

It appears that the order in which you carry out your functions might make everything fall apart, thus defeating the purpose of a user-defined function.

And we can even name our user-defined function (the one that works, not the one that starts with `abs`), and have it stored in the computer language, available anytime we need it. For instance, we might call the new function `RealSqrt`, meaning that our new function will always result in a real number and never produce an Error message — a good thing for a programmer to have. Depending on the computer language, it might be as simple to create as this:

```
Function RealSqrt(x)
    RealSqrt(x) = sqrt(abs(x))
End Function
```

Our new function, `RealSqrt`, is actually *composed* of two functions, `abs` and `sqrt`, performed one after the other. As such, we say that our new function is the ***composition*** of the two functions. So, if you wanted to use the function with an input of –100, you could write

`RealSqrt(–100)`, and the output would be `10`.

❑ COMPOSITION OF FUNCTIONS

Here's the mathematical approach to the composition of functions. A function takes an input and produces exactly one output. Isn't it possible that this output might itself be the input to some other function? For example, suppose we have the two functions

$$f(x) = x^2 \quad \text{and} \quad g(x) = x - 10$$

Our starting point will be to use the input 7 and place it into the function f. This yields an output of **49**. Now, declare the 49 to be the input to the function g. The output at this point is **39**. Skipping the middle step, the original input of 7 produced — by performing f and then g — an output of **39**.

Another way to describe this "composing" of functions is the following: The starting number of 7 was put into the "squaring machine," creating the number 49. Then the 49 was put into the "subtract 10 machine," giving the final answer of **39**.

Since the function f was done first, and the function g was done second, and the starting number was 7, we can consider "g of f of 7," written

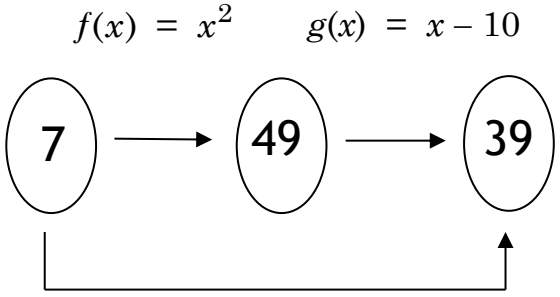
$$g\big(f(7)\big) = \mathbf{39}$$

Showing all the details, we can write

$$g\big(f(7)\big) = g\big(7^2\big) = g(49) = 49 - 10 = \mathbf{39}$$

Using the output of one function as the input to a second function is called "composing" the functions. The function that takes the 7 directly to the 39 is called the "composition" of the two functions. It's a single function which has the same effect as the two original functions, one followed by the other.

In summary,



f takes 7 to 49, and then g takes 49 to 39. We can write

$$g\big(f(7)\big) = 39$$

EXAMPLE 2: **If $f(x) = x^3 - x$ and $g(x) = 2x + 10$, calculate**

a. $f\big(g(-3)\big)$ **b.** $g\big(f(2)\big)$

Solution:

a. $f\big(g(-3)\big) = f\big(2(-3)+10\big) = f(4) = 4^3 - 4 = 64 - 4 = \boxed{60}$

b. $g\big(f(2)\big) = g\big(2^3 - 2\big) = g(6) = 2(6)+10 = \boxed{22}$

❑ *OFFICIAL NOTATION FOR THE COMPOSITION OF FUNCTIONS*

Calculating an expression of the form $f\big(g(x)\big)$ is important enough to warrant its own notation. We write

$$\boxed{(f \circ g)(x) = f\big(g(x)\big)}$$

and call $f \circ g$ the ***composition*** (or composite) of f and g. That is, given two functions f and g, we can create a third function called the composition of f and g, denoted by $f \circ g$, defined by the above equation. We can read $f\big(g(x)\big)$ as "f of g of x." Notice that the function g is done

first, followed by f. Thus, function composition is done from right to left.

EXAMPLE 3: **If $f(x) = x^3 - x$ and $g(x) = 2x + 10$, find $(g \circ f)(x)$.**

Solution: By the definition of the composition of two functions,

$$(g \circ f)(x) = g\big(f(x)\big) = g(x^3 - x) = 2(x^3 - x) + 10 =$$

$$\boxed{2x^3 - 2x + 10}$$

Function Composition is NOT Commutative

Remember the commutative property for multiplication? It says that if a and b are real numbers, then their products are equal: $ab = ba$. We'll now demonstrate that the composition of functions is a non-commutative operation. In other words, the order of the functions may make a difference in the final answer.

Let $f(x) = x^2$ and $g(x) = x + 1$. Let $x = 10$; we'll calculate

$$(f \circ g)(10) = f\big(g(10)\big) = f(10 + 1) = f(11) = 11^2 = \mathbf{121},$$

but

$$(g \circ f)(10) = g\big(f(10)\big) = g(10^2) = g(100) = 100 + 1 = \mathbf{101}.$$

Homework

2. Let $u(x) = x^2$ and $w(x) = 3x - 4$. Calculate:
   a. $u(w(3))$ b. $w(u(3))$

3. Let $f(x) = \sqrt{x}$ and $h(x) = 3x$.
   a. Calculate $(f \circ h)(1)$
   b. Explain why $(f \circ h)(-3)$ does not exist (as a real number).

4. Let $f(x) = \sqrt{x}$ and $g(x) = x + 10$.
   a. Find $(f \circ g)(-1)$ b. $(f \circ g)(-10)$ c. $(f \circ g)(-11)$

5. If $g(x) = x^2 + 1$ and $f(x) = 5x - 9$, calculate formulas for $(f \circ g)(x)$ and $(g \circ f)(x)$.

6. If $f(x) = 3x$ and $g(x) = \sqrt{x+1}$, calculate formulas for $(f \circ g)(x)$ and $(g \circ f)(x)$.

7. If $g(x) = (x + 1)^2$ and $h(x) = \sqrt{x}$, calculate formulas for $(g \circ h)(x)$ and $(h \circ g)(x)$.

8. If $f(x) = 7x - 5$, calculate a formula for $(f \circ f)(x)$.

9. If $h(x) = x^2 + x - 1$, calculate a formula for $(h \circ h)(x)$.

10. Let $f(x) = \sqrt{x+4}$. Find a formula for $(f \circ f \circ f)(n)$.

    Now calculate $(f \circ f \circ f)(437)$.

❑ *To ∞ and Beyond*

A. Consider the following five functions:

$f(x) = x^2$ $\quad g(x) = x + 6$ $\quad h(x) = \sqrt{x+17}$

$j = \left\{\left(\pi, \sqrt{2}\right), \left(\sqrt{2}, \pi\right)\right\}$ $\quad k = \{ (a, \pi), (b, e) \}$

Calculate: $(h \circ g \circ f \circ j \circ k)(a)$

B. Solve for *n*: $\sqrt{\sqrt{\sqrt{n+4}+4}+4} = 3$

C. Solve for *n*: $\sqrt{\sqrt{\sqrt{n+4}+4}+4} = w$

Solutions

1. a. $f(7) = 7^2 + 3(7) = 49 + 21 = 70$ b. $g(7) = 5 - 7 = -2$

c. 0 d. 5 e. 130 f. –5 g. 10

h. 10 i. 1 j. –3 k. 0 l. –2

2. a. $u(w(3)) = u(5) = 25$ b. 23

3. a. $f(h(1)) = f(3) = \sqrt{3}$ b. It would be $\sqrt{-9}$, not in $\mathbb{R}$

4. a. 3 b. 0 c. $\sqrt{-1}$, not in $\mathbb{R}$

5. $(f \circ g)(x) = f(g(x)) = f(x^2+1) = 5(x^2+1) - 9 = 5x^2 - 4$

$(g \circ f)(x) = g(f(x)) = (5x-9)^2+1 = 25x^2-90x+82$

6. $(f \circ g)(x) = 3\sqrt{x+1}$ $\qquad$ $(g \circ f)(x) = \sqrt{3x+1}$

7. $(g \circ h)(x) = (\sqrt{x}+1)^2 = x+2\sqrt{x}+1$

$(h \circ g)(x) = \sqrt{(x+1)^2} = |x+1|$

8. $(f \circ f)(x) = f(f(x)) = f(7x-5) = 7(7x-5)-5 = 49x-40$

9. $x^4 + 2x^3 - x - 1$

10. $\sqrt{\sqrt{\sqrt{n+4}+4}+4}$

Evaluating the function at n = 437 gives a result of 3.

"One child, one teacher, one book, one pen can change the world."



- Malala Yousafzai

PIECEWISE (BRANCH) FUNCTIONS



A shipping company might offer a discount if the shipping weight exceeds a certain minimum. For example, the shipping fee might be \$3/lb for a shipment weighing less than 20 lbs, and \$2/lb for a weight of 20 lb or more.

Our federal income tax system uses a progressive tax; higher-income earners pay a higher *percentage* of their taxable income than lower-income earners. For instance (and this is an oversimplification), those who earn under \$50,000/yr might pay at a rate of 12% of their income, those who earn between \$50,000 and \$90,000 pay 15%, and those who earn more than \$90,000 pay 20%.



EXAMPLE 1: **For the shipping company in the Introduction,**

A. Find the cost to ship 12 lbs.

B. Find the cost to ship 35 lbs.

C. Express the shipping cost, *S*, as a branch function of the shipping weight, *w*.

Solution:

A. Since $w = 12$, which is less than 20, it costs \$3/lb, for a total of 12 lbs × \$3/lb = **\$36**.

B. 35 lbs at \$2/lb (because 35 lbs > 20 lbs) comes to **\$70**.

C. In English, we could say:

If $w < 20$, then $S = 3w$.

If $w \geq 20$, then $S = 2w$.

In our new notation, this is written

$$S = \begin{cases} 3w & \text{if } w < 20 \\ 2w & \text{if } w \geq 20 \end{cases}$$

EXAMPLE 2: **Find the piecewise (branch) function for the income tax in the Introduction, and then use the function to find the tax on an income of \$88,000.**

Solution: Here's the branch function, where I = income and T = tax:

$$T = \begin{cases} 12\% \times I & \text{if } I < 50{,}000 \\ 15\% \times I & \text{if } 50{,}000 \leq I \leq 90{,}000 \\ 20\% \times I & \text{if } I > 90{,}000 \end{cases}$$

To find the tax on an income of \$88,000, we find which branch of the tax function the number 88,000 lies in. It lies in the middle branch, so we use the formula 15% × I, giving a tax of

$$T = 15\% \times \mathrm{I} = 0.15(88{,}000) = \textbf{\$13,200}$$

EXAMPLE 3: Let's look more closely at the absolute value function:

$$f(x) = |x|$$

We know, for example, that $f(9) = 9$, $f(-17) = 17$, and $f(0) = 0$. Here's one way we can define **absolute value,** using the **if/then** form:

If $x \geq 0$, then $|x| = x$

If $x < 0$, then $|x| = -x$

In programming, it might be written:

```
If x >= 0 Then
    Abs(x) = x
Else
    Abs(x) = -x
Endif
```

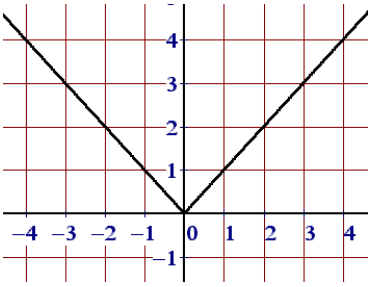
As a branch function, we write

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

In summary, we call a function described in the three examples above a ***branch function*** (or piecewise function), because the formula we use to calculate the output depends upon which branch of the domain the input lies in.

For example, in the *absolute value* branch function, suppose $x = 9$. Since $9 \geq 0$, we use the formula in the top branch; so $|9| = 9$. If we take x to be 0, we are again in the top branch, and thus $|0| = 0$. But if we take $x = -17$, then x falls in the bottom branch, whose formula is $-x$. So $|-17| = -(-17) = 17$, just as it should be.

Note: The definition of absolute value we've just written tells us that the domain of f is $\mathbb{R}$, all real numbers. Here's how you can tell: Choose <u>any</u> real number, and then see that it <u>must</u> fall into exactly one of the two "if's"; that is, the real number you chose is either **≥ 0** or it's **< 0**. All the bases are covered, so the absolute value function is defined for all real numbers.



<u>EXAMPLE 4:</u> **Consider the branch function defined by**

$$f(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ 2x & \text{if } x > 3 \end{cases}$$

I. Calculate: **a.** $f(-8)$ **b.** $f(0)$ **c.** $f(3)$

d. $f(200)$ **e.** $f\left(\frac{\pi}{2}\right)$

II. What is the domain of f?

III. Graph f.

IV. Analyze all the interesting limits.

Solution: To calculate function values, we first need to determine whether the input we're looking at falls in the first branch ($x \le 0$), in the second branch ($x > 3$), or possibly in no branch at all.

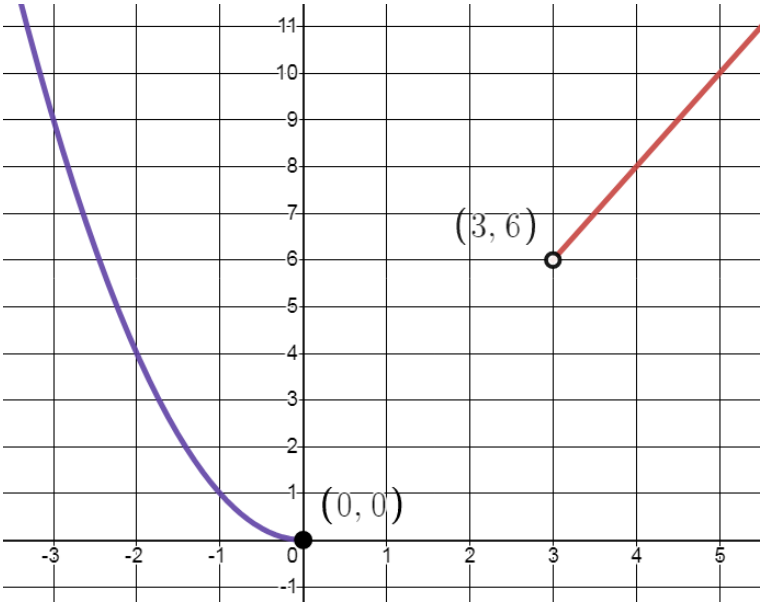
I. a. Since $-8 \le 0$, $f(8) = 8^2 = \mathbf{64}$.

b. Also, $0 \le 0$, so $f(0) = 0^2 = \mathbf{0}$.

c. The value $x = 3$ is not in either branch. Thus, $f(3)$ is **undefined**.

d. Because $200 > 3$, $f(200) = 2(200) = \mathbf{400}$.

e. Note that $\frac{\pi}{2}$ lies between 0 and 3 — it falls through the cracks of the domain. Therefore, $f\left(\frac{\pi}{2}\right)$ is **undefined**.



II. The domain

III. For $x \le 0$, th

Note that there's a solid dot at the origin, because f is a parabola for any $x \le 0$. In other

words, 0 is in the domain; thus the point (0, 0) is on the graph. On the other hand, the point (3, 6) is not on the graph, since 3 is not in the domain — hence the open dot.

IV. Now for some limits:

As $x \to 3$ (from the right), $f(x) \to 6$,

but x cannot approach 3 from the left.

As $x \to 0$ (from the left), $f(x) \to 0$,

yet x cannot approach 0 from the right.

As $x \to \infty$, $f(x) \to \infty$ (up the straight line).

As $x \to -\infty$, $f(x) \to \infty$ (up the parabola).

Final Comment: To say that $x \le 0$ means that $x \in (-\infty, 0]$. Also, if $x > 3$, then $x \in (3, \infty)$. Thus, an equivalent form of the original branch function f in this example is

$$f(x) = \begin{cases} x^2 & \text{if } x \in (-\infty, 0] \\ 2x & \text{if } x \in (3, \infty) \end{cases}$$

The notation $x \in A$ means that x is an **element** of A.

Also, while I use the word "if," some books use "for."

Homework

1. Let h be the function defined by $h(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0 \end{cases}$

   A. What is the domain of h?

   B. Find $h(13)$, $h(0)$, and $h(-\pi)$.

   C. Graph h.

2. Consider the branch function defined by $f(x) = \begin{cases} x^2 & \text{for } x \le 0 \\ 3x & \text{for } x > 2 \end{cases}$

   A. Calculate: a. $f(-8)$ b. $f(0)$ c. $f(2)$

d. $f(200)$ e. $f(\pi/3)$

B. What is the domain of f?

C. Graph f.

3. Let g be the piecewise function defined by

$$g(x) = \begin{cases} x^3 & \text{if } x \in (-\infty, -2) \\ -x^2 & \text{if } x \in [-1, 1] \\ 2 & \text{if } x \in (1, \infty) \end{cases}$$

A. Calculate: a. $g(1/2)$ b. $g(99)$ c. $g(-4)$ d. $g(-1.5)$
e. $g(-5)$ f. $g(-2)$ g. $g(-1)$ h. $g(1)$

B. What is the domain of g?

C. Graph g.

Check out:

https://www.mathsisfun.com/sets/functions-piecewise.html

❑ *TO ∞ AND BEYOND*

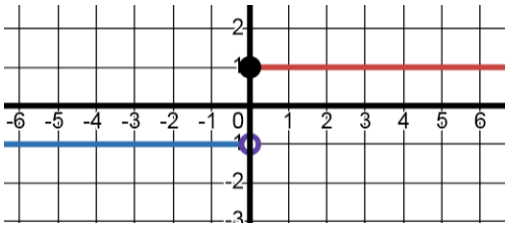
Define the function f by $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \pi & \text{if } x \text{ is rational} \end{cases}$

Find $\left(f \circ f\right)\left(\frac{2}{3}\right)$.

Solutions

1. A. The domain is $\mathbb{R}$, all real numbers, since every real number falls in exactly one of the two branches of the domain.

B. 1; 1; –1

C. 

2. A. a. 64 b. 0 c. Undefined d. 600 e. Undefined

B. $(-\infty, 0] \cup (2, \infty)$

C.



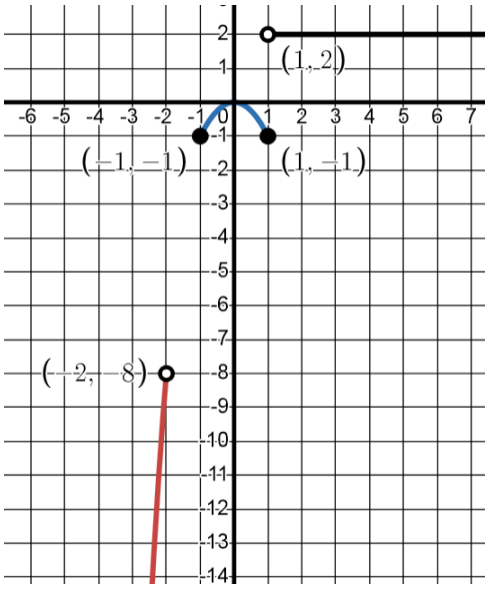
3. A. a. $-\frac{1}{4}$ b. 2 c. –64 d. Undefined
e. –125 f. Undefined g. –1 h. –1

B. It's just the union of the three intervals given in the function:

$$(-\infty, -2) \cup [-1, 1] \cup (1, \infty)$$

But note that the last two intervals can be combined into $[-1, \infty)$.

C.



FUNCTIONS – REWRITING FORMULAS

The formula for the area of a square is $A = s^2$, where s is the length of a side. Since A is based on s, we can say that A is a function of s, insofar as A depends on s. We can even use the terminology: s is the input and A is the output.



The volume of a cylinder is given by $V = \pi r^2 h$, where r is the radius and h is the height. Here we say that V is a function of both r and h.

To express X as a function of b means to write an equation where X is on the left side of the equation, and the right side consists of some formula containing only the variable b (and perhaps some constants).

EXAMPLE 1: **Express the side of a square as a function of its area.**

Solution: The formula $A = s^2$ expresses A as a function of s. To express the side s as a function of the area A, all we need to do is solve the formula for s:

$$\boxed{s = \sqrt{A}}$$

Now s is a function of A.

NOTES: First, this may seem like a trivial problem, but it's crucial when using computers. If you want to find the side of a square given the area, then you must do the algebra to solve for s — the computer cannot do it for you.

The second note is that technically, solving $A = s^2$ for s should result in $s = \pm\sqrt{A}$. But the side of a square can never be negative, so in this context it's perfectly fine to drop the "±" sign and consider only the non-negative square root of A.

EXAMPLE 2: **Express the area of a circle as a function of its circumference.**

Solution: Normally, we think of the area of a circle as a function of its radius, namely $A = \pi r^2$, and we also consider the circumference as a function of the radius, as in $C = 2\pi r$.

To express the area, A, as a function of the circumference, C, we need to write a formula with A on the left side and some formula with C (but not r) on the right side. To accomplish this, we will remove the r from the formula $A = \pi r^2$ and insert the C from the formula $C = 2\pi r$. It's a substitution trick.

$$C = 2\pi r \qquad \text{(usual circumference formula)}$$

$$\Rightarrow \quad r = \frac{C}{2\pi} \qquad \text{(divide each side by } 2\pi\text{)}$$

Now substitute this version of r into the area formula:

$$A = \pi r^2 \qquad \text{(usual area formula)}$$

$$\Rightarrow \quad A = \pi\left(\frac{C}{2\pi}\right)^2 \qquad \text{(since } r = \frac{C}{2\pi} \text{ from above)}$$

$$\Rightarrow \quad A = \pi\left(\frac{C^2}{4\pi^2}\right) \qquad \text{(square the fraction)}$$

$$\Rightarrow \quad \boxed{A = \frac{C^2}{4\pi}} \qquad \text{(cross-cancel)}$$

EXAMPLE 3: **Express the surface area of a sphere as a function of its diameter.**

Solution: The standard formula for the surface area of a sphere is $S = 4\pi r^2$, a formula in which the surface area S is written as a function of the radius r. To express the area as a function of the diameter, we need to get rid of the variable r and replace it with d.

From the formula $d = 2r$, we solve for r: $r = \frac{d}{2}$. Thus,

$$S = \underbrace{4\pi r^2 = 4\pi\left(\frac{d}{2}\right)^2}_{\text{since } r = \frac{d}{2}} = 4\pi\left(\frac{d^2}{4}\right) = \pi d^2$$

Hence, $\boxed{S = \pi d^2}$

To help you absorb the concept of "something as a *function* of something else," let's summarize the formulas we've just seen.

$A = s^2$	A is a function of s
$V = \pi r^2 h$	V is a function of r and h
$s = \sqrt{A}$	s is a function of A
$r = \frac{C}{2\pi}$	r is a function of C
$A = \pi r^2$	A is a function of r
$A = \frac{C^2}{4\pi}$	A is a function of C
$S = 4\pi r^2$	S is a function of r
$d = 2r$	d is a function of r
$r = \frac{d}{2}$	r is a function of d
$S = \pi d^2$	S is a function of d

Homework

1. Express the area of a square as a function of its side.
2. Express the area of a square as a function of its perimeter.
3. Express the perimeter of a square as a function of its area.
4. Express the leg of a right triangle as a function of its other leg and hypotenuse.
5. Express the length of a rectangle as a function of its width and area.
6. Express the circumference of a circle as a function of its diameter.
7. Express the diameter of a circle as a function of its circumference.
8. Express the area of a circle as a function of its diameter.
9. Express the circumference of a circle as a function of its area.
10. Express the volume of a cylinder as a function of its height and radius.
11. Express the radius of a cylinder as a function of its volume and height.
12. Express the radius of a sphere as a function of its surface area.
13. Express the volume of a sphere as a function of its diameter.
($V = \frac{4}{3}\pi r^3$)



14. Express the radius of a sphere as a function of its volume.
15. Express the volume of a sphere as a function of its surface area.
16. Express the surface area of a sphere as a function of its volume.

Review Problems

17. a. Express the volume of a cube as a function of its surface area.

    b. Express the surface area of a cube as a function of its volume.

18. In the formula for the mean, $\bar{x} = \frac{x_1 + x_2}{2}$, express x_1 as a function of x_2 and $\bar{x}$.

19. True/False:

    a. $A = \pi r^2$ gives the radius of a circle as a function of its area.

    b. $S = \pi d^2$ gives the surface area of a sphere as a function of its diameter.

    c. If $G = \sqrt{2}\,h^5$, it's possible to write h as a function of G.

Solutions

1. $A = s^2$

2. $A = s^2$; but $P = 4s \Rightarrow s = \frac{P}{4}$; substituting into the area formula:

$A = \left(\frac{P}{4}\right)^2 \Rightarrow A = \frac{P^2}{16}$.

3. $P = 4s$; but $A = s^2 \Rightarrow s = \sqrt{A}$; substituting gives $P = 4\sqrt{A}$.

4. Since, $(\text{leg1})^2 + (\text{leg2})^2 = h^2$, it follows that $\text{leg1} = \sqrt{h^2 - (\text{leg2})^2}$.

5. $l = \frac{A}{w}$

6. $C = 2\pi r;\ d = 2r \Rightarrow r = \frac{d}{2}$. Therefore, $C = 2\pi\left(\frac{d}{2}\right) \Rightarrow C = \pi d$.

7. $d = \frac{C}{\pi}$

8. $A = \pi r^2;\ r = \frac{d}{2}$. Therefore, $A = \pi\left(\frac{d}{2}\right)^2 \Rightarrow A = \frac{\pi d^2}{4}$.

9. $C = 2\pi r;\ A = \pi r^2 \Rightarrow r = \sqrt{\frac{A}{\pi}}$. Thus, $C = 2\pi\sqrt{\frac{A}{\pi}}$.

10. $V = \pi r^2 h$ **11**. $r = \sqrt{\frac{V}{\pi h}}$ **12**. $r = \sqrt{\frac{S}{4\pi}}$

13. $V = \frac{4}{3}\pi r^3;\ r = \frac{d}{2}$. Therefore, $V = \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 \Rightarrow V = \frac{\pi d^3}{6}$.

14. $r = \sqrt[3]{\frac{3V}{4\pi}}$ **15**. $V = \frac{4}{3}\pi\left(\frac{S}{4\pi}\right)^{3/2}$ **16**. $S = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$

17. a. $V = \left(\frac{A}{6}\right)^{3/2}$ b. $A = 6V^{2/3}$

18. Solving the mean formula for x_1 gives $x_1 = 2\bar{x} - x_2$.

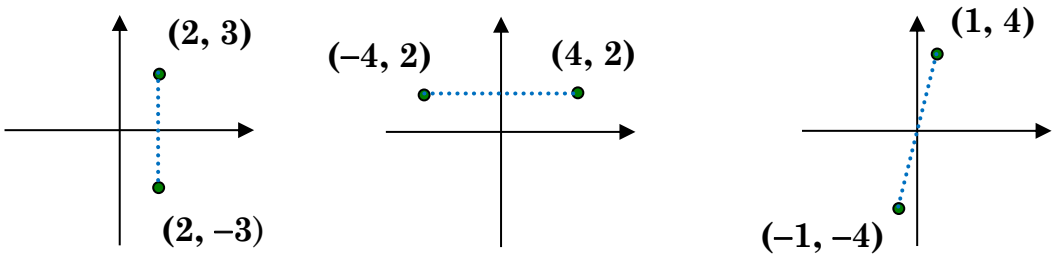
19. a. F b. T c. T

“On matters of style, swim with the current; on matters of principle, stand like a rock.”

Thomas Jefferson

SYMMETRY – EVEN AND ODD FUNCTIONS

❑ *SYMMETRY OF A PAIR OF POINTS*



***x*-axis** symmetry ***y*-axis** symmetry **origin** symmetry

In the first graph, notice that (2, 3) is directly above (2, –3), and that the two points are equidistant (equally distant) from the *x*-axis; that is, the dotted line is bisected by the *x*-axis. It's as though the bottom point is a reflection in the water of the top point. Also, note that the *x*-coordinates of the two points are the same and their *y*-coordinates are opposites. The two points are ***symmetric with respect to the x-axis***.

In the second graph, we see that (4, 2) is directly to the right of (–4, 2), and that the two points are equidistant from the *y*-axis; that is, the dotted line is bisected by the *y*-axis. It's like each point is a mirror image of the other. The two points are ***symmetric with respect to the y-axis.*** Also, can you see that the *x*-coordinates of the two points are opposites, and their *y*-coordinates are the same?

In the third graph, note that the two points are equidistant from the origin, and the dotted line is bisected by the origin. The two points are ***symmetric with respect to the origin***. Lastly, it should be clear that the *x*-coordinates of the two points are opposites, and their *y*-coordinates are also opposites.

Homework

1. For each of the following points, find the point that has ***x*-axis** symmetry with it:

   a. $(2, 7)$ b. $(-3, 4)$ c. $(5, \pi)$ d. $(4, 0)$ e. $(0, -3)$

2. For each of the following points, find the point that has ***y*-axis** symmetry with it:

   a. $(-2, 5)$ b. $(-1, -2)$ c. $(7, 0)$ d. $(\pi, -\sqrt{2})$ e. $(0, -2)$

3. For each of the following points, find the point that has **origin** symmetry with it:

   a. $(-3, 5)$ b. $(8, -7)$ c. $(0, 0)$ d. $(6, 0)$ e. $(0, -\pi)$

4. The line segment connecting the points P and Q is bisected by the y-axis. Does this imply that P and Q must be symmetric?

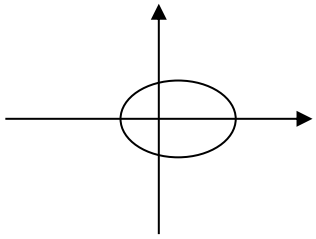
5. Let P be a point in the first quadrant and let Q be a point such that P and Q have x-axis symmetry. Then Q must lie in which quadrant?

6. Let (x, y) be a point in the third quadrant. Name the point such that the pair of points has origin symmetry.

❑ *SYMMETRY OF A GRAPH*

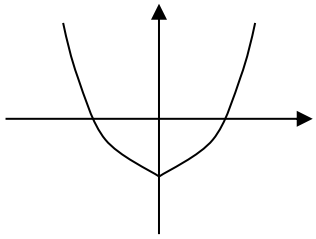
In the previous section, we discussed what it meant for a pair of points to have some kind of symmetry. Now we extend this concept for any graph in the plane.

x-axis symmetry



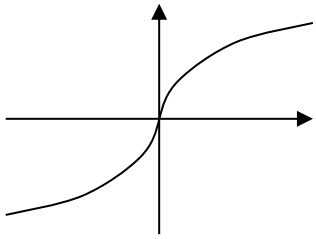
Given any point on the graph, there's an equidistant point on the other side of the x-axis.

y-axis symmetry



Given any point on the graph, there's an equidistant point on the other side of the y-axis.

origin symmetry



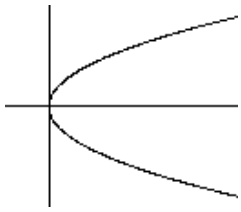
Given any point on the graph, there's an equidistant point on the other side of the origin.

The Litmus Tests for Symmetry

Now that we can visualize the three types of symmetry, we need a way to test a given equation for symmetry without having to graph it.

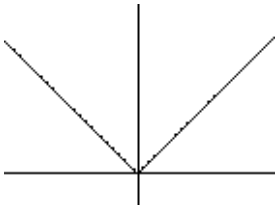
x-axis Replace y with $-y$ everywhere in the given equation. If the new equation is equivalent to the old one, then the graph of the equation has ***x*-axis symmetry**.

For example, consider the sideways-opening parabola $x = y^2$. Replacing y with $-y$ gives the equation $x = (-y)^2$, which is the same as $x = y^2$. Since the replacement didn't really change the equation, we have x-axis symmetry.



y-axis Replace x with $-x$ everywhere in the given equation. If the new equation is equivalent to the old one, then the graph of the equation has ***y*-axis symmetry**.

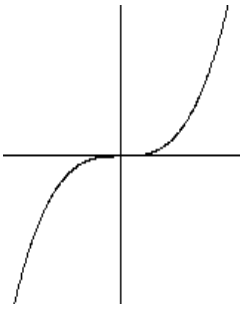
For instance, look at the absolute value function $y = |x|$. Replacing x with $-x$ gives the equation $y = |-x|$, which is actually the same



as $y = |x|$ (since the absolute values of a number and its opposite are the same). Thus, the graph has y-axis symmetry.

origin Replace $\boldsymbol{x}$ with $\boldsymbol{-x}$ and replace $\boldsymbol{y}$ with $\boldsymbol{-y}$ everywhere in the equation. If the new equation is equivalent to the old one, then the graph of the equation has **origin symmetry**.

Consider the cubic function $y = x^3$. Replacing x with $-x$ and y with $-y$ gives the equation $-y = (-x)^3$, which implies that $-y = -x^3$, which implies that $y = x^3$, so there was no net change in the equation. Therefore, the graph has origin symmetry.



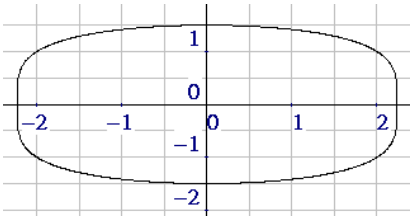
EXAMPLE 1: **Find all the symmetries of $x^2 + y^4 = 5$.**

Solution:

x-axis Replace y with $-y$: $x^2 + (-y)^4 = 5 \Rightarrow$ $x^2 + y^4 = 5$, the same as the original equation; therefore the graph has x-axis symmetry.

y-axis Replacing x with $-x$ will again result in the same equation; thus the graph has y-axis symmetry.

origin Replace both variables with their negatives: $(-x)^2 + (-y)^4 = 5 \Rightarrow x^2 + y^4 = 5$, the original equation; we conclude that the graph has origin symmetry as well.



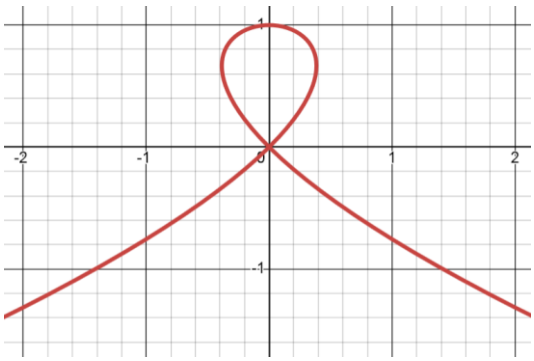
EXAMPLE 2: **Find all the symmetries of $y^2 = x^2 + y^3$.**

Solution:

x-axis Replace y with $-y$: $(-y)^2 = x^2 + (-y)^3$
$\Rightarrow y^2 = x^2 - y^3$, which is not the same as the original equation, nor can it be made to be the same. Thus, the graph does not have x-axis symmetry.

y-axis Replace x with $-x$: $y^2 = (-x)^2 + y^3$
$\Rightarrow y^2 = x^2 + y^3$, which is the same as the original equation. Therefore, the graph has y-axis symmetry.

origin Replace x with $-x$ and y with $-y$: $(-y)^2 = (-x)^2 + (-y)^3$
$\Rightarrow y^2 = x^2 - y^3$, which is not the same as the original equation — and will never be — no matter what we do to each side of the equation. We conclude that the graph does not have origin symmetry.



In summary, the graph has y-axis symmetry only.

Homework

7. Sketch a line with x-axis symmetry, a line with y-axis symmetry, and a line with origin symmetry.

8. a. Sketch a parabola with y-axis symmetry.
   b. Sketch a parabola with x-axis symmetry.

9. Sketch a circle that has x-axis symmetry, but neither of the other two.

10. Sketch a circle that has y-axis symmetry, but neither of the other two.

11. Sketch a circle that has origin symmetry, but neither of the other two.

12. A graph with origin symmetry contains some points in the 2nd quadrant. Which other quadrant must contain some points of the graph?

13. Use the litmus tests to find the symmetries of each graph:

a. $x^2 - y^2 = 1$ b. $y = x^2 + x - 1$

c. $y = x^2 - 8$ d. $x = |y|$

e. $y = \frac{1}{x}$ f. $y = x^6 - x^4 + x^2$

14. If a graph has x-axis symmetry and y-axis symmetry, does that imply that it has origin symmetry? What about the converse? [See Chapter 1, *Logic*.]

❑ *EVEN AND ODD FUNCTIONS*

We end this chapter by categorizing functions based upon their symmetries.

If a function has y-axis symmetry, we call it an **even function**.

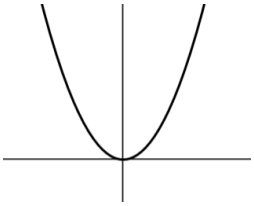
If a function has origin symmetry, we call it an **odd function**.

That's it — just two new terms that refer to the symmetries we already know.

Where do these two new terms come from?

The classic function with ***y*-axis** symmetry is our friend the parabola:

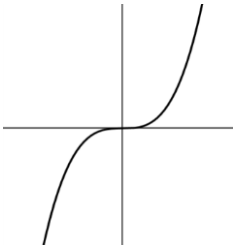
$y = x^2$



Look at the exponent of the function; it's an <u>even</u> number.

The classic function with **origin** symmetry is the cubic function:

$y = x^3$



Look at the exponent of the function; it's an <u>odd</u> number.

<u>Final Comment:</u> Note that if a graph has x-axis symmetry, it's NOT a function at all, so it certainly can't be an even or odd function.

Solutions

1. a. (2, –7) b. (–3, –4) c. (5, –π) d. (4, 0) e. (0, 3)

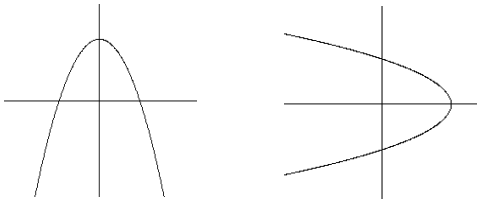
2. a. (2, 5) b. (1, –2) c. (–7, 0) d. $(-\pi, -\sqrt{2})$ e. (0, –2)

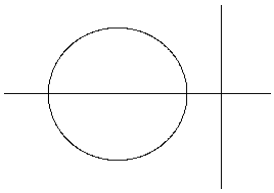
3. a. (3, –5) b. (–8, 7) c. (0, 0) d. (–6, 0) e. (0, π)

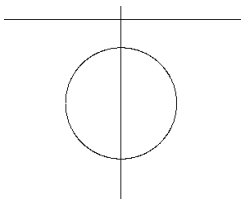
4. No **5**. IV **6**. $(-x, -y)$

7. any vertical line; any horizontal line; any line through the origin

8.



9. 

10. 

11. Impossible **12**. Quadrant IV

13. a. all b. none c. y-axis d. x-axis e. origin f. y-axis

14. Yes; but the converse is false — consider $y = 2x$. It has origin symmetry but neither of the other two.

"Education is simply the soul of a society as it passes from one generation to another."

– **Gilbert Chesterton**

FUNCTIONS – TABLES AND MAPPINGS

The notion of a *function* is much more abstract than most of the algebra concepts you've seen thus far, so we'll start with three specific non-math examples.

❑ *THE MEANING OF A FUNCTION VIA TABLES*



First Example: A football game is divided into four quarters. The following table shows each quarter together with the total number of points scored during that quarter.

Quarter	Number of Points Scored
1st	21
2nd	9
3rd	0
4th	25

Inputs Outputs

This table is a ***function***, and here's why:

First, we have a set of **inputs**, the four quarters:

1st 2nd 3rd 4th

Second, we have a set of **outputs**, the total points scored in each quarter:

21 9 0 25

Third, there's a definite connection, or correspondence, between the inputs and the outputs. For example, the input "2nd" is associated with the output "9." What about the input "4th"? The output is "25." Similarly, "21" is the output for the input "1st," while "3rd" produces the output "0."

Fourth, and most importantly, notice that each input has exactly one output. For instance, the input "2nd" has exactly one output, namely "9." There can be no argument that the output is 9 — just look at the table. Also, the input "4th" clearly has the output "25" and nothing else. This is the essence of a function, so our table of football quarters and points scored is a function.

Second Example: So, what isn't a function? Consider the following table:

City	Major League Baseball Teams
San Francisco	Giants
Atlanta	Braves
New York	Yankees
New York	Mets
Inputs	Outputs

The input San Francisco has exactly one output, Giants, because the city of San Francisco is home to exactly one major league baseball team. The input Atlanta also has exactly one output associated with it. So far, so good – in terms of being a function.

But check out New York: It has two baseball teams, the Yankees and the Mets. We thus have an input (New York) with more than one output (Yankees and Mets). This kills the function concept, and we conclude that our table of cities and baseball teams is not a function.

To summarize this example, for something to be a function, there must be exactly one output for each input; this example has an input with two outputs. That's why the city/team pairings do NOT constitute a function.

Third Example:

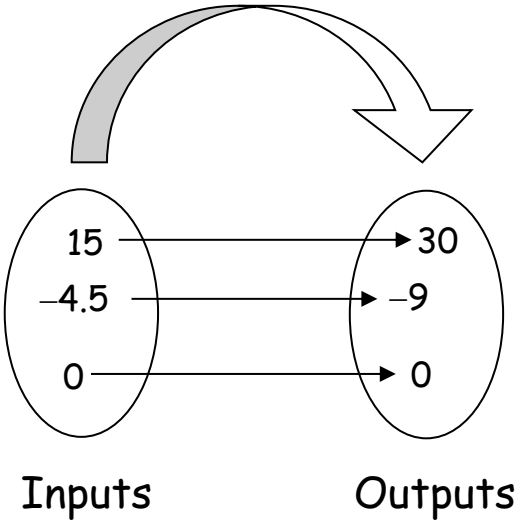
City	Population
Elmville	35,000
Gomerville	47,500
Moeville	35,000
Inputs	Outputs

Is this table a function? Does each input have exactly one output? YES, and I'll prove it to you. The input Elmville has one output (35,000), the input Gomerville has one output (47,500), and the input Moeville has one output (35,000). Does it matter that Elmville and Moeville have the same population? Not at all — it's just a coincidence. The fact is, every city has exactly one population associated with it; that is, each input has exactly one output. We must conclude that the population table is a function — although this function somehow differs from the function in the first example.

❑ *MATHEMATICAL EXAMPLES OF FUNCTIONS VIA MAPPINGS*

EXAMPLE 1: **Analyze the *doubling* function.**

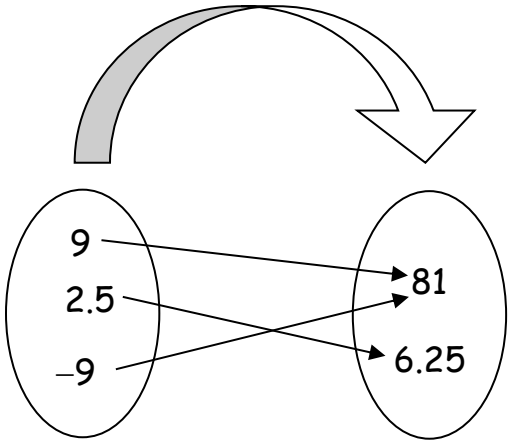
Solution: Consider the ***doubling function***. This function takes an input, and produces an output that is double the input. For example, if the input is 15, then the output is double 15, or 30. That is, this function takes an input of 15 and produces an output of 30. In the diagram below we see three inputs (15, –4.5, and 0) producing three outputs (30, –9, and 0).



The DOUBLING Function

EXAMPLE 2:

A. Look at the ***squaring*** function.



The SQUARING Function

Why is this a function? Because, given an input, there is exactly one output for it. Notice that even though both 9 and –9 produce the same output of 81, it is still the case that 9 squared is 81, and only 81 — and that –9 squared is 81, and only 81. Each input has exactly one output: *squaring* is a function.

B. Consider the ***square root*** function, $\sqrt{x}$. An input of 49 produces an output of 7, and an input of 2 produces an output of $\sqrt{2}$. Recall that the formula represents the non-negative square root only; so, for instance $\sqrt{100} = 10$, and <u>not</u> −10. [If you want both square roots, write: $\pm\sqrt{100}$.]

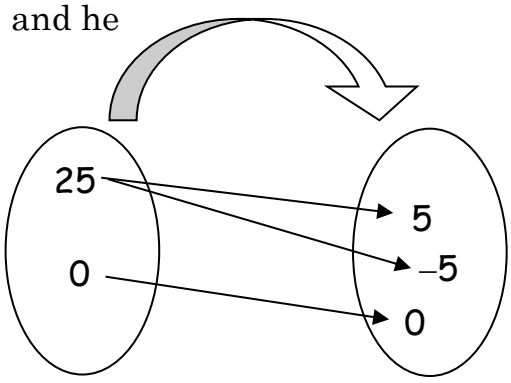
What inputs are allowed in this function? Since negative numbers don't have real-number square roots, all inputs must be at least zero, i.e., zero or bigger.

To reiterate, this is a function because given any legal input ($x \geq 0$), we obtain exactly <u>one</u> output.

C. Now for the ***reciprocal*** function; that is, $\frac{1}{x}$. An input of 7 produces $\frac{1}{7}$, an input of $-\frac{2}{3}$ produces $-\frac{3}{2}$, and an input of 0 produces nothing at all, since 0 has no reciprocal. That is, 0 is not a legal input to the reciprocal function. What can x be in this function? Well, the only number we can't divide by is 0, so x can be any real number except 0.

Starting with any legal input ($x \neq 0$), we obtain exactly one output — we conclude that the reciprocal is a function.

E. Now consider the operation "**take <u>both</u> square roots**": $\pm\sqrt{x}$. Is this a function? Suppose we take the input $x = 25$. Ask a person what the output is, and he might say 5, since 5 is indeed a square root of 25. But ask another person, and she might say −5, since −5 is also a square root of 25. Who's right? They're both right! There's nothing illegal or immoral about our formula. The fact is, it's simply <u>not</u> a function — there's an input that produces more than one output.



Homework

1. Construct a table of inputs and outputs that represents a function.

2. Construct a table of inputs and outputs that does not represent a function.

3. Construct a mapping (the circles and arrows) demonstrating "absolute value." Is it a function? Explain.

4. Construct a mapping that represents a non-function. Explain precisely why it's not a function.

5. Using the functions in Example 2, calculate the output for the three given inputs:

   a. The *squaring* function: $\frac{2}{3}$ -15 π
   Is there any number that cannot be used as an input?

   b. The (non-negative) *square root* function: 144 1 5
   Is there any number that cannot be used as an input?

   c. The *reciprocal* function: $\frac{1}{4}$ -3 1
   Is there any number that cannot be used as an input?

6. True/False:

   a. In the tripling function, if the input is 12, the output is 36.

   b. In the cubing function, if the output is 125, the input is 5.

Solutions

1. Be sure that every input has exactly one output.

2. Be sure that at least one input has more than one output.

3. I'd like to see what you came up with.

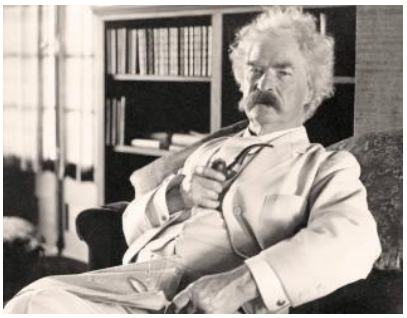
4. Be sure you have at least one input with 2 or more arrows diverging from it.

5.

a.	4/9	225	π^2	Every number can be used as an input.
b.	12	1	$\sqrt{5}$	No negative number can be square-rooted.
c.	4	$-\frac{1}{3}$	1	The only number without a reciprocal is 0.

6. a. T b. T

"The secret of getting ahead is getting started. The secret of getting started is breaking your complex overwhelming tasks into small manageable tasks, and then starting on the first one."



– Mark Twain

INTRO TO GEOMETRY

❑ *INTRODUCTION*

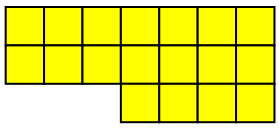


Geo: Greek for earth, and *metros*: Greek for measure. These roots are the origin of the word ***geometry***, which literally means "earth measurement." The study of geometry has gone way beyond the notions of triangles and circles – from the shapes of molecules to the structure of 4-dimensional space-time.

Area

The ***area*** of a geometric figure is a measure of how big a region is enclosed <u>inside</u> the figure.

For example, the area of the region below is 18 square units. Just count the number of little squares, and you've got the **area**.

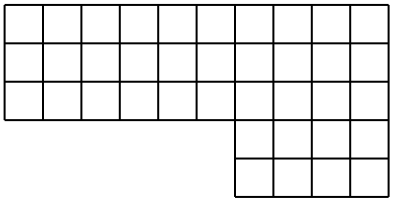


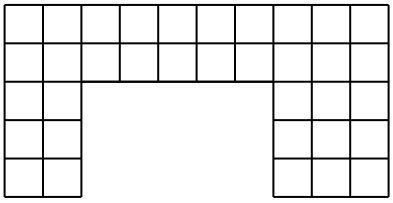
Perimeter

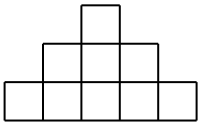
The distance <u>around</u> a geometric figure is called its ***perimeter***. For example, the perimeter of the region above is 20 units. Start anywhere you like, and march along the edge of the region until you reach your starting point. The distance you've traveled is the **perimeter**. To help you remember this, note that *peri* means "around," as in *peri*scope or *peri*pheral vision.

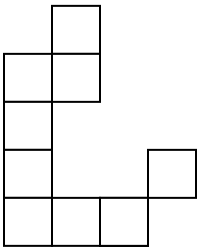
Homework

Find the area and perimeter of each geometric shape:

1. 

2. 

3. 

4. 

❑ *RECTANGLES AND SQUARES*

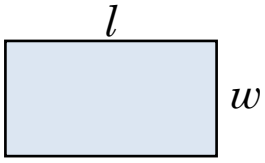
In Manhattan, Fifth Avenue (which runs north-south) meets 42nd Street (which runs east-west) at a **90° angle**. The floor and the wall also meet at a 90° angle.

A ***rectangle*** is a four-sided figure with all inside angles equal to 90°. This implies that adjacent sides (sides next to each other) are *perpendicular* and opposite sides are *parallel*. Notice that a ***square*** (where all four sides have the same length) is also a four-sided figure with all four inside angles equal to 90°. Therefore, by definition, a square is a special kind of rectangle. We can conclude that every square is a rectangle, but certainly not every rectangle is a square.



Note also that because of all the 90° angles, it follows that the top and bottom of the rectangle are the same length, and the left and right sides are the same length. In other words, the sides opposite each other in a rectangle have the same length.

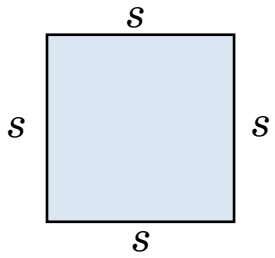
We know that the distance around the rectangle (the sum of all four of its sides) is called its ***perimeter***. If l is the length of the rectangle and w is the width, then the perimeter is $P = l + l + w + w = 2l + 2w$. The ***area*** of a rectangle is a measure of the size of the region enclosed by the rectangle. The formula for the area is $A = lw$.



Perimeter: $P = 2l + 2w$

Area: $A = lw$

As for the square, we can see that the perimeter, the distance around, is simply four s's added together: $s + s + s + s$, or $\mathbf{4}\boldsymbol{s}$. The area of a square, since it's a special rectangle, is just the length times the width; but the length and the width are both s, so the area is $s \times s$, or $\boldsymbol{s}^{\mathbf{2}}$.



Perimeter: $P = 4s$

Area: $A = s^2$

Homework

5. Find the perimeter of each rectangle given the length and width:

a. $l = 23 \quad w = 42$

b. $l = 34.7 \quad w = 1.22$

c. $l = 7 \quad w = \frac{4}{5}$

d. $l = 2\frac{3}{4} \quad w = 4\frac{1}{2}$

6. Find the area of each rectangle given the length and width:

a. $l = 23 \quad w = 42$

b. $l = 34.7 \quad w = 1.22$

c. $l = 7 \quad w = \frac{4}{5}$

d. $l = 2\frac{3}{4} \quad w = 4\frac{1}{2}$

7. Each side of a square is 22. Find the square's perimeter and its area.

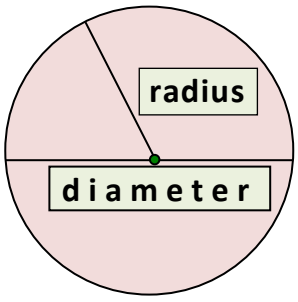
8. If each side of a square is 23.7, find its perimeter and area.

9. If each side of a square is $\frac{4}{5}$, find its perimeter and area.

❑ *CIRCLES*

Basic Definitions

The distance from any point on a circle to the center of the circle is called its ***radius***. The distance from one point on a circle to another point on the circle, through the center of the circle, is called its ***diameter***. It should be clear that the diameter is twice the radius:



Diameter: $\boldsymbol{d = 2r}$

Equivalently, the radius is half the diameter $r = \frac{1}{2}d$, which can also be written as

Radius: $\boldsymbol{r = \frac{d}{2}}$

The ***circumference*** of a circle is the distance all the way around the circle — it's the circle's *perimeter*. The ***area*** of a circle is a measure of the region inside the circle. The formulas for circumference and area will be stated after a discussion of a very important number.

A Very Special Number

Choose any circle at all — tiny, mid-sized, or gigantic. When you divide its circumference by its diameter — no matter the circle — you always end up with the same number, a constant a little bigger than 3. This quotient, (the circumference divided by the diameter), which is the same for all circles, is denoted by the Greek letter pi, "π", from the Greek word "perimetros" — though the Greeks themselves did not use the symbol π for this special number. In other words,

The modern symbol for pi, π was first used in 1706 by William Jones.

$$\pi = \frac{C}{d} \qquad \text{(definition of the number } \pi\text{)}$$

We also say that π is the ***ratio*** of the circumference to the diameter of any circle. To reiterate:

Given any circle, of any size, anywhere in the whole universe, the ratio of its circumference to its diameter is always the same number: π.

The decimal version of π has an infinite number of digits in it (very difficult to prove), and therefore any decimal number we write for π will simply be an approximation. The calculator gives something like 3.141592654; two useful approximations of π

The circumference of any circle is a little over 3 times its diameter.

are 3.14 and $\frac{22}{7}$. But since this is an algebra class, we'll usually stick with the exact value of π, which is, of course, simply written π.

Formulas for Circumference and Area

The circumference of a circle can be found by calculating 2 times π times the radius (this formula is derived *here*):

Circumference: $C = 2\pi r$

As for area, the formula can be derived in a later course, so we simply state it and use it:

Area: $A = \pi r^2$

Note #1: Since the Order of Operations specifies that exponents have priority over multiplication, we note that πr^2 tells us to first square the r, and then multiply by π.

Note #2: Students sometimes mix up the formulas for circumference and area. Here's a way to remember which is which. The area of a geometric shape is always in square units, for example, square feet — and which quantity, $2\pi r$ or πr^2, has the "square" in it? The πr^2 does, of course. So area is πr^2, and circumference is the one without the square in it, $2\pi r$.

Note #3: No teacher intends to confuse you, but it happens. Your Pre-Algebra teacher may have taught you that the circumference of a circle is given by the formula $C = \pi d$, instead of the formula $C = 2\pi r$ stated above. They are really the same formula, since $d = 2r$, so you can certainly use whichever one you like, though in this and future courses, you'll almost always use $C = 2\pi r$.

EXAMPLE 1:

A. **The radius of a circle is 89. Find the diameter.**

Since $d = 2r$, it follows that $d = 2(89) = \mathbf{178}$

B. **The diameter of a circle is 13. Find the radius.**

The formula $r = \frac{d}{2}$ tells us that $r = \frac{13}{2} = \mathbf{6\frac{1}{2}}$, or **6.5**

EXAMPLE 2: **The radius of a circle is 17.5. Find the circumference.**

Solution: Using the formula for the circumference of a circle, we proceed as follows:

	$C = 2\pi r$	(the circumference formula)
$\Rightarrow$	$C = 2\pi$ (**17.5**)	(plug in the given radius for r)
$\Rightarrow$	$C = 2(17.5)\ \pi$	(rearrange the factors)
$\Rightarrow$	$C = \boxed{35\pi}$	(multiply 2 by 17.5)

Note that this is the exact answer because the symbol π is in the final answer. An approximation could easily be found by using 3.14, for instance, for π, and then multiplying 35 by 3.14 to get **109.9**.

EXAMPLE 3: **The radius of a circle is 17. Find the area.**

Solution: The relevant formula is $A = \pi r^2$. Notice that only the r is being squared in this formula, since the Order of Operations specifies that exponents have priority over multiplication. We therefore calculate the area as follows:

$A = \pi r^2$ (the area of a circle formula)

$\Rightarrow \quad A = \pi (\mathbf{17})^2$ (plug in the given radius for r)

$\Rightarrow \quad A = \pi (289)$ (square the 17, and we're basically done)

$\Rightarrow \quad A = \boxed{289\pi}$ (it looks prettier this way)

Homework

The Circle Formulas

$$\pi = \frac{C}{d} \qquad d = 2r \qquad r = \frac{d}{2} \qquad C = 2\pi r \qquad A = \pi r^2$$

10. a. The radius of a circle is 0.023. What is the ratio of its circumference to its diameter?

b. The diameter of a circle is $\frac{123}{967}$. What is the ratio of its circumference to its diameter?

c. The circumference of a circle is 2134.909. What is the ratio of its circumference to its diameter?

d. The area of a circle is $11.178\pi^3$. What is the ratio of its circumference to its diameter?

11. For each problem, find the diameter if the radius is given, and find the radius if the diameter is given:

a. $r = 20$	b. $r = 7.6$	c. $r = \frac{1}{7}$	d. $r = 0.5$
e. $d = 44$	f. $d = 0.25$	g. $d = 17$	h. $d = \frac{1}{5}$

12. Find the **circumference** of the circle with the given radius; leave your answers in exact form (that is, leave π in the answer):

a. $r = 8$	b. $r = 33.4$	c. $r = 0.07$	d. $r = 89$
e. $r = 13$	f. $r = 0.5$	g. $r = 100$	h. $r = 77.5$

13. Find the **area** of the circle with the given radius – leave your answers in exact form (that is, leave π in the answer):

a. $r = 9$	b. $r = 100$	c. $r = 0.3$	d. $r = 3.5$
e. $r = 10$	f. $r = 2.5$	g. $r = 1$	h. $r = 0.08$

❑ *A Biblical View of π*

From I Kings 7:23 comes the sentence:

> "*Then He made the molten sea, ten cubits from brim to brim, while a line of 30 cubits measured it around.*"

The word "sea" refers to a large container for holding water. "Brim to brim" refers to the diameter of 10 cubits (the cubit was an ancient measurement of about 20 inches), while the 30 cubits refers to the circumference. Thus, from this quote we derive the value $\pi = \frac{C}{d} = \frac{30 \text{ cubits}}{10 \text{ cubits}} = 3$, which is quite a good estimate, given how old the description is.

Review Problems

14. The length of a rectangle is 55 and its width is 44. Find the perimeter of the rectangle.
15. The length of a rectangle is 34 and its width is 22. Find its area.
16. Find the perimeter and area of a square each of whose sides is 25.
17. Given any circle of any size, what is the ratio of its circumference to its diameter? That is, what is $\frac{C}{d}$ for any circle?
18. The radius of a circle is 15. What is its diameter?
19. The diameter of a circle is 88. Find the radius.
20. The radius of a circle is 7. Find the area in exact form.
21. The radius of a circle is 12. Find the circumference in exact form.

Solutions

1. $A = 38$; $P = 30$

2. $A = 35$; $P = 36$

3. $A = 9$; $P = 16$

4. $A = 9$; $P = 22$

5. a. 130 b. 71.84 c. $15\frac{3}{5}$ d. $14\frac{1}{2}$

6. a. 966 b. 42.334 c. $5\frac{3}{5}$ d. $12\frac{3}{8}$

7. $P = 88$ $A = 484$

8. $P = 94.8$ $A = 561.69$

9. $P = 3\frac{1}{5}$ $A = \frac{16}{25}$

10. a. π b. π c. π d. π

11. a. $d = 40$ b. $d = 15.2$ c. $d = \frac{2}{7}$ d. $d = 1$
e. $r = 22$ f. $r = 0.125$ g. $r = 8\frac{1}{2}$ h. $r = \frac{1}{10}$

12. a. 16π b. 66.8π c. 0.14π d. 178π
e. 26π f. π g. 200π h. 155π

13. a. 81π b. $10{,}000\pi$ c. 0.09π d. 12.25π
e. 100π f. 6.25π g. π h. 0.0064π

14. 198 **15**. 748 **16**. $P = 100$; $A = 625$ **17**. π

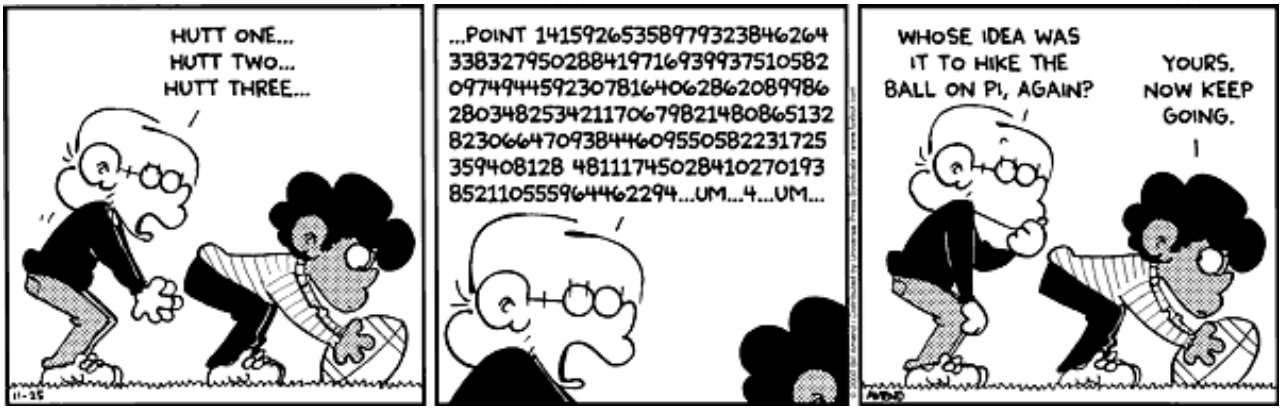
18. 30 **19**. 44 **20**. 49π **21**. 24π

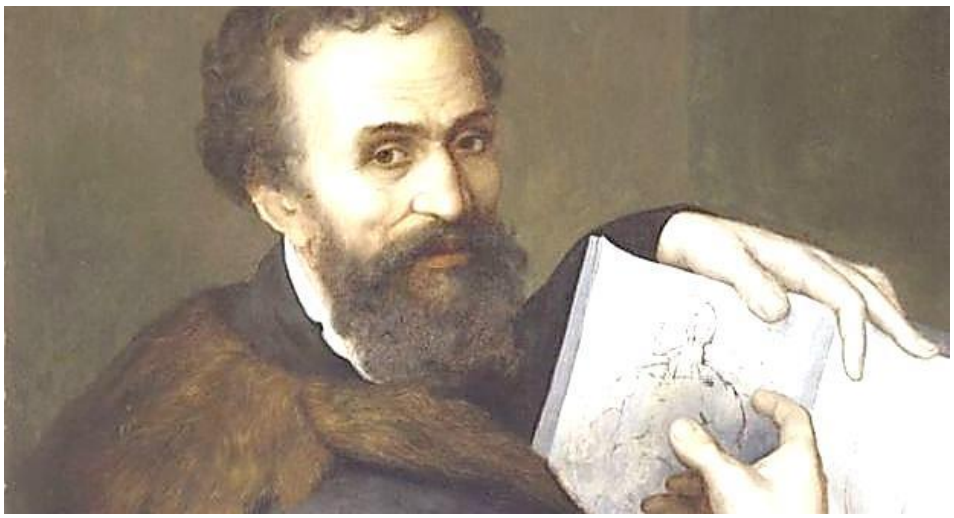
❑ *TO ∞ AND BEYOND*

A. Which is a better approximation of π: 3.14 or $\frac{22}{7}$?

B. A circle has an area of $56{,}987\pi^5$ square kilometers. Find the exact *ratio* of its diameter to its circumference.

C. "*A 20-inch-diameter pizza contains <u>twice</u> as much pizza as a 10-inch-diameter pizza.*" Any comments on this statement?



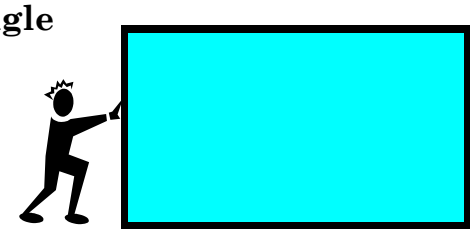


"I am still learning."

– **Michelangelo, at age 87**

AREA OF A RECTANGLE

EXAMPLE: **The length of a rectangle is 6 less than 4 times its width. If the <u>area</u> is 18, find the dimensions of the rectangle.**



Solution: Letting w represent the width of the rectangle, the length l must be $l = 4w - 6$. We'll start our solution with the formula for the area of a rectangle, $\boldsymbol{A = lw}$, substitute the given area of 18 for A, and substitute the expression $4w - 6$ for l.

	$A = lw$	(Area = length × width)
$\Rightarrow$	$18 = (4w - 6)w$	(substitute 18 for A and $4w - 6$ for l)
$\Rightarrow$	$18 = w(4w - 6)$	(turn the factors around)
$\Rightarrow$	$18 = 4w^2 - 6w$	(distribute . . . look, it's quadratic!)
$\Rightarrow$	$0 = 4w^2 - 6w - 18$	(bring all terms to same side)
$\Rightarrow$	$4w^2 - 6w - 18 = 0$	(turn it around for standard form)
$\Rightarrow$	$\frac{4w^2 - 6w - 18}{2} = \frac{0}{2}$	(divide each side by 2)
$\Rightarrow$	$2w^2 - 3w - 9 = 0$	
$\Rightarrow$	$(2w + 3)(w - 3) = 0$	
$\Rightarrow$	$2w + 3 = 0$ OR $w - 3 = 0$	
$\Rightarrow$	$w = -\frac{3}{2}$ OR $w = 3$	

We must discard the first solution (because a width cannot be negative) and thus take the width to be **3**. The length is found as follows:

$$l = 4w - 6 = 4(\mathbf{3}) - 6 = 12 - 6 = \mathbf{6}$$

The dimensions of the rectangle are therefore

$$\boxed{6 \times 3}$$

Let's make sure these are the correct dimensions. First, is the length equal to 6 less than 4 times the width? $4w - 6 = 4(\mathbf{3}) - 6 = 6$, which is the length. Second, is the area of the rectangle 18? Of course: $6 \times 3 = 18$.

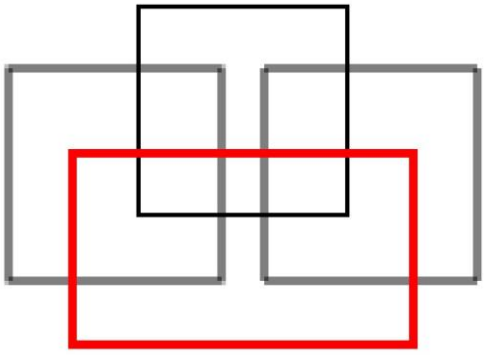
Homework

1. The length of a rectangle is 3 more than twice its width. If the area is 65, find the dimensions of the rectangle.

2. The length of a rectangle is 2 less than 10 times its width. If the area is 36, find the dimensions of the rectangle.

3. The length of a rectangle is 1 more than its width. If the area is 156, find the dimensions of the rectangle.

4. The length of a rectangle is 11 less than 3 times its width. If the area is 190, find the dimensions of the rectangle.



5. The length of a rectangle is 12 less than twice its width. If the area is 144, find the dimensions of the rectangle.

6. The length of a rectangle is 9 more than 10 times its width. If the area is 19, find the dimensions of the rectangle.

Solutions

1. 13×5

2. 18×2

3. 13×12

4. 19×10

5. 12×12

6. 19×1

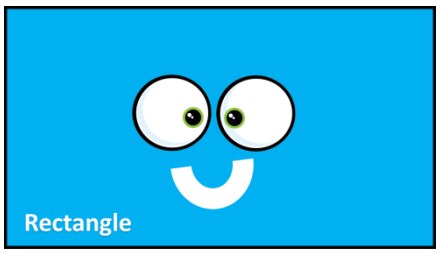
"Knowledge is the eye of desire and can become the pilot of the soul."

Will Durant

PERIMETER OF A RECTANGLE

❑ *INTRODUCTION*

We need only a couple of things to review in order to work this chapter's problems.



First, recall that the ***perimeter*** of a rectangle — being that it's the distance all the way around the rectangle — consists of adding together the two lengths and the two widths:

> If l is the length of a rectangle and w is its width, then its perimeter is given by the formula
>
> $$P = 2l + 2w$$

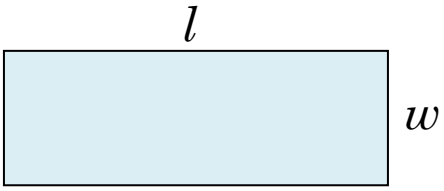
Note that, using the Distributive Property, we can also write the perimeter as $P = 2(l + w)$.

Second, we need to be able to translate some English phrases and equalities to Algebra:

> 4 more than $x \Rightarrow x + 4$
>
> 7 less than $w \Rightarrow w - 7$
>
> 3 more than 8 times $y \Rightarrow 8y + 3$
>
> 13 less than 5 times $z \Rightarrow 5z - 13$
>
> The length is 4 more than 6 times the width $\Rightarrow l = 6w + 4$
>
> The width is 5 less than twice the length $\Rightarrow w = 2l - 5$

EXAMPLE 1: **The length of a rectangle is 4 less than 3 times its width. If the perimeter is 32, find the dimensions of the rectangle.**

Solution: Ignoring details for a moment, we see that the problem is talking about the perimeter of a rectangle:



$$P = 2l + 2w$$

Since the length is 4 less than 3 times the width, we can write

$$l = 3w - 4$$

We also know that the perimeter is 32; that is, $P = 32$. Thus, the general formula

$$P = 2l + 2w$$

becomes

	$32 = 2(3w - 4) + 2w$	(since $P = 32$ and $l = 3w - 4$)
$\Rightarrow$	$32 = 6w - 8 + 2w$	(distribute)
$\Rightarrow$	$32 = 8w - 8$	(combine like terms)
$\Rightarrow$	$40 = 8w$	(add 8 to each side)
$\Rightarrow$	$w = \mathbf{5}$	(divide each side by 8)
$\Rightarrow$	length $= 3w - 4$	
	$= 3(\mathbf{5}) - 4$	
	$= 15 - 4$	
	$= \mathbf{11}$	

Therefore, the dimensions of the rectangle are | 11 by 5 |

Homework

1. The length of a rectangle is 30 more than 2 times the width. If the perimeter is 66, find the dimensions of the rectangle.

2. Find the dimensions of a rectangle with a perimeter of 44 if its length is 6 more than 3 times its width.

3. The width of a rectangle is 23 less than 2 times the length. If the perimeter is 62, find the dimensions of the rectangle.

4. The width of a rectangle is 18 less than 2 times the length. If the perimeter is 48, find the dimensions of the rectangle.

5. Find the dimensions of a rectangle with a perimeter of 42 if its width is 15 less than 2 times its length.

6. The width of a rectangle is 26 less than 3 times the length. Find the dimensions of the rectangle if its perimeter is 28.

7. The length of a rectangle is 539 less than 6 times its width. If the perimeter is 2254, find the dimensions of the rectangle.

8. The length of a rectangle is 30 more than 5 times the width. If the perimeter is 120, find the dimensions of the rectangle.

Solutions

1. 32 by 1 **2**. 18 by 4 **3**. 18 by 13 **4**. 14 by 10

5. 12 by 9 **6**. 10 by 4 **7**. 889 by 238 **8**. 55 by 5

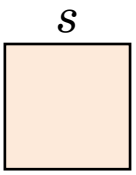
“They cannot stop me. I will get my education, if it is in the home, school, or anyplace.”



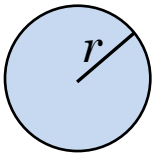
Malala Yousafzai

GEOMETRY – SQUARES AND CIRCLES

❑ *REVIEW OF FORMULAS*



If s is the length of each side of a **square**, then its perimeter is $\mathbf{4}\boldsymbol{s}$ and its area is $\boldsymbol{s}^{\mathbf{2}}$.



If r is the radius of a **circle**, then its diameter is $\mathbf{2}\boldsymbol{r}$, its circumference is $\mathbf{2}\pi\boldsymbol{r}$, and its area is $\pi\boldsymbol{r}^{\mathbf{2}}$.

Homework

1. True/False: Every square is a rectangle.
2. True/False: Every rectangle is a square.
3. Each side of a square is 17. Find the square's perimeter and area.
4. The radius of a circle is 10. Find the diameter, the circumference, and the area in exact form (that means leave π as π).
5. Each side of a square is 25. Find the square's perimeter and area.
6. The radius of a circle is 1. Find the diameter, the circumference, and the area.
7. The radius of a circle is $\frac{1}{2}$. Find the diameter, the circumference, and the area.

❑ *SQUARES*

EXAMPLE 1: **The perimeter of a square is 94. What is the length of each side of the square?**

Solution: The formula for the perimeter of a square is

$$P = 4s$$

Substituting the given information, we get

$$\mathbf{94} = 4s$$

Dividing each side of the equation by 4 gives

$$\frac{94}{4} = \frac{4s}{4}, \text{ or } \boxed{s = 23.5}$$

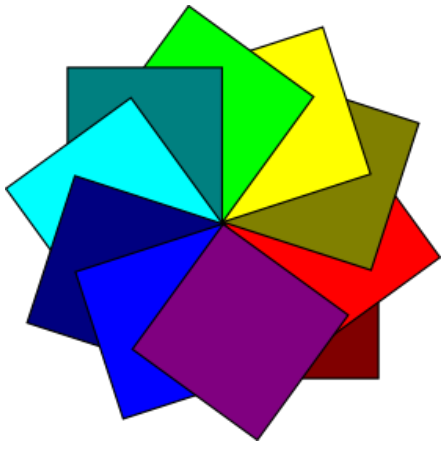
EXAMPLE 2: **The area of a square is 361. How long is each side of the square?**

Solution: The area of a square is given by $A = s^2$. Plugging 361 in for A yields the equation

$$\mathbf{361} = s^2$$

We need to take the square root of 361 in order to solve for s. Using a calculator (or clever guessing), we conclude that

$$\boxed{s = 19}$$



EXAMPLE 3: **The perimeter of a square is 50.4. Find the area of the square.**

Solution: We're being asked for the area of a square. But the area formula, $A = s^2$, requires that we know the side, and we don't. However, we are given the perimeter. Do you see what we're going to do? We will use the perimeter to find the side, and then use the side to find the area.

$P = 4s$ (perimeter of a square)

$\Rightarrow$ $\mathbf{50.4} = 4s$ (put in the given area)

$\Rightarrow$ $\underline{s = 12.6}$ (divide each side by 4)

Therefore,

$A = s^2$ (area of a square)

$= (\mathbf{12.6})^2$ (put in the s just calculated)

$= 158.76$ (multiply 12.6 by itself)

We conclude that the area is $\boxed{158.76}$

EXAMPLE 4: **The area of a square is 625. What is the square's perimeter?**

Solution: This is the reverse of the previous example, but we solve it in essentially the same way: We'll use the given area to determine the side, and then use the side to find the perimeter.

$A = s^2 \Rightarrow 625 = s^2 \Rightarrow \underline{s = 25}$

And so,

$P = 4s = 4(\mathbf{25}) = \boxed{100}$

Homework

8. For each problem, the perimeter of a square is given. Find the **side** of the square. You may use a calculator.

a. $P = 144$ b. $P = 49.2$ c. $P = 210$ d. $P = 0.092$

e. $P = 92$ f. $P = 10.4$ g. $P = 504$ h. $P = 0.04$

9. For each problem, the area of a square is given. Find the **side** of the square. You may use a calculator.

a. $A = 144$ b. $A = 6.76$ c. $A = 0.0625$ d. $A = 6{,}241$

e. $A = 529$ f. $A = 6.25$ g. $A = 0.0016$ h. $A = 1{,}764$

10. For each problem, the perimeter of a square is given. Find the **area** of the square. You may use a calculator.

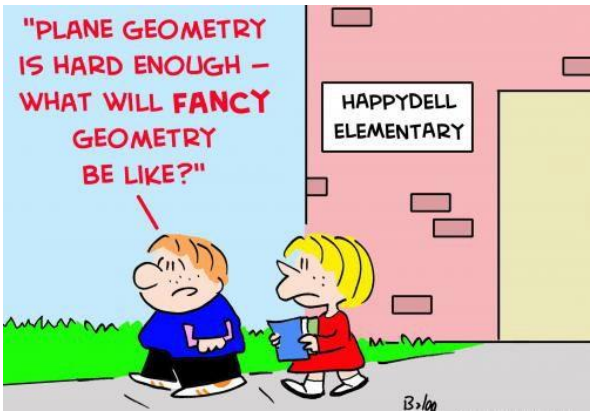
a. $P = 48$ b. $P = 50$ c. $P = 9.6$ d. $P = 201.2$

e. $P = 100$ f. $P = 30$ g. $P = 20.8$ h. $P = 165.2$

11. For each problem, the area of a square is given. Find the **perimeter** of the square. You may use a calculator.

a. $A = 121$ b. $A = 2116$ c. $A = 18.49$ d. $A = 0.09$

e. $A = 196$ f. $A = 841$ g. $A = 12.25$ h. $A = 0.0025$



❑ *CIRCLES*

Deriving the Circumference Formula: $C = 2\pi r$

The next problem we're going to solve using our algebra skills is to derive the formula $C = 2\pi r$ for the circumference of a circle whose radius is r.

We begin with the definition of π from earlier in the class, namely that π is the ratio of the circumference of any circle to its diameter:

$$\pi = \frac{C}{d}$$

Multiplying each side of this equation by d gives:

$$\pi[d] = \frac{C}{\cancel{d}}[\cancel{d}]$$

Simplifying gives

$$\pi d = C$$

Now we turn the equation around and also change d into $2r$, since the diameter is twice the radius:

$$C = \pi(2r)$$

Applying the fact that multiplication is an associative operation, we can "shift" the parentheses to associate the π and the 2:

$$C = (\pi 2)r$$

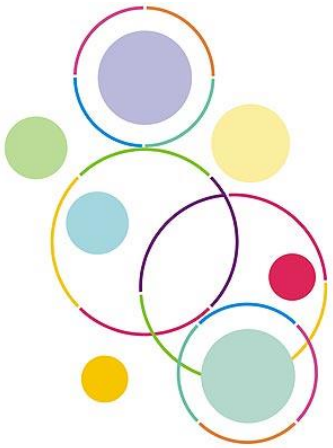
Since multiplication is also a commutative operation, we can switch the π and the 2 to get:

$$C = (2\pi)r$$

Again, since multiplication is an associative operation, the parentheses are redundant (unnecessary) and we can remove them. Our final formula:

$$\boxed{C = 2\pi r}$$

EXAMPLE 5: **The diameter of a circle is 12. Find the area.**



Solution: The formula for the area of a circle, $A = \pi r^2$, requires that we have the radius, which was not explicitly given to us in the problem. But the diameter of the circle was given, so we can use this diameter to find the radius, and then use the radius to find the area:

given in the problem

$$r = \frac{d}{2} = \frac{12}{2} = \mathbf{6}$$

Now we know that $r = 6$, so the area is calculated as follows:

$$A = \pi r^2 = \pi \cdot 6^2 = \boxed{36\pi}$$

EXAMPLE 6: **Find the circumference of a circle whose diameter is 9.**

Solution: The process is analogous to the previous problem.

$r = \frac{d}{2} = \frac{\mathbf{9}}{\mathbf{2}}$, and so

$$C = 2\pi r = 2\pi\left(\frac{9}{2}\right) = \cancel{2}\pi\left(\frac{9}{\cancel{2}}\right) = \boxed{9\pi}$$

EXAMPLE 7: **The circumference of a circle is 24π. Find the radius.**

Solution: The circumference of a circle is given by $C = 2\pi r$. Also, the circumference is given to be 24π. So if we set the

circumference formula ($2\pi r$) equal to the given circumference (24π), we should be able to solve for r:

$2\pi r = 24\pi$ ($C = 2\pi r$)

$\Rightarrow \quad \frac{2\pi r}{2\pi} = \frac{24\pi}{2\pi}$ (divide each side by 2π)

$\Rightarrow \quad \frac{\cancel{2\pi}\, r}{\cancel{2\pi}} = \frac{\overset{12}{\cancel{24}}\,\cancel{\pi}}{\underset{1}{\cancel{2}}\,\cancel{\pi}}$ (divide out common factors)

$\Rightarrow \quad \boxed{r = 12}$ (and we've isolated the r)

EXAMPLE 8: **Find the radius of a circle if it's known that its area is 121π.**

Solution: We set the area formula (πr^2) equal to the given area (121π), and solving for r will be as easy as π.

$$\pi r^2 = 121\pi \quad \Rightarrow \quad \frac{\cancel{\pi} r^2}{\cancel{\pi}} = \frac{121\cancel{\pi}}{\cancel{\pi}} \quad \Rightarrow \quad r^2 = 121$$

Using a calculator, or better yet, your brain, we see that $\boxed{r = 11}$

Homework

12. Find the circumference and the area of the circle whose diameter is given:

    a. $d = 10$ b. $d = 13$ c. $d = 2$ d. $d = 200$

13. Find the radius of the circle given its circumference or area:

    a. $C = 30\pi$ b. $C = 19\pi$ c. $A = 100\pi$ d. $A = \pi$

EXAMPLE 9: **The circumference of a circle is 4.6π. Find the area.**

Solution: First ask yourself, What do I need in order to find the area? The radius, of course. But was the radius given to us? No, but the circumference was, so we can use the circumference to find the radius, and then use that radius to find the area.

Step 1: Use the circumference to find the radius

The circumference is given to be 4.6π.

$\Rightarrow \quad 2\pi r = 4.6\pi$ (circumference = $2\pi r$)

$\Rightarrow \quad \frac{2\pi r}{2\pi} = \frac{4.6\pi}{2\pi}$ (divide each side by 2π)

$\Rightarrow \quad r = 2.3$ (simplify each side)

Step 2: Use the radius to find the area

$$A = \pi r^2 = \pi(2.3)^2 = \boxed{5.29\pi}$$



EXAMPLE 10: **The area of a circle is 2.25π. Find the circumference.**

Solution: Reversing the logic in the previous problem, we use the given area to calculate the radius, and then use that radius to find the circumference.

$$\pi r^2 = 2.25\pi \Rightarrow \frac{\pi r^2}{\pi} = \frac{2.25\pi}{\pi} \Rightarrow r^2 = 2.25 \Rightarrow r = \mathbf{1.5}$$

It thus follows that $C = 2\pi r = 2\pi(1.5) = 2(1.5)\pi = \boxed{3\pi}$

Homework

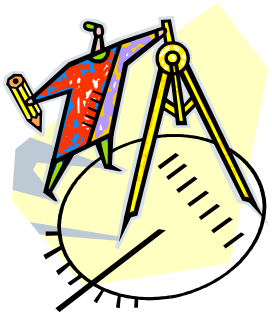
14. Find the area of the circle with the given circumference:

    a. $C = 10\pi$ b. $C = 2\pi$ c. $C = \pi$ d. $C = 5\pi$

15. Find the circumference of the circle with the given area:

    a. $A = 25\pi$ b. $A = \pi$ c. $A = 81\pi$ d. $A = 225\pi$

16. a. Find the circumference of a circle with diameter 2.
    b. Find the area of a circle with diameter 30.
    c. Find the radius of a circle with circumference 38π.
    d. Find the diameter of a circle with circumference 8π.
    e. Find the radius of a circle with area 576π.
    f. Find the diameter of a circle with area 400π.
    g. Find the area of a circle with circumference 2π.
    h. Find the circumference of a circle with area 16π.
    i. Find the circumference of a circle with diameter 28.
    j. Find the area of a circle with diameter 28.
    k. Find the radius of a circle with circumference 34π.
    l. Find the diameter of a circle with circumference 20π.
    m. Find the radius of a circle with area 529π.
    n. Find the diameter of a circle with area 81π.
    o. Find the area of a circle with circumference 18π.
    p. Find the circumference of a circle with area 169π.



Review Problems

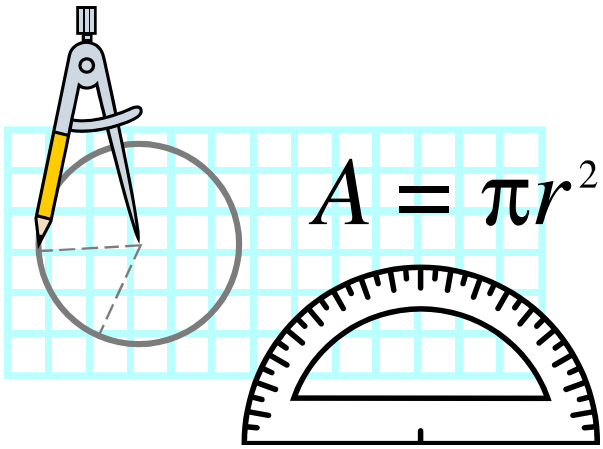
17. The perimeter of a square is 73.6. Find the side of the square.

18. The area of a square is 10,000. Find the side of the square.

19. Find the area of a square whose perimeter is 180.

20. Find the perimeter of a square whose area is 6.25.

21. The circumference of a circle is 20π. Find the area.



22. The area of a circle is 49π. Find the circumference.

23. True/False:

    a. If the area of a square is 25, its perimeter is 20.

    b. If the perimeter of a square is 16, its area is 16.

    c. If the diameter of a circle is 12, its area is 144π.

    d. If the diameter of a circle is 20, its circumference is 20π.

    e. If the circumference of a circle is 20π, its area is 100π.

    f. If the area of a circle is 49π, its diameter is 14.

    g. If the radius of a circle is 11, its area is 121.

    h. If the area of a circle is 36π, its circumference is 36π.

Solutions

1. True **2**. False **3**. $P = 68$; $A = 289$

4. $d = 20$; $C = 20\pi$; $A = 100\pi$ **5**. $P = 100$; $A = 625$

6. $d = 2$; $C = 2\pi$; $A = \pi$ **7**. $d = 1$; $C = \pi$; $A = \frac{1}{4}\pi$

8. a. $s = 36$ b. $s = 12.3$ c. $s = 52.5$ d. $s = 0.023$
e. $s = 23$ f. $s = 2.6$ g. $s = 126$ h. $s = 0.01$

9. a. $s = 12$ b. $s = 2.6$ c. $s = 0.25$ d. $s = 79$
e. $s = 23$ f. $s = 2.5$ g. $s = 0.04$ h. $s = 42$

10. a. $A = 144$ b. $A = 156.25$ c. $A = 5.76$ d. $A = 2530.09$
e. $A = 625$ f. $A = 56.25$ g. $A = 27.04$ h. $A = 1705.69$

11. a. $P = 44$ b. $P = 184$ c. $P = 17.2$ d. $P = 1.2$
e. $P = 56$ f. $P = 116$ g. $P = 14$ h. $P = 0.2$

12. a. $C = 10\pi$; $A = 25\pi$ b. $C = 13\pi$; $A = 42.25\pi$
c. $C = 2\pi$; $A = \pi$ d. $C = 200\pi$; $A = 10{,}000\pi$

13. a. $r = 15$ b. $r = 9.5$ c. $r = 10$ d. $r = 1$

14. a. $A = 25\pi$ b. $A = \pi$ c. $A = 0.25\pi$ d. $A = 6.25\pi$

15. a. $C = 10\pi$ b. $C = 2\pi$ c. $C = 18\pi$ d. $C = 30\pi$

16. a. $C = 2\pi$ b. $A = 225\pi$ c. $r = 19$ d. $d = 8$
e. $r = 24$ f. $d = 40$ g. $A = \pi$ h. $C = 8\pi$
i. $C = 28\pi$ j. $A = 196\pi$ k. $r = 17$ l. $d = 20$
m. $r = 23$ n. $d = 18$ o. $A = 81\pi$ p. $C = 26\pi$

17. 18.4 **18**. 100 **19**. 2025

20. 10 **21**. 100π **22**. 14π

23. a. T b. T c. F d. T
e. T f. T g. F h. F

❑ *TO ∞ AND BEYOND*

A. Find a square whose area is equal to its perimeter (ignoring the fact that the units of area differ from the units of perimeter).

B. Find a rectangle (which is not a square) whose area is equal to its perimeter (ignoring the fact that the units of area differ from the units of perimeter).

C. Find a circle whose area is equal to its circumference (even though the units must be different).

D. Find the radius of a circle whose circumference is 20.

E. Find the radius of a circle whose area is 10.

F. By what factor must the radius of a circle be increased in order to increase the circumference by a factor of 4?

G. By what factor must the radius of a circle be increased in order to increase the area by a factor of 9?

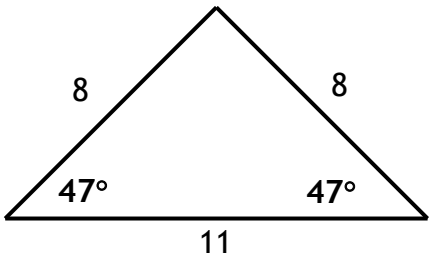
"The wisest mind has something yet to learn."

George Santayana (1863 - 1952)

GEOMETRY – TRIANGLES AND ANGLES

❑ *INTRODUCTION*

An ***isosceles triangle*** has at least two equal sides. It's also the case that an isosceles triangle must have at least two equal angles in it, namely the angles that are opposite the equal sides.



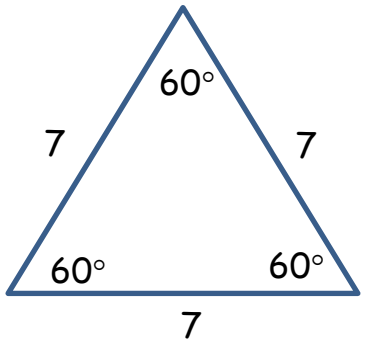
An Isosceles Triangle

It's a fact from geometry that the sum of the three inside angles of any triangle is always 180°. For example, in the isosceles triangle above, the top angle must be 86°, because

$$47° + 47° + 86° = 180°$$

Because an ***equilateral triangle*** has three equal sides, all the angles must be the same — namely 60° each — since

$$60° + 60° + 60° = 180°$$



An Equilateral Triangle
(It's also isosceles)

Homework

1. Two of the angles in a triangle are 100° and 30°. What is the measure of the third angle?

2. Two of the angles in an isosceles triangle are 35° and 35°. Find the measure of the third angle.

3. Name the angles in any equilateral triangle.

4. True/False:

   a. The two acute angles (the ones that are not 90°) in a right triangle must be the same.

   b. The two acute angles in a right triangle have a sum of 90°.

❑ *ANGLE PROBLEMS*

Note that the following problems deal with the three ***angles*** in a triangle, not their sides. Notice that the problems do not specify what the sum of the three angles is — that's because we know that **the sum of the inside angles in <u>any</u> triangle is 180°** (and it's your job to remember that fact!).

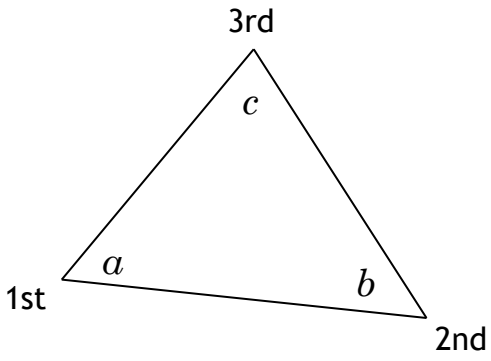
<u>EXAMPLE 1:</u> **The second angle of a triangle is 12° less than 4 times the first angle. The third angle is 30° less than the second angle. What is the measure of each angle?**

<u>Solution:</u> First, since the problem talks about the three angles of a triangle, the key to the equation will be that the angles must total 180°. We start by giving names to the three angles inside the triangle:

a = 1st angle

b = 2nd angle

c = 3rd angle



Now we parse the given problem phrase by phrase:

The second angle of a triangle is 12° less than 4 times the first angle translates to

$b = 4a - 12$

The third angle is 30° less than the second angle becomes

$c = b - 30$

Our equation is

$a + b + c = 180$ (angles total 180°)

$\Rightarrow \quad a + \boxed{4a - 12} + \boxed{b - 30} = 180$ (substitute for b & c)

$\Rightarrow \quad 5a - 42 + b = 180$ (combine like terms)

$\Rightarrow \quad 5a - 42 + \boxed{4a - 12} = 180$ (substitute for b)

$\Rightarrow \quad 9a - 54 = 180$ (combine like terms)

$\Rightarrow \quad 9a = 234$ (add 54 to each side)

$\Rightarrow \quad \boldsymbol{a = 26}$ (divide by 9)

We have our final answers:

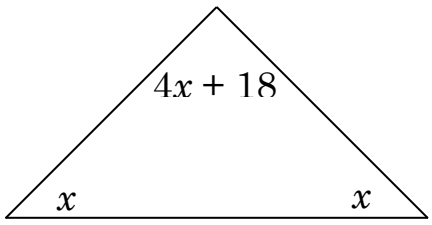
The first angle is $a = \boxed{26°}$

The second angle $= b = 4a - 12 = 4(\mathbf{26}) - 12 = \boxed{92°}$

The third angle $= c = b - 30 = \mathbf{92} - 30 = \boxed{62°}$

EXAMPLE 2: **The third angle of an isosceles triangle is 18° more than 4 times one of the equal angles. What is the measure of each angle?**

Solution: For variety, let's try solving this problem using just one variable. Since two of the angles must be equal,



let's call each of them x. So the third angle, which is 18° more than 4 times x, can be written $4x + 18$. The sum of the three angles must be 180°:

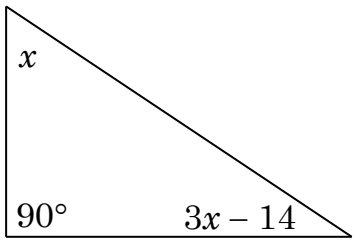
	$x + x + (4x + 18) = 180$	(3 angles total 180°)
$\Rightarrow$	$x + x + 4x + 18 = 180$	(remove parentheses)
$\Rightarrow$	$6x + 18 = 180$	(combine like terms)
$\Rightarrow$	$6x = 162$	(subtract 18 from each side)
$\Rightarrow$	$\boldsymbol{x = 27}$	(divide each side by 6)

and so $4x + 18 = 4(\mathbf{27}) + 18 = 126$

Hence, the three angles are | 27°, 27°, and 126° |

EXAMPLE 3: **One of the acute angles of a right triangle is 14° less than 3 times the other acute angle. Find the measure of the larger acute angle.**

Solution: Again, let's use just one variable to solve this problem. Since one of the angles in a right triangle is 90°, we need only concern ourselves with the two acute angles. Calling one of them x means that the other one must be called $3x - 14$. Since the sum of the three angles must be 180°, we get the equation



	$x + (3x - 14) + 90 = 180$	(3 angles total 180°)
$\Rightarrow$	$x + 3x - 14 + 90 = 180$	(drop the parentheses)
$\Rightarrow$	$4x + 76 = 180$	(combine like terms)
$\Rightarrow$	$4x = 104$	(subtract 76 from each side)
$\Rightarrow$	$\boldsymbol{x = 26}$	(divide each side by 4)
$\Rightarrow$	$3x - 14 = 3(\mathbf{26}) - 14 = 64$	

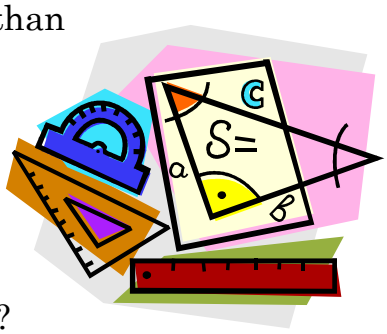
Therefore, the larger acute angle is | 64° |

Homework

5. One of the acute angles of a right triangle is 18° more than 2 times the other acute angle. Find the measures of the two acute angles.

6. The third angle of an isosceles triangle is 15° more than 3 times one of the equal angles. What is the measure of each angle?

7. The second angle of a triangle is 16° more than 3 times the first angle. The third angle is 125° less than the second angle. What is the measure of each angle?



8. The third angle of an isosceles triangle is 4° less than 2 times one of the equal angles. What is the measure of each angle?

9. The second angle of a triangle is 38° less than 4 times the first angle. The third angle is 13° more than the second angle. What is the measure of each angle?

10. One of the acute angles of a right triangle is 6° less than 5 times the other acute angle. Find the measures of the two acute angles.

11. One of the acute angles of a right triangle is 10° less than 3 times the other acute angle. Find the measures of the two acute angles.

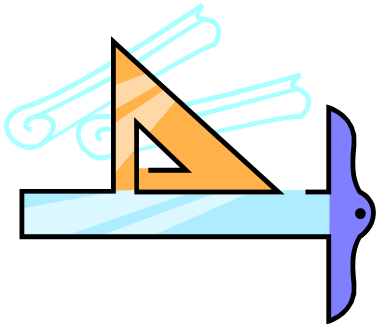
12. The third angle of an isosceles triangle is 8° less than 2 times one of the equal angles. What is the measure of each angle?

13. The second angle of a given triangle is 66° more than its first angle, while its third angle is 18° less than 4 times the first angle. Find the measure of each angle.

14. One of the acute angles of a right triangle is 18° less than 2 times the other acute angle. Find the measures of the two acute angles.

15. One of the acute angles of a right triangle is 60° less than 2 times the other acute angle. Find the measures of the two acute angles.

16. The third angle of an isosceles triangle is 68° less than 2 times one of the equal angles. What is the measure of each angle?



17. The second angle of a triangle is 20° less than the first angle. The third angle is 20° more than twice the second angle. What is the measure of each angle?

Review Problems

18. The second angle of a triangle is 30° more than the first angle, while the first angle is 26° more than 3 times the third angle. What is the measure of each angle?

19. One of the acute angles of a right triangle is 18° more than 5 times the other acute angle. What is the measure of each acute angle?

20. The first angle of a triangle is 1° more than 8 times the third angle, while the second angle is 47° more than 3 times the third angle. What is the measure of each angle?

21. The third angle of a triangle is 90° less than 10 times the second angle, while the first angle is 50° less than 9 times the second angle. What is the measure of each angle?

22. One of the acute angles of a right triangle is 10° less than the other acute angle. What is the measure of each acute angle?

23. The second angle of a triangle is 96° less than 3 times the third angle, while the first angle is 25° less than 3 times the third angle. What is the measure of each angle?

24. The third angle of a triangle is 38° more than 3 times the second angle, while the first angle is 43° less than the third angle. What is the measure of each angle?

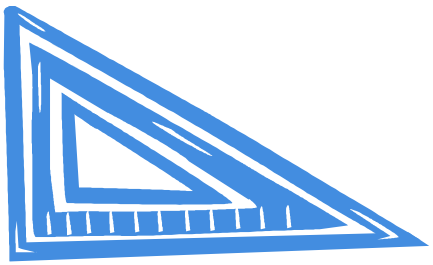
25. The second angle of a triangle is 11° more than 2 times the third angle, while the first angle is 55° less than 5 times the third angle. What is the measure of each angle?

26. One of the acute angles of a right triangle is 14° less than the other acute angle. What is the measure of each acute angle?

27. The second angle of a triangle is 3° less than 4 times the first angle, while the third angle is 42° more than the second angle. What is the measure of each angle?

28. The second angle of a triangle is 32° less than 6 times the first angle, while the third angle is 42° less than the second angle. What is the measure of each angle?

29. The third angle of an isosceles triangle is 12° more than 5 times one of the equal angles. What is the measure of each angle?



30. The second angle of a triangle is 28° more than 5 times the first angle, while the third angle is 42° more than 4 times the first angle. What is the measure of each angle?

31. The second angle of a triangle is 75° more than 5 times the first angle, while the third angle is 21° less than 3 times the first angle. What is the measure of each angle?

32. The third angle of an isosceles triangle is 4° more than 20 times one of the equal angles. What is the measure of each angle?

33. The second angle of a triangle is 29° more than the first angle, while the first angle is 15° more than 5 times the third angle. What is the measure of each angle?

Solutions

1. 50°

2. 110°

3. 60°, 60°, 60°

4. a. F b. T

5. 24°, 66°

6. 33°, 33°, 114°

7. 39°, 133°, 8°

8. 46°, 46°, 88°

9. 27°, 70°, 83°

10. 16°, 74°

11. 25°, 65°

12. 47°, 47°, 86°

13. 22°, 70°, 88°

14. 36°, 54°

15. 40°, 50°

16. 56°, 62°, 62°

17. 35°, 55°, 90°

18. 14°, 68°, 98°

19. 12°, 78°

20. 11°, 89°, 80°

21. 16°, 70°, 94°

22. 50°, 40°

23. 43°, 33°, 104°

24. 21°, 101°, 58°

25. 28°, 67°, 85°

26. 52°, 38°

27. 16°, 61°, 103°

28. 22°, 100°, 58°

29. 24°, 24°, 132°

30. 11°, 83°, 86°

31. 14°, 145°, 21°

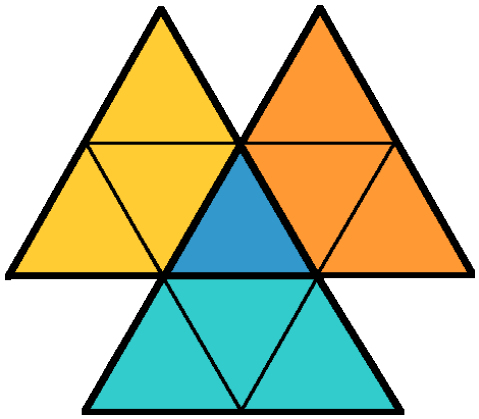
32. 8°, 8°, 164°

33. 11°, 70°, 99°

"The beginning of knowledge is the discovery of something we do not understand."

Frank Herbert

GEOMETRY – TRIANGLES AND PERIMETER

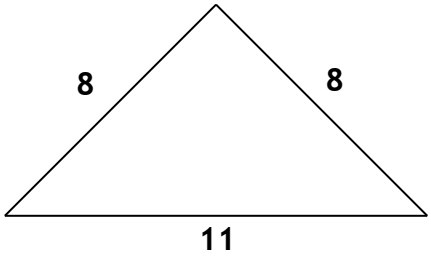


❑ *INTRODUCTION*

We continue our study of using algebra to solve geometry problems, but now we focus exclusively on triangles.

❑ *TERMINOLOGY*

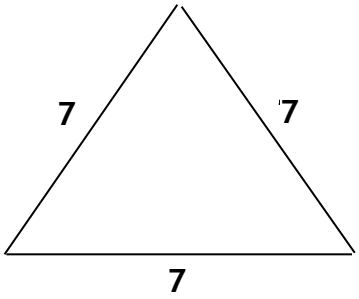
A triangle with <u>at least</u> two equal sides is called an ***isosceles triangle***.



An Isosceles Triangle

Since an isosceles triangle must have <u>at least</u> two equal sides, it follows that an equilateral triangle is <u>also</u> isosceles, since it has three equal sides, which is <u>at least</u> two equal sides. Thus, based on our definitions, every equilateral triangle is isosceles, but certainly not every isosceles triangle is equilateral.

An ***equilateral triangle*** has three equal sides.



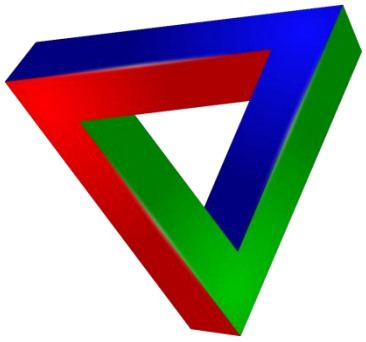
An Equilateral Triangle, (which is also isosceles)

Homework

1. Two sides of an isosceles triangle are 14 and 20. The third side must be either _____ or _____.
2. One side of an equilateral triangle is 52. What are the other two sides of the triangle?
3. Two sides of a triangle are 23 and 99. If the perimeter is 200, find the length of the third side.
4. Two sides of an isosceles triangle are 17 and 17. Must the third side be different from 17?

5. True/False:
   a. Every equilateral triangle is isosceles.
   b. Every isosceles triangle is equilateral.

6. Translate to algebra (and simplify if possible):
   a. 3 more than n
   b. 13 less than a
   c. 7 more than twice T
   d. 12 less than 3 times Q
   e. 5 more than $n + 8$
   f. 10 less than $3a + 6$
   g. 3 more than $4x - 13$
   h. 8 less than $2b - 20$
   i. 17 more than $9y + 18$
   j. 23 less than $5w + 20$



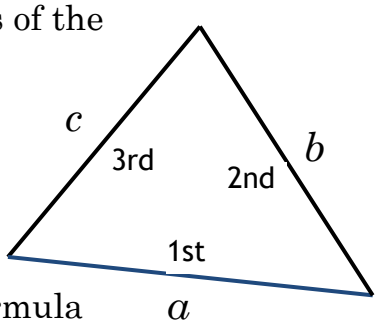
❑ *PERIMETER PROBLEMS*

EXAMPLE 1: **The first side of a triangle is 2 more than twice the second side, while the third side is 3 more than the second side. If the perimeter is 21, what is the length of each side?**

Solution: First, notice that this problem deals with the perimeter of a triangle, so the main formula we'll deal with is the fact that the perimeter is the sum of the lengths of the three sides.

We start by giving names to the three sides of the triangle:

a = 1st side
b = 2nd side
c = 3rd side



The first equation we might write is the formula for the perimeter of the triangle:

$a + b + c = 21$ (the perimeter is 21)

Here's the dilemma in a nutshell: too many variables for a single equation. Let's read the question again, looking for more information.

Check out the phrase: *The first side of the triangle is 2 more than twice the second.* We can translate this into Algebra like this:

$a = 2b + 2$

Similarly, *the third side is 3 more than the second side* becomes

$c = b + 3$

Let's rewrite the perimeter formula we formed above, and then substitute our little formulas above for a and c:

$$\begin{array}{l} a + b + c = 21 \\ \downarrow \quad \downarrow \quad \downarrow \\ \boxed{2b+2} + \boxed{b} + \boxed{b+3} = 21 \end{array}$$

$\Rightarrow \quad 2b + 2 + b + b + 3 = 21$

$\Rightarrow \quad 4b + 5 = 21$ (combine like terms)

$\Rightarrow \quad 4b = 16$ (subtract 5 from each side)

$\Rightarrow \quad \underline{b = 4}$ (divide each side by 4)

Now that we have b, the second side, we deduce that:

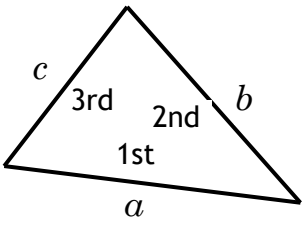
The first side is $a = 2b + 2 = 2(4) + 2 = \underline{10}$.
The second side is, of course, $b = \underline{4}$.
The third side is $c = b + 3 = 4 + 3 = \underline{7}$.

We're done. The sides of the triangle are

10, 4, and 7

EXAMPLE 2: **The perimeter of a triangle is 38. Its first side is 3 less than 2 times the third side, and its second side is 6 less than the first side. What are the lengths of its sides?**

Solution: As in the previous example, let's label the 1st, 2nd, and 3rd sides as a, b, and c.



A perimeter of 38 means that

$$a + b + c = 38$$

And we have the same problem ... too many variables.

The phrase *Its first side is 3 less than 2 times the third side* translates to

$$a = 2c - 3$$

The phrase *its second side is 6 less than the first side* gives us the additional equation:

$$b = a - 6$$

We rewrite the perimeter formula, and then substitute:

$$a + b + c = 38$$

$$\Rightarrow \quad \boxed{2c - 3} + \boxed{a - 6} + \boxed{c} = 38 \qquad \text{(substitute)}$$

$$\Rightarrow \quad 3c - 9 + a = 38 \qquad \text{(combine like terms)}$$

We have a little problem here. Our perimeter equation has <u>two</u> variables in it, a and c. We need to substitute again, using the fact that $a = 2c - 3$:

$$\Rightarrow \quad 3c - 9 + \boxed{2c - 3} = 38 \qquad \text{(substitute } 2c - 3 \text{ for } a\text{)}$$

$$\Rightarrow \quad 5c - 12 = 38 \qquad \text{(combine like terms}$$

$$\Rightarrow \quad 5c = 50 \qquad \text{(add 12 to each side)}$$

$$\Rightarrow \quad c = \mathbf{10} \qquad \text{(divide each side by 5)}$$

We can now find all three sides:

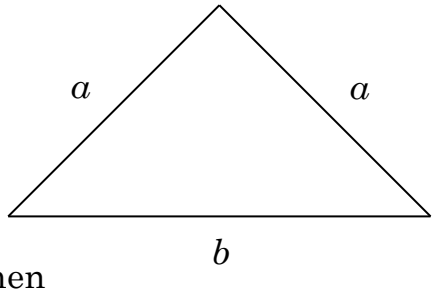
The third side is $c = \mathbf{10}$.
The first side is $a = 2c - 3 = 2(\mathbf{10}) - 3 = 17$.
The second side is $a - 6 = \mathbf{17} - 6 = 11$.

The three sides of the triangle are

17, 11, and 10

EXAMPLE 3: **An isosceles triangle has a perimeter of 24. If its third side is 4 less than 2 times one of the equal sides, what is the longest side of the triangle?**

Solution: An isosceles triangle has (at least) two equal sides, but we certainly can't assume that all three sides are equal. So we'll let a represent the length of each of the equal sides. Then the third side, call it b, must be expressed as



$$b = 2a - 4$$

Since the sum of these three quantities must equal the given perimeter, our equation is

	$a + a + b = 24$	(perimeter formula)
$\Rightarrow$	$a + a + 2a - 4 = 24$	(since $b = 2a - 4$)
$\Rightarrow$	$4a - 4 = 24$	(combine like terms)
$\Rightarrow$	$4a = 28$	(add 4 to each side)
$\Rightarrow$	$\boldsymbol{a = 7}$	(divide each side by 4)

So the two equal sides, a, are each 7, while the third side, b, is $2(\mathbf{7}) - 4 = 10$. Hence, the longest side of the triangle (go back and read what's being asked for) has a length of

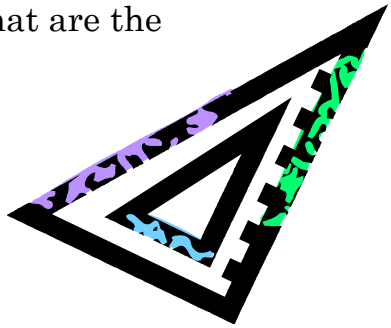
$$\boxed{10}$$

Homework

7. An isosceles triangle has a perimeter of 30. If its third side is 2 less than 2 times one of the equal sides, what are the lengths of its sides?

8. The perimeter of a triangle is 59. Its first side is 4 less than 2 times the second, and its third side is 13 less than the first side. What are the lengths of its sides?

9. An isosceles triangle has a perimeter of 9. If its third side is 3 less than 2 times one of the equal sides, what are the lengths of its sides?



10. The perimeter of a triangle is 40. Its first side is 4 less than 2 times the second, and its third side is 2 less than the first side. What are the lengths of its sides?

11. An isosceles triangle has a perimeter of 18. If its third side is 2 less than 2 times one of the equal sides, what are the lengths of its sides?

12. The perimeter of a triangle is 12. Its first side is 1 less than 2 times the second, and its third side is 1 less than the first side. What are the lengths of its sides?

13. An isosceles triangle has a perimeter of 47. If its third side is 2 more than one of the equal sides, what are the lengths of its sides?

14. The perimeter of a triangle is 32. Its first side is 3 less than the second, and its third side is 11 less than twice the first side. What are the lengths of its sides?

15. The third side of a triangle is 16 more than 3 times the first side, while the second side is 8 more than the third side. If the perimeter is 187, what is the length of each side?

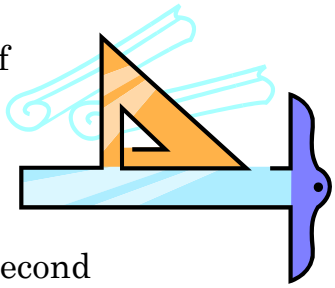
16. The second side of a triangle is 15 less than the third side, while the third side is 4 less than 3 times the first side. If the perimeter is 117, what is the length of each side?

17. The second side of a triangle is 14 more than 3 times the third side, while the first side is 12 more than 3 times the third side. If the perimeter is 166, what is the length of each side?

18. The second side of a triangle is 5 less than 2 times the first side, while the third side is 2 less than 2 times the first side. If the perimeter is 48, what is the length of each side?

19. An isosceles triangle has a perimeter of 91. If its third side is 57 less than 2 times one of the equal sides, what are the lengths of its sides?

Review Problems



20. An isosceles triangle has a perimeter of 80. If its third side is 24 less than 2 times one of the equal sides, what are the lengths of its sides?

21. The first side of a triangle is 2 less than the second side, while the second side is 9 more than 4 times the third side. If the perimeter is 205, what is the length of each side?

22. The first side of a triangle is 14 less than 4 times the third side, while the second side is 16 less than 4 times the third side. If the perimeter is 159, what is the length of each side?

23. The third side of a triangle is 3 less than the first side, while the first side is 12 less than 2 times the second side. If the perimeter is 58, what is the length of each side?

24. The third side of a triangle is 11 more than 3 times the first side, while the second side is 7 less than 4 times the first side. If the perimeter is 108, what is the length of each side?

25. The third side of a triangle is 3 less than the second side, while the second side is 6 more than 3 times the first side. If the perimeter is 121, what is the length of each side?

26. An isosceles triangle has a perimeter of 139. If its third side is 41 less than 2 times one of the equal sides, what are the lengths of its sides?

27. The second side of a triangle is 12 less than 2 times the first side, while the third side is 10 less than the second side. If the perimeter is 81, what is the length of each side?

28. The second side of a triangle is 16 less than the third side, while the third side is 17 less than 3 times the first side. If the perimeter is 97, what is the length of each side?

Solutions

1. 14, 20

2. 52 and 52

3. 78

4. No

5. a. T b. F

6. a. $n + 3$ b. $a - 13$
c. $2T + 7$ d. $3Q - 12$
e. $(n + 8) + 5 = n + 13$ f. $(3a + 6) - 10 = 3a - 4$

g. $(4x - 13) + 3 = 4x - 10$ h. $(2b - 20) - 8 = 2b - 28$
i. $(9y + 18) + 17 = 9y + 35$ j. $(5w + 20) - 23 = 5w - 3$

7. 8, 8, 14 **8**. 28, 16, 15 **9**. 3, 3, 3 **10**. 16, 10, 14

11. 5, 5, 8 **12**. 5, 3, 4 **13**. 15, 15, 17 **14**. 9, 10, 13

15. 21, 79, 87 **16**. 20, 56, 41 **17**. 20, 74, 72 **18**. 11, 17, 20

19. 37, 37, 17 **20**. 26, 26, 28 **21**. 21, 93, 91 **22**. 21, 70, 68

23. 17, 22, 19 **24**. 13, 50, 45 **25**. 16, 54, 51 **26**. 45, 45, 49

27. 23, 34, 24 **28**. 21, 46, 30

"A life spent making mistakes is not only more honorable, but more useful than a life spent doing nothing."

– George Bernard Shaw

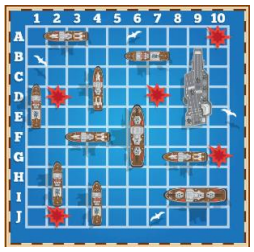
THE COORDINATE PLANE

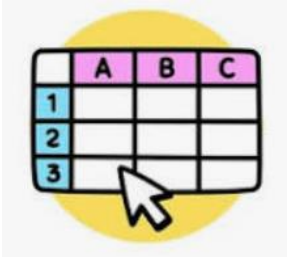
Someone once said, "A picture is worth a thousand words." This is especially true in math, where many ideas are very abstract. The French mathematician-philosopher René Descartes devised a way for us to visually represent the solutions of many kinds of equations. It's called the ***Cartesian Coordinate System***. [The term *Cartesian* is the Latin form of the name Descartes.]



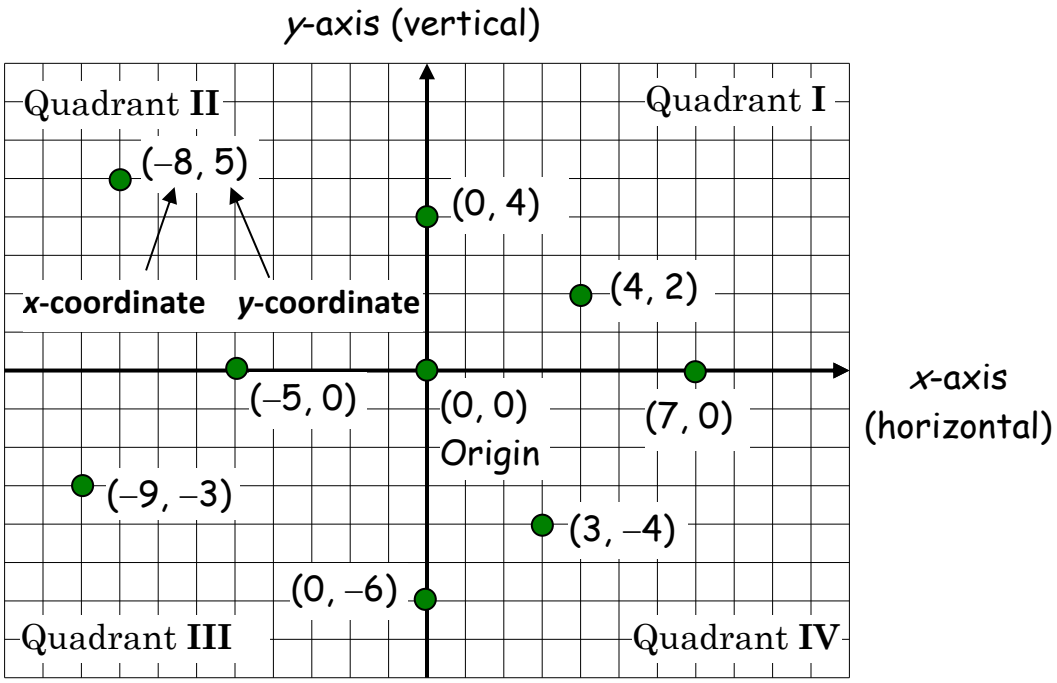
"I think, therefore I am"

We've seen one-dimensional number lines before. In this chapter, we take two number lines (one called the *x*-axis and one called the *y*-axis), line them up perpendicular to each other (90° angle between them), and we have a two-dimensional system sometimes called **2-*space***. It's a fancy form of the "Battleship" game, or the basic layout of a spreadsheet.





❑ THE CARTESIAN COORDINATE SYSTEM (2-SPACE)



Observations on 2-Space

1. A two-dimensional coordinate system represents a ***plane***. The horizontal axis is called the ***x*-axis** in math, but will be called other things in other subjects. Similarly, the vertical ***y*-axis** will be called something else in other subjects.



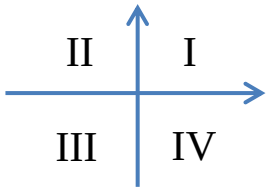
2. The ***ordered pair* (*x*, *y*)** represents a single ***point*** in the plane. The numbers ***x*** and ***y*** in the ordered pair are the ***coordinates*** of that point. Notice that a single point (ordered pair) consists of two coordinates. [The coordinates of a point on the Earth are called its longitude and latitude.]

3. The point **(0, 0)**, where the axes intersect (cross), is called the ***origin***.

4. The first coordinate of the point **(*x*, *y*)** represents the distance to the right or left from the origin. The second coordinate represents the distance up or down. For example, the point (3, –4) is plotted by starting at the origin, moving 3 units to the *right*, and then moving 4 units *down*.

5. The ***quadrants*** are numbered I (one) through IV (four), starting in the upper-right region and going counterclockwise.

In Quadrant I, both coordinates (the x and y) are positive.
In Quadrant II, x is negative and y is positive.
In Quadrant III, both coordinates are negative.
In Quadrant IV, x is positive and y is negative.



6. Points on the x-axis or the y-axis are not in any quadrant.

7. Every point on the ***x***-axis has a ***y***-coordinate of 0.
Every point on the ***y***-axis has an ***x***-coordinate of 0.

Homework

1. Is the Cartesian coordinate system described in this chapter 1-, 2-, or 3-dimensional?

2. a. Does the notation (x, y) represent one point or two points?
   b. Does the notation (x, y) represent one coordinate or two coordinates?

3. a. The point $(7, -\sqrt{3})$ lies in Quadrant ____.
   b. The point $(-\pi, -9)$ lies in Quadrant ____.
   c. The point $(-1, \sqrt{2})$ lies in Quadrant ____.
   d. The point $(\pi, \sqrt{5})$ lies in Quadrant ____.

4. a. The point (17, 0) lies on the ___ axis.

   b. The point (0, –20) lies on the ___ axis.

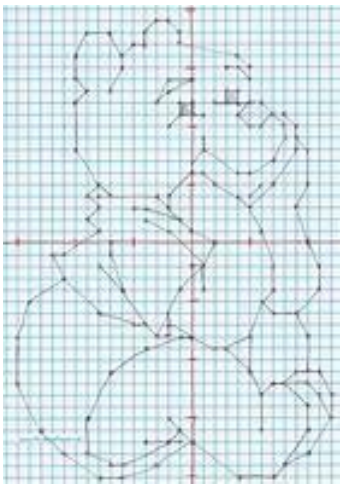
   c. The point (0, 0) is called the ____ and lies on the ___ axis.

5. a. In Quadrant I, x is _____ and y is _____.

   b. In Quadrant II, x is _____ and y is _____.

   c. In Quadrant III, x is _____ and y is _____.

   d. In Quadrant IV, x is _____ and y is _____.



6. a. In which quadrants are the signs of x and y the same?

   b. In which quadrants are the signs of x and y opposites?

7. a. A point lies on the x-axis. What can you say for sure about the coordinates of that point? Hint: The following are points on the x-axis: (7, 0), (–23, 0), and (π, 0).

   b. A point lies on the y-axis. What can you say for sure about the coordinates of that point?

   c. A point lies on both axes. What can you say for sure about the coordinates of that point?

❑ *GEOGRAPHY*

One last connection between the *x*-*y* coordinate system and the longitude/latitude system used on the Earth: The *x*-axis can be thought of as the *Equator* (0° latitude), and the *y*-axis can be viewed as the *Prime Meridian* (0° longitude), a line of longitude going from the North Pole to the South Pole and passing through Greenwich, England, just outside London.

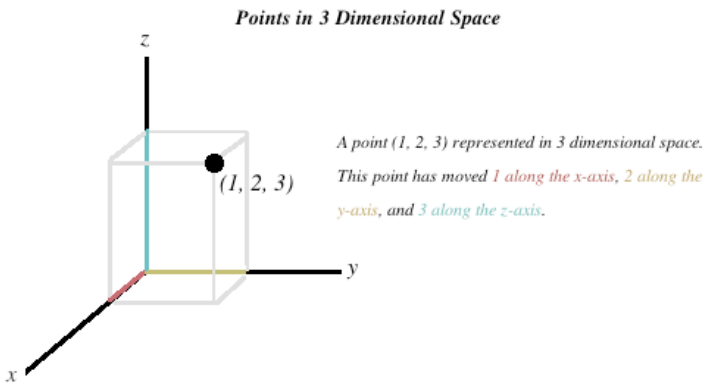


Review Problems

8. a. The y-axis is the (horizontal, vertical) axis.
   b. In the point (7, 9), the x-coordinate is _____.
   c. The point (0, 0) is called the _____.
   d. In which quadrant is x positive and y negative?
   e. Every point on the y-axis has an x-coordinate of _____.
   f. Which quadrant is the point $(-\pi, \sqrt{17})$ in?

Solutions

1. The Cartesian coordinate system in this course is **2**-dimensional. A **1**-dimensional coordinate system might be the x-axis. In later courses, you might encounter a **3**-dimensional coordinate system, and if you major in math or physics, you will find even higher-dimension coordinate systems, including an infinite-dimensional coordinate system!



2. a. one b. two

3. a. IV b. III c. II d. I

4. a. x b. y c. origin; it's on both axes.

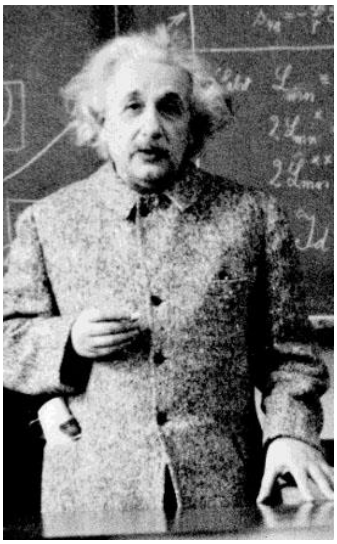
5. a. positive; positive b. negative; positive
c. negative; negative d. positive; negative

6. a. I and III b. II and IV

7. a. The y-coordinate must be 0.
b. The x-coordinate must be 0.
c. Both coordinates must be 0, since it's the origin.

8. a. vertical b. 7 c. origin
d. IV e. 0 f. II

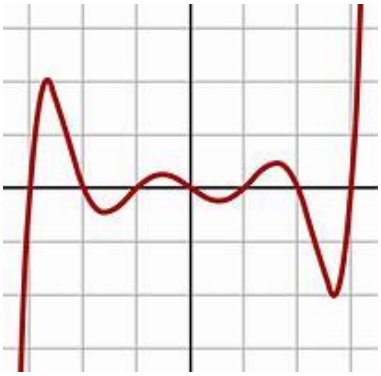
"Where the world ceases to be the scene of our personal hopes and wishes, where we face it as free beings admiring, asking and observing, there we enter the realm of Art and Science."



Albert Einstein
(1879-1955)

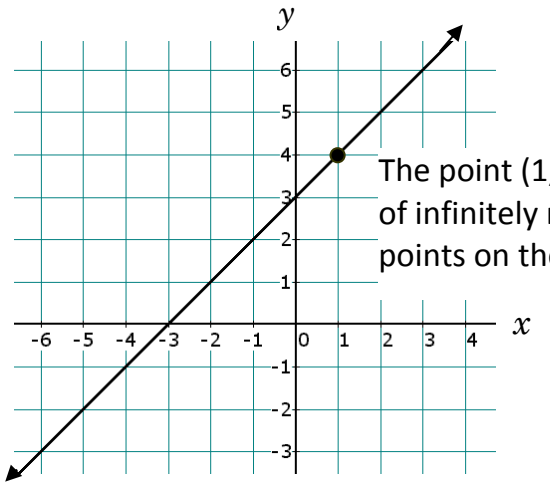
FROM GRAPH TO EQUATION

In this chapter, I'm going to give you a graph, but only a small portion of it, generally around the origin. We will first analyze points we can actually see (for instance, I will ask what *y*-value is associated with a certain *x*-value). Then I will ask you what *y*-value is associated with an *x*-value that you can't even see on the graph! How will you do this? You will use the given graph to create an algebraic formula to describe the graph, and then use that formula to predict *y*-values from <u>any</u> *x*-value I choose to give you.



❑ *LINES*

<u>EXAMPLE 1:</u> **Consider the following graph:**



The point (1, 4) is one of infinitely many points on the line.

a. Calculate the *y*-value when $x = 100$.

b. Calculate the *y*-value when $x = 2.7$.

c. Calculate the *x*-value when $y = 1000$.

Solution: We certainly don't want to extend (draw) the graph all the way to $x = 100$. So our approach, described in the Introduction, will be two-fold:

1) Determine the equation of the given graph by analyzing the relationship between x and y.

2) Use that equation to deduce the y-value for the given x-value.

The first step will be to create an x-y table by reading the coordinates off the graph. For example, on the given graph, we see that when $x = 1$, $y = 4$; thus the point (1, 4) is on the graph.

x	–4	–3	–2	–1	0	1	2	3
y	–1	0	1	2	3	4	5	6

Remember that the arrowheads on the graph indicate that the graph (the line) goes forever in both directions. It then follows that x can be any number at all. For example, although it's not easy to see from the graph, if we choose x to be 1.5, it can be guessed that $y = 4.5$. Similarly, we could let $x = \pi$ and then see that y would be a little bigger than 6.

Guessing certainly isn't very accurate, and that's even when we have a graph to look at. It would be even more difficult to find points on the line if we had to extend the graph ourselves. This is why it's so very important, when possible, to create an *equation* from the given graph. Assuming that we end up with the right equation, we are guaranteed exact y-values for any given x-value.

Now let's examine the equation of the line. Looking at the x-values and y-values in the table, we might see that the y-value is always **3 more than** the x-value. We are therefore led to the formula

$$y = x + 3$$

Check this formula against all the x-y pairs in the table and you will see that it really works. Now we can answer the three questions:

a. If $x = 100$, then $y = x + 3 = \mathbf{100} + 3 = \mathbf{103}$.

b. If $x = 2.7$, then $y = x + 3 = \mathbf{2.7} + 3 = \mathbf{5.7}$.

c. Be careful here; note that y is given and we want to find x.

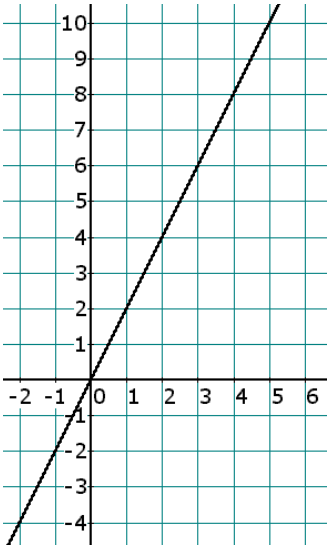
	$y = x + 3$	(the formula for the line)
$\Rightarrow$	$1000 = x + 3$	(let $y = 1000$)
$\Rightarrow$	$1000 - 3 = x + 3 - 3$	(subtract 3 from each side)
$\Rightarrow$	$\mathbf{997} = x$	(solve for x)

In summary, the following three points lie on the line:

(100, 103)	(2.7, 5.7)	(997, 1000)

Homework

1. Given the graph, create a formula and then use that formula to answer the questions:



a. When $x = 3$, $y =$ ___.

b. When $x = 0$, $y =$ ___.

c. When $x = -4$, $y =$ ___.

d. When $x = 278$, $y =$ ___.

e. When $x = -900$, $y =$ ___.

f. When $x = 18\pi$, $y =$ ___.

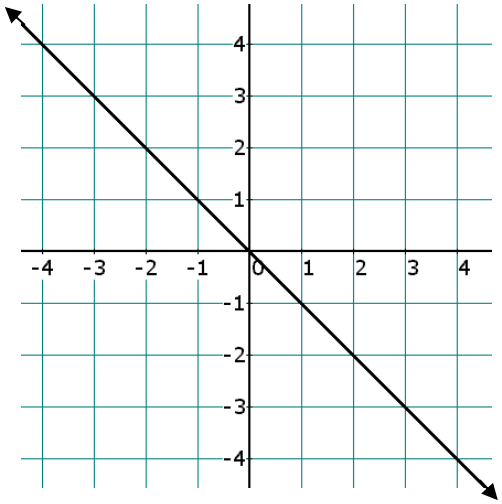
g. When $y = 10$, $x =$ ___.

d. W

h. When $y = 250$, $x =$ ___.

i. When $y = 22\pi$, $x =$ ___.

EXAMPLE 2: **Consider the following graph:**



a. Calculate the *y*-value when $x = 1200$.

b. Calculate the *y*-value when $x = -123$.

c. Calculate the *x*-value when $y = -25\pi$.

Solution: Again, the points in which we're interested are not visible on the graph, so let's create a table based on the graph, and then use that table to construct a formula.

x	−4	−3	−2	−1	0	1	2	3
y	4	3	2	1	0	−1	−2	−3

We see that the *y*-value is simply the ***opposite*** of the *x*-value:

$$y = -x$$

Notice that when $x = 0$, $y = -0 = 0$ (which yields the origin); therefore, 0 has an opposite, namely 0. We conclude that every

number has an opposite. We can now answer the three questions:

a. If $x = 1200$, then $y = -x = \mathbf{-1200}$.

b. If $x = -123$, then $y = -x = -(-123) = \mathbf{123}$.

c. As in the previous example, the y-value is given and we're asked for the x-value. So let's begin with the formula, plug in the given value of y, and then solve for x:

$y = -x$ (the formula for the line)

$\Rightarrow \quad -25\pi = -x$ (let $y = -25\pi$)

$\Rightarrow \quad 25\pi = x$ (divide or multiply each side by -1)

Therefore, the following three points lie on the line:

(1200, – 1200) (–123, 123) (25π, –25π)

Homework

2. The graph of $y = -x$ lies within Quadrants ___ and ___ and at the origin.

3. What is the **opposite** of each number?

   a. 17 b. 0 c. –3.5 d. 8π e. $-\sqrt{2}$

4. a. T/F: Every number has an opposite.

   b. The opposite of 0 is ____.

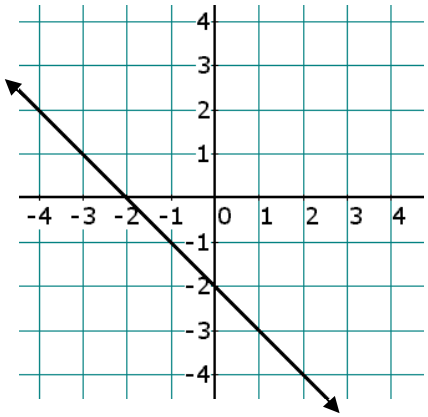
   c. The opposite of a negative number is always _____.

   d. The opposite of a positive number is always _____.

5. Consider the formula $y = -x + 5$. Calculate the y-value for the given x-value:

   a. 8 b. 99 c. –10 d. –5 e. 0 f. π g. –π

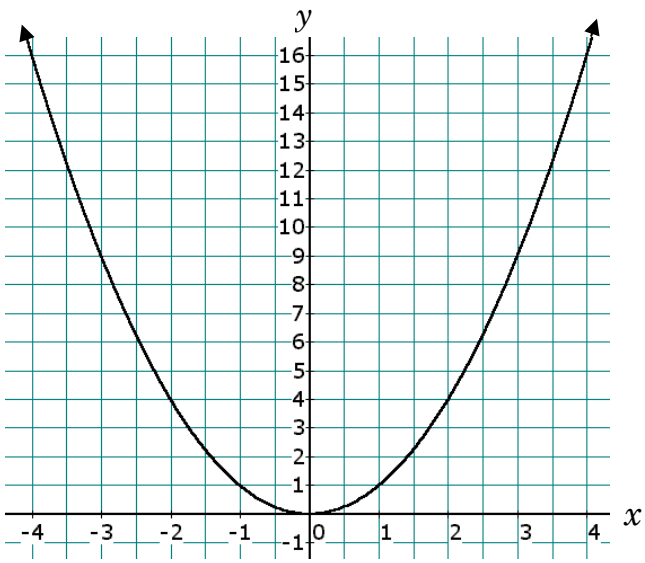
6. Given the graph, create a formula and then use that formula to answer the questions:



a. When $x = 2$, $y =$ ___.
b. When $x = 0$, $y =$ ___.
c. When $x = -4$, $y =$ ___.
d. When $x = 25$, $y =$ ___.
e. When $x = -200$, $y =$ ___.
f. When $y = 32$, $x =$ ___.
g. When $y = -300$, $x =$ ___.

❑ *BEYOND LINES*

EXAMPLE 3: **Consider the following graph:**



a. Calculate the *y*-value when $x = 1{,}000$.

b. Calculate the *y*-value when $x = 1.2$.

c. Calculate the *x*-values when $y = 625$.

Solution: This graph is much different from the previous graph. It passes through the origin, and the rest of the graph resides only in Quadrants I and II. Let's try to read off some ordered pairs from the graph:

x	–4	–3	–2	–1	0	1	2	3	4
y	16	9	4	1	0	1	4	9	16

Now, how is the *y*-value related to the *x*-value? When $x = 0$, $y = 0$; this is rather useless, as is the pair (1, 1). But look at the ordered pairs (2, 4), (3, 9), and (4, 16). There's something going on here — it appears that the *y*-value is simply the square of the *x*-value:

$$y = x^2$$

Lest we jump to conclusions, let's double-check the rest of the pairs in the table. For example, is the square of –4 equal to 16? Yes. Is the square of –1 equal to 1? Yes. Is the square of 0 equal to 0? Yes. Our formula $y = x^2$ is looking better and better. Notice that even though *x* can be any number at all, the *y*-value is never negative; the *y*-values start at 0 and go toward positive infinity. Now to answer the original questions:

a. If $x = 1{,}000$, then $y = x^2 = 1{,}000^2 =$ **1,000,000**.

b. If $x = 1.2$, then $y = x^2 = 1.2^2 =$ **1.44**.

c. Notice that this part of the problem asks for the *x*-values, plural. I guess this means that a *y*-value of 625 might have two *x*-values associated with it. This seems reasonable from the table — if, for instance, given a *y*-value of 9, we notice that both 3 and –3 are *x*-values that yield a *y*-value of 9. In other words, if $y = 9$, then $x = 3$ or –3. So, what two

numbers, when squared, would yield a result of 625? Well, $25^2 = 625$, and of course $(-25)^2 = 625$.

In short, the following four ordered pairs lie on the curve:

(1000, 1000000)	(1.2, 1.44)	(25, 625)	(–25, 625)

Homework

7. The graph of $y = x^2$ lies within Quadrants ___ and ___ and at the origin.

8. Using the formula $y = x^2$, answer each question:

   a. If $x = 50$, then $y =$ ___.

   b. If $x = -25$, then $y =$ ___.

   c. If $x = 0$, then $y =$ ___.

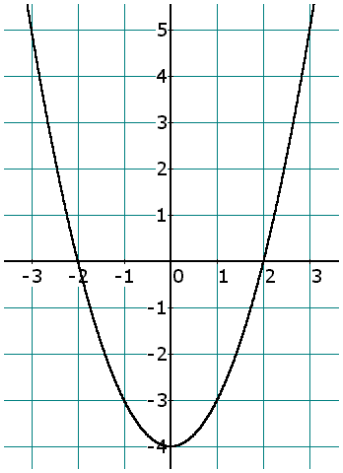
   d. If $y = 49$, then $x =$ ___ or ___.

   e. If $y = 144$, then $x =$ ___ or ___.

   f. If $y = -9$, then $x =$ ___.

9. If $y = 2x^2 - 3x - 1$, then what is y if $x = -5$?
   Recall: The Order of Operations requires that exponents be done before multiplication, which is to be done before any adding or subtracting.

10. Consider the graph at the right:



a. Fill in the following x-y table.

x	–3	–2	–1	0	1	2	3
y							

b. Find a formula that relates x and y.
Hint: There's an x^2 in the formula.

c. When $x = 100$, $y =$ ___.

d. When $x = -20$, $y =$ ___.

e. What is the lowest point on the graph?

f. How many y-values are associated with the x-value 1,000?

g. How many x-values are associated with the y-value 1,000?

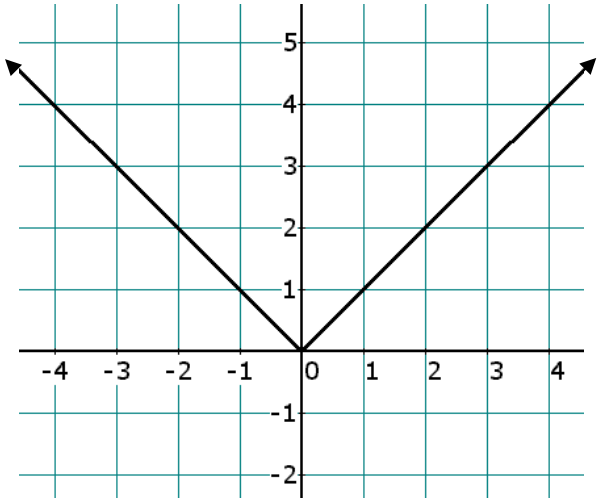
h. If $y = -3$, then $x =$ ___ or ___.

i. If $y = 96$, then $x =$ ___ or ___.

j. If $y = 221$, then $x =$ ___ or ___.

k. If $y = -6$, then $x =$ ___.

EXAMPLE 4: **Consider the following graph:**



a. Calculate the *y*-value when $x = 239$.

b. Calculate the *y*-value when $x = -777$.

c. Calculate the *x*-values when $y = 250$.

Solution: It's not a line and it's not curvy, so what is it? As usual, we begin with a table whose entries are read from the graph:

x	−4	−3	−2	−1	0	1	2	3	4
y	4	3	2	1	0	1	2	3	4

How is the *y*-value connected to the *x*-value? Well, when *x* is 0 or positive ($x \geq 0$), the *y*-value is the same as the *x*-value. But when *x* is negative ($x < 0$), the *y*-value is the opposite of the *x*-value.

Though the graph is straightforward (it's just the shape of the letter "V"), there's no simple, recognizable formula for the relationship between *x* and *y*, or is there? You might have learned about the notion of absolute value; we say that

y is the ***absolute value*** of *x*, and we write $y = |x|$.

Therefore, the vertical bar symbol around the *x* means "compute the absolute value of *x*," which means:

1. If $x \geq 0$, then $|x| = x$.
   The absolute value of a positive number or 0 is the number itself.

2. If $x < 0$, then $|x| = -x$.
   The absolute value of a negative number is the opposite of the number (thus making it positive).

We should now be ready to answer the following questions:

a. For $x = 239$, $y = |x| = |239| = \mathbf{239}$.

b. For $x = -777$, $y = |x| = |-777| = \mathbf{777}$.

c. We are asked for the x-values (plural) when $y = 250$. In other words, we have to solve the equation $\mathbf{250 = |x|}$, which asks: What numbers have an absolute value of 250? Well, since $|250| = 250$ and $|-250| = 250$, the two x-values are **250** and **–250**. We conclude that the following four points lie on the graph:

(239, 239)	(–777, 777)	(250, 250)	(–250, 250)

Be sure you can visualize the location of these four points on the graph.

Homework

11. The graph of $y = |x|$ lies within Quadrants ___ and ___ and at the origin.

12. Find the **absolute value** of each number:

a. 72 b. –99 c. 0 d. π e. –π f. $-\sqrt{2}$

13. Evaluate each expression:

a. $|17-7|$ b. $|3-25|$ c. $|2(3)-6(1)|$ d. $|2\pi+3\pi|$

14. Using the formula $y = |x|$, answer each question:

a. If $x = 33$, then $y =$ ____.

b. If $x = 0$, then $y =$ ____.

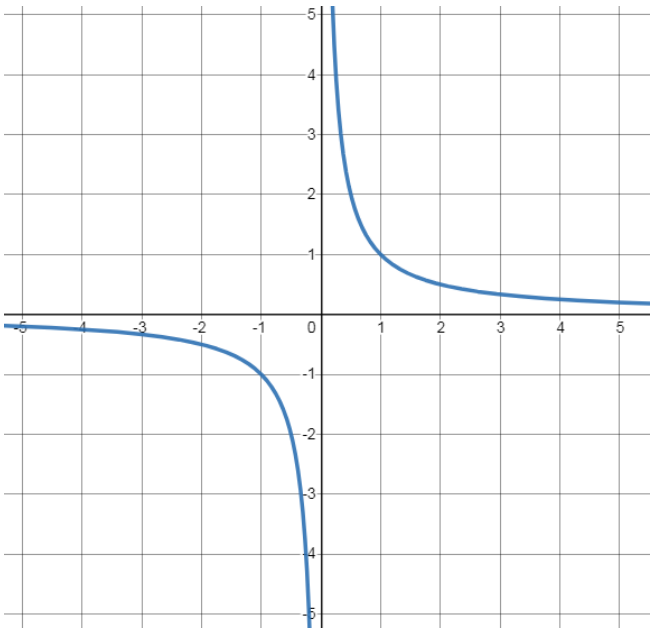
c. If $x = -25$, then $y =$ ____.

d. If $y = 17$, then $x =$ ____ or ____.

e. If $y = 0$ then $x =$ ____.

f. If $y = -5$, then $x =$ ____.

EXAMPLE 5: **Consider the following graph:**



a. Calculate the *y*-value when $x = 50$.

b. Calculate the *y*-value when $x = -1/10$.

c. Calculate the *y*-value when $x = 0$.

Solution: We have another curvy graph, but this one consists of two distinct pieces. The most important characteristic of this graph is that x is *never* 0; that is, the graph never touches the y-axis (even though it does get infinitely close!).

As in the two previous examples, the questions asked of us cannot be answered by looking at the graph — we must find a formula using the points we can see on the graph, and then use that formula to predict the y-values for the given x-values. To find some points on this graph, you'll have to trust me to a certain extent, because it's not easy to read fractional numbers from such a rough picture. See if you can agree with the following claims:

When $x = 1$, it's pretty clear that $y = 1$. When $x = 2$, y looks like it's about 1/2 (or 0.5). And trust me, when $x = 3$, $y = 1/3$, and when $x = 4$, $y = 1/4$.

Now let $x = 1/2$; do you see that $y = 2$? How about when $x = 1/3$, then $y = 3$? This quite nicely analyzes the first quadrant. Note again that x cannot be 0, because if you start at the origin (where $x = 0$), you can go up or down the y-axis as far as you'd like and never run into the graph. It's time for a summary:

x	1/3	1/2	1	2	3	4
y	3	2	1	1/2	1/3	1/4

What is going on here? Using the point (3, 1/3) as a guide, we conjecture that the y-value is found by dividing 1 by the x-value; in other words, flip over the x-value to get the y-value:

$$y = \frac{1}{x}$$

Let's check our formula for $x = 1$ and $x = 1/3$:

$$x = 1 \Rightarrow y = \frac{1}{x} = \frac{1}{1} = 1 \quad \checkmark$$

$$x = \frac{1}{3} \Rightarrow y = \frac{1}{x} = \frac{1}{\frac{1}{3}} = \frac{1}{1} \times \frac{3}{1} = 3 \quad \checkmark$$

Now for a check of a negative x-value. Consider $x = -3$; the formula gives

$$y = \frac{1}{-3} = -\frac{1}{3}, \text{ yielding the point } (-3, -1/3),$$

which seems very reasonable from the graph. We are now convinced that we have the right formula: $y = \frac{1}{x}$. We call this the ***reciprocal*** formula, and we'll use it to answer the original questions:

a. If $x = 50$, then $y = \frac{1}{x} = \mathbf{\frac{1}{50}}$, or **0.02**.

b. If $x = -1/10$, then $y = \frac{1}{x} = \frac{1}{-\frac{1}{10}} = \frac{1}{1} \times -\frac{10}{1} = \mathbf{-10}$.

c. If $x = 0$, then $y = \frac{1}{x} = \frac{1}{0}$, which is **undefined.** Notice that this is perfectly consistent with the earlier observation that x can never be 0 on the graph, and we see here that y can never be 0 in the formula, either.

Homework

15. The graph of $y = \frac{1}{x}$ lies entirely within Quadrants ___ and ___.

16. As x grows larger and larger, y is always (positive, negative), but getting (larger, smaller).

17. Using the formula $y = \frac{1}{x}$, answer each question:

a. If $x = 14$, then $y =$ ____.

b. If $x = \frac{2}{3}$, then $y =$ ____.

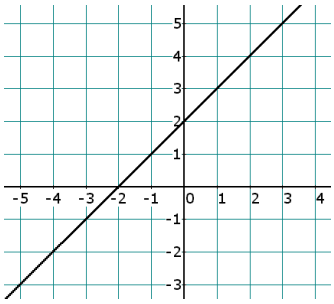
c. If $x = -99$, then $y =$ ____.

d. If $x = -\frac{5}{4}$, then $y =$ ____.

e. If $x = 0$, then $y =$ ____.

Review Problems

18. Consider the following graph:



a. When $x = 3$, $y =$ _____.

b. When $x = 99$, $y =$ _____.

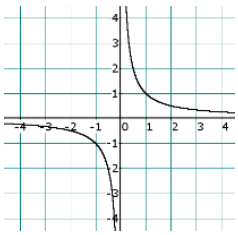
c. When $x = -45$, $y =$ _____.

d. When $y = 132$, $x =$ _____.

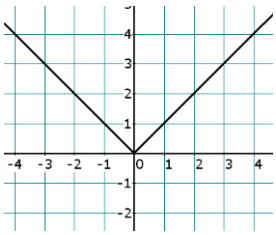
e. When $y = -33$, $x =$ _____.

19. What is the equation for each graph?

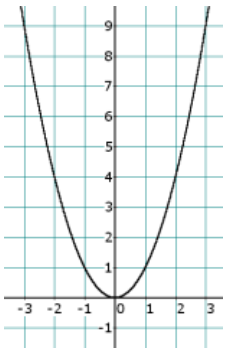
a.



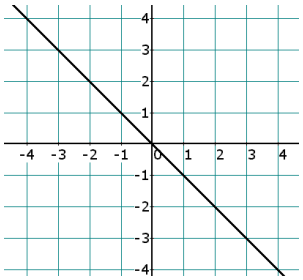
b.



c.



d.



20. Let $y = x^2$.

a. If $x = 17$, $y =$ _____

b. If $x = -13$, $y =$ _____

c. If $y = 16$, $x =$ _____

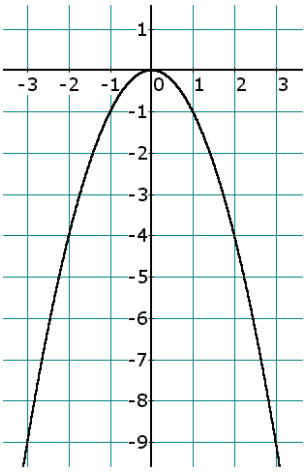
d. If $y = 0$, $x =$ _____

e. If $y = -9$, $x =$ _____

f. If $x = \pi$, $y =$ _____

g. Find the points on the graph where the two coordinates match.

21. Consider the graph:



a. If $x = 0$, then $y =$ _____

b. If $x = 10$, then $y =$ _____

c. If $x = -12$, then $y =$ _____

d. If $y = -225$, then $x =$ _____

e. If $y = 9$, then $x =$ _____

22. Evaluate: a. $|17|$ b. $|-\pi|$ c. $|0|$

23. Evaluate: $\dfrac{|-7|+|7|}{\frac{1}{2}+\frac{1}{3}}$

24. If $y = \dfrac{1}{x}$, and if $x = \dfrac{1}{10}$, what is y?

Solutions

1. a. 6 b. 0 c. –8 d. 556 e. –1800
f. 36π g. 5 h. 125 i. 11π

2. II and IV

3. a. –17 b. 0 c. 3.5 d. -8π e. $\sqrt{2}$

4. a. True b. 0 c. positive d. negative

5. a. –3 b. –94 c. 15 d. 10
e. 5 f. $-\pi + 5$ g. $\pi + 5$

6. a. –4 b. –2 c. 2 d. –27
e. 198 f. –34 g. 298

7. I and II

8. a. 2,500 b. 625 c. 0 d. 7, –7 e. 12, –12
f. No answer (no real number squared could be negative)

9. 64

10. a. 5 0 –3 –4 –3 0 5 b. $y = x^2 - 4$ c. 9,996
d. 396 e. (0, –4) f. 1 g. 2 h. –1 1 i. 10 –10
j. 15 –15 k. No answer

11. I and II

12. a. 72 b. 99 c. 0 d. π e. π f. $\sqrt{2}$

13. a. 10 b. 22 c. 0 d. 5π

14. a. 33 b. 0 c. 25 d. 17, –17 e. 0 f. No answer

15. I and III

16. positive; smaller

17. a. $\frac{1}{14}$ b. $\frac{3}{2}$ c. $-\frac{1}{99}$ d. $-\frac{4}{5}$ e. Undefined

18. a. 5 b. 101 c. –43 d. 130 e. –35

19. a. $y = \frac{1}{x}$ b. $y = |x|$ c. $y = x^2$ e. $y = -x$

20. a. 289 b. 169 c. ±4 d. 0 e. Does not exist
f. π^2 g. Hint: There are two such points.

21. a. 0 b. −100 c. −144 d. ±15 e. Does not exist

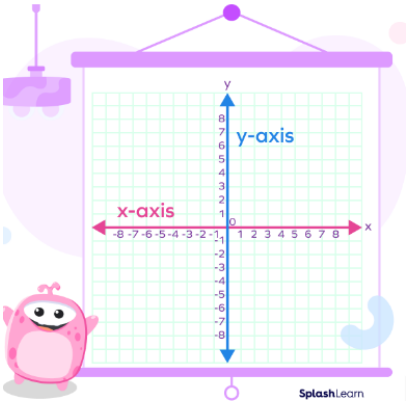
22. a. 17 b. π c. 0

23. $\frac{84}{5}$ **24**. 10

"The greatest use of life is to spend it for something that will outlast it."

William James (the father of modern psychology)

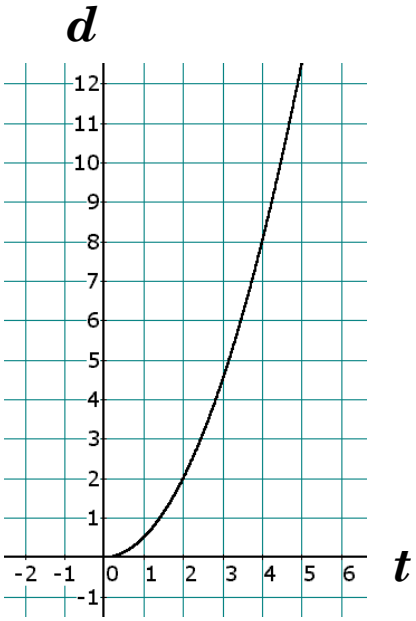
INTRO TO GRAPHING



We call the horizontal and vertical axes the *x*-axis and the *y*-axis. Why use these letters? Because ***x*** and ***y*** are classic algebra letters. They don't stand for anything in particular, so they can represent anything. But outside this classroom, people need better variables for the things they're analyzing.

For example, when graphing degrees Fahrenheit and degrees Celsius, we might use *F* and *C* as the labels for the axes. For a supply and demand curve from economics, why use *x* and *y* when we can use *S* and *D*?

EXAMPLE 1: **The graph shows the distance in meters (*d*) a car has traveled based on how long it's been traveling in seconds (*t*).**



a. Is the horizontal axis time or distance? time

b. As time increased, did the distance the car traveled increase or decrease? increase

c. What distance did the car travel after 4 sec? When $t = 4$, $d = 8$, so the car traveled 8 m.

d. About how many seconds did it take the car to travel 6 m? Looking at the d-axis and locating the value $d = 6$, we see that the t-value is about 3.4. So it took about 3.4 sec to travel the 6 m.

Homework

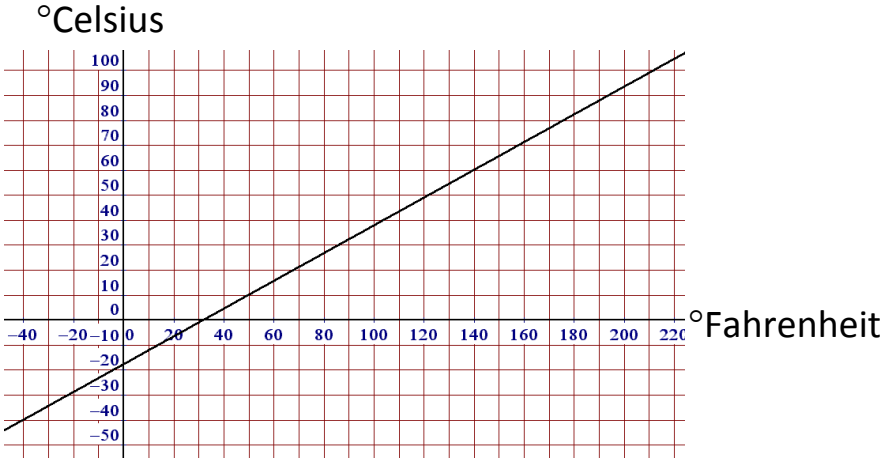
1. This problem is a continuation of Example 1.

   a. After 1 second, did the car move more than 1 m or less than 1 m?

   b. Why do you think there is no graph in the second quadrant?

   c. How far did the car travel between $t = 3$ and $t = 5$?

2. The graph below depicts the relationship between two temperature scales, Fahrenheit and Celsius.



   a. The Fahrenheit temperatures are on the (horizontal, vertical) axis.

   b. The Celsius temperatures are on the (horizontal, vertical) axis.

c. The graph passes through every quadrant except _____.

d. A temperature of 140°F corresponds to what temperature on the Celsius scale? Hint: Locate 140°F on the horizontal axis; then go straight up until you hit the line. Now go to the left until you hit the vertical axis. What Celsius temperature do you see?

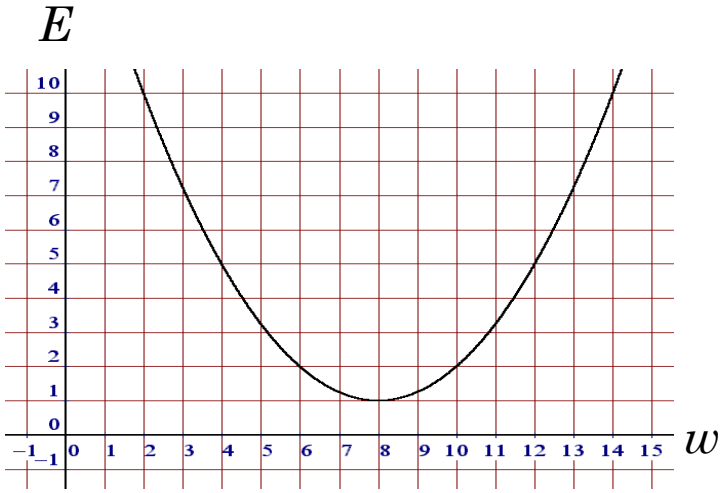
e. What Celsius temperature corresponds to 50°F?

f. 212°F is the boiling point of water (assuming pure water and normal atmospheric pressure). Use the graph to estimate the boiling point of water on the Celsius scale.

g. 0°C is the freezing point of water. Use the graph to estimate the freezing point of water on the Fahrenheit scale.

h. This one's kind of hard: There's exactly one point on the line where the Fahrenheit and Celsius temperatures match. What temperature is that?

3. The following graph shows the relationship between E, the expenses, and w, the number of widgets sold.



a. Find the expense when 2 widgets are sold.

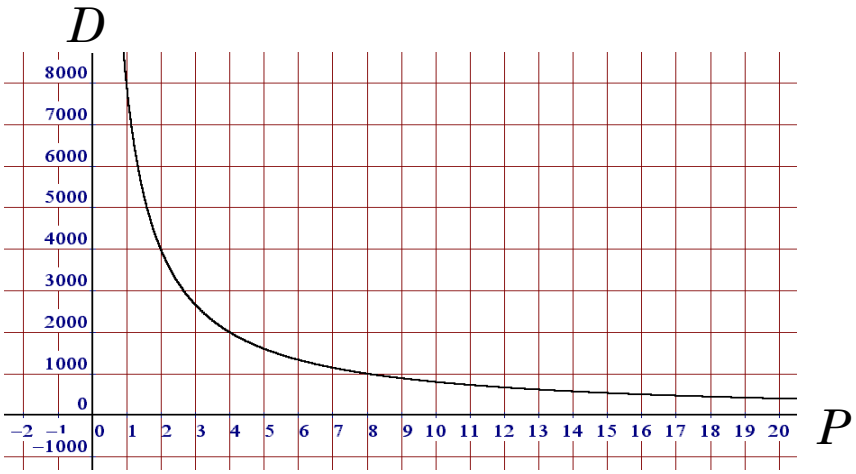
b. Find the expense when 8 widgets are sold.

c. Find the expense when 12 widgets are sold.

d. If the expense is \$2, what are the two possible values for the number of widgets sold?

e. What production level (what number of widgets) would produce the least expense?

4. This graph shows the relationship between the demand for a deluxe widget and its price:



a. If the price of a deluxe widget is \$4, what is the demand?

b. When the price is just \$1, how many widgets are expected to be sold?

c. If the price jumps to \$16, what is the expected demand?

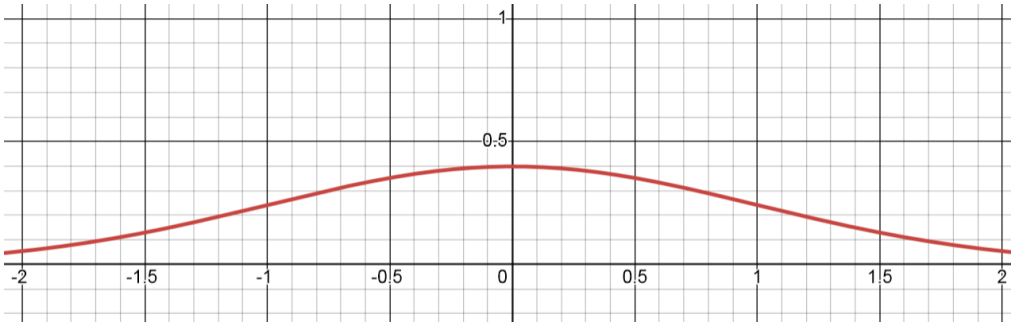
d. If the demand is 8000 widgets, what is the price of a widget?

e. As the price increases, the demand _____.

f. As the price decreases, the demand _____.

g. The entire graph is contained within which quadrant?

5. Now we come to the famous *bell-shaped curve*, one of the most important graphs in probability, statistics, business, economics, psychology, biology, and all the physical sciences.



a. When $x = 0$, y is a approximately ___.

b. The graph reaches its maximum y-value when $x =$ ___.

c. When $x = 0.5$, y is approximately ___.

d. When $x = -0.5$, y is approximately ___.

e. Assume that the point (3, 0.006) lies on the bell-shaped curve. What is the y-value when $x = -3$?

f. Assume that the graph goes forever to the right and forever to the left. As x moves farther and farther to the right, the y-value is always (positive, negative) and gets infinitely close to ____.

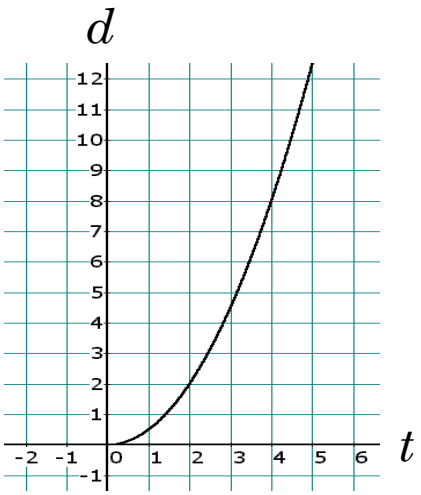
g. As x moves farther and farther to the left, the y-value is always (positive, negative) and gets infinitely close to ____.

h. The graph has symmetry with respect to which axis?

Review Problems

6. The point $(\pi, -\pi)$ lies in which quadrant?

7. The graph below shows the distance in meters a car has traveled based on how long it's been traveling in seconds.



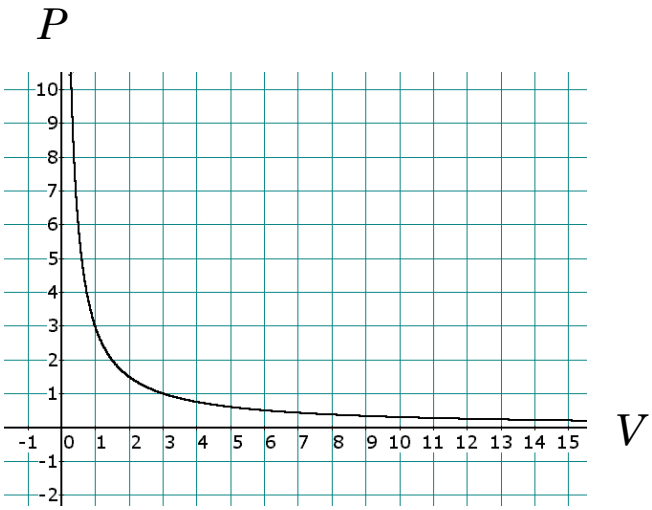
a. After 5 seconds, the car had traveled _____ meters.

b. Is the vertical axis time or distance?

c. After 3 seconds, the car had traveled _____ meters.

d. Between $t = 2$ and $t = 4$, how far did the car travel?

8. The following graph shows the relationship between the pressure and the volume of a gas:



a. When the volume is 3, the pressure is ____.
b. When the volume is 6, the pressure is ____.
c. As the volume increases, the pressure _____.
d. As the volume decreases, the pressure _____.

Solutions

1. a. Looking at the point where $t = 1$, we see that the d-value is between 0 and 1, so the car moved less than 1 m.

b. A point in the second quadrant would indicate a negative time value, and we usually measure time starting at zero.

c. The distance traveled at $t = 3$ is around 4.5 m, and the distance at $t = 5$ is about 12.5. Thus, the distance traveled between the two times was about 8 m.

2. a. horizontal b. vertical c. II d. 60°C e. 10°C f. 100°C
g. 32°F h. –40°

3. a. \$10 b. \$1 c. \$5 d. 6, 10 e. 8

4. a. 2000 widgets b. 8000 widgets c. 500 widgets d. \$1
e. decreases f. increases g. I

5. a. 0.4 b. 0 c. 0.35 d. 0.35 e. 0.006
f. positive; 0 g. positive; 0 h. y-axis

6. IV

7. a. about 12 b. distance c. 4.5 d. 6 m

8. a. 1 b. 0.5 c. decreases d. increases

"A well-educated mind will always have more questions than answers."



Helen Keller (1880–1968)

GRAPHING LINES – REVISITED

❑ *INTRODUCTION*

Now that we've studied intercepts, slopes, and the equations of lines, we have a couple of new ways to graph lines.

❑ *USING THE INTERCEPTS*

❑ *USING THE SLOPE AND Y-INTERCEPT*

Homework

Review Problems

Solutions

- ***To ∞ and Beyond!***

GRAPHING LINES



Consider the equation

$y = 2x - 1$

If we let $x = 10$, for instance, we can calculate the corresponding value of y:

$$y = 2(\mathbf{10}) - 1 = 20 - 1 = \underline{19}$$

Thus, a solution of the equation is the pair $x = 10$, $y = 19$, which we can write as the ordered pair **(10, 19)**. Another solution to this equation is **(0, –1)**; in fact, there are infinitely many solutions. Our goal now? Make a graph (a picture) of ALL the solutions of the equation.

❑ *GRAPHING A LINE*

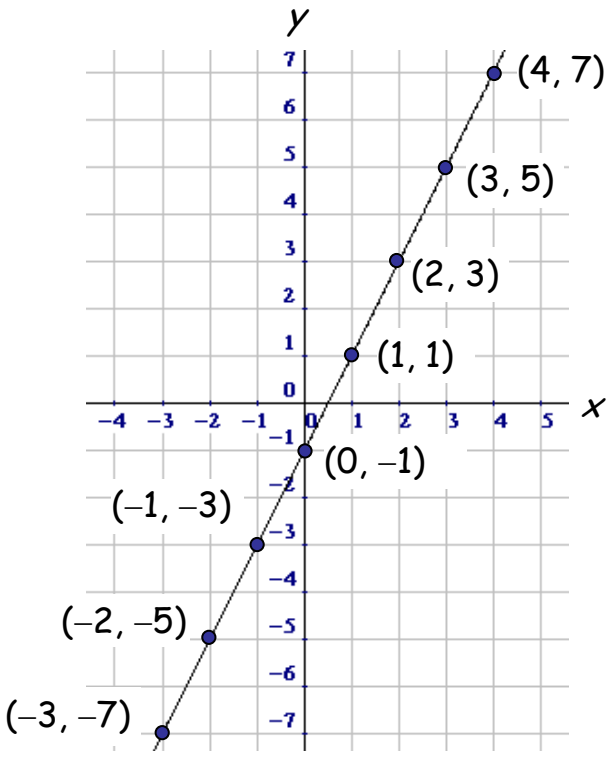
EXAMPLE 1: **Graph the line $y = 2x - 1$.**

Solution: The most important aspect of graphing is to learn where the x-values come from. They basically come from your head — you get to make them up! Certainly, this process will become more sophisticated later in your math studies, but for now, just conjure them up from your imagination.

I'm going to choose x-values of –3, –2, –1, 0, 1, 2, 3, and 4. For each of these x-values, I will calculate the corresponding y-value using the given formula, $y = 2x - 1$. Organizing all this information in a table is useful:

x	$y = 2x - 1$	(x, y)
–3	$2(\mathbf{-3}) - 1 = -6 - 1 = -7$	(–3, –7)
–2	$2(\mathbf{-2}) - 1 = -4 - 1 = -5$	(–2, –5)
–1	$2(\mathbf{-1}) - 1 = -2 - 1 = -3$	(–1, –3)
0	$2(\mathbf{0}) - 1 = 0 - 1 = -1$	(0, –1)
1	$2(\mathbf{1}) - 1 = 2 - 1 = 1$	(1, 1)
2	$2(\mathbf{2}) - 1 = 4 - 1 = 3$	(2, 3)
3	$2(\mathbf{3}) - 1 = 6 - 1 = 5$	(3, 5)
4	$2(\mathbf{4}) - 1 = 8 - 1 = 7$	(4, 7)

Now I will plot each of the eight points just calculated in 2-space, that is, on an x-y coordinate (Cartesian) system. Then the points will be connected with the most reasonable graph, a straight line.



Notice that each (x, y) point is just one of the many solutions of the equation $y = 2x - 1$. Thus, each time we plot one of these points, we're plotting a solution of the equation. Also, whatever graph we get when we're done plotting all possible points is simply a picture of all of the solutions of the equation. So, in essence, we'll have a graph of the equation $y = 2x - 1$.

Notice that the graph passes through every quadrant except the second; also notice that as we move from left to right (that is, as the x's grow larger) the graph rises (increases).

Final Comments on Example 1:

We could let $x = 1{,}000{,}000$ for this equation, in which case y would equal $2(\mathbf{1{,}000{,}000}) - 1 = 1{,}999{,}999$. That is, the point (1000000, 1999999) is on the line. Can we graph it? Not with the scale we've selected for our graph. But if we traveled along the line — up and up and up and way up — we would eventually run into the point (1000000, 1999999).

We could also have used fractions like 2/7 for x. This would have given us the point (2/7, –3/7), which is in Quadrant IV. This point, too, is definitely on the line.

And we may even have used a number like π for x, in which case we would obtain the point $(\pi, 2\pi - 1)$. This point is impossible to plot precisely, but I guarantee that it's in the first quadrant (both coordinates are positive), and it's on the line we've drawn.

In short, every solution of the equation $y = 2x - 1$ is a point on the line, and every point on the line is a solution of the equation.

EXAMPLE 2: **Graph the line $y + x = 2$.**

Solution: The equation of the line will be easier to work with if we first solve for y by subtracting x from each side of the equation:

$$y = 2 - x$$

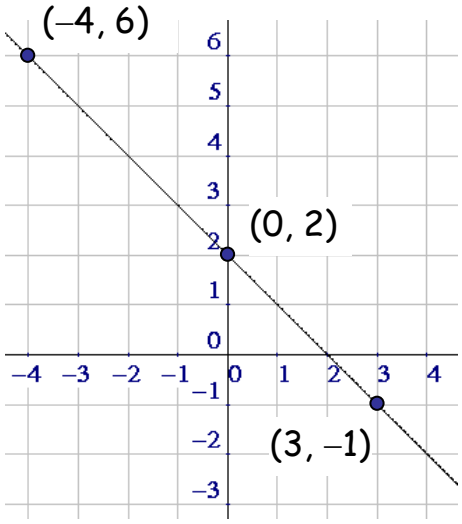
It's also better form (you'll see why later) if we put the $-x$ term right after the equal sign. The line is now

$$y = -x + 2$$

If $x = 0$, then $y = -\mathbf{0} + 2 = 2$. Thus, **(0, 2)** is on the line.

If $x = 3$, then $y = -\mathbf{3} + 2 = -1$. So **(3, –1)** is on the line.

If $x = -4$, then $y = -(-4) + 2 = 4 + 2 = 6$. Therefore, **(–4, 6)** is on the line. If we plot these three points, and then connect them together, we get a graph like the following:



Notice that the graph passes through every quadrant except the third. Also note that as we move from left to right (that is, as the *x*'s grow larger) the graph falls (decreases).

Note: It's clear that two distinct (different) points completely determine a line, and so some students plot exactly two points, connect them with a straight line, and they're done. Warning: You're taking a big gamble when you plot just two points. If you make an error with either point, you'll get the wrong line, and there will be no way for you to know that you goofed. Even worse, what if the equation isn't even a line in the first place? Plotting just two points (even if they're both correct) will not suffice to graph a curve that is not a simple straight line (like a circle, for instance).

The more points you plot, the better!

Homework

1. a. Does the point (3, 19) lie on the line $y = 6x + 1$?
   b. Does the point (–1, 5) lie on the line $y = -2x + 4$?
   c. Does the point (4, –3) lie on the line $2x + 5y = 3$?
   d. Does the point (–2, –4) lie on the line $x - 3y = 10$?

2. Graph each of the following lines:

   a. $y = x$ b. $y = -x$ c. $y = 2x$

   d. $y = -3x$ e. $y = 1.5x$ f. $y + 0.5x = 0$

3. Graph each of the following lines:

   a. $y = x + 3$ b. $y = x - 2$ c. $y = -x + 1$

   d. $y = 2x + 3$ e. $y = 3x - 4$ f. $y = -2x - 1$

   g. $y + 2x = 1$ h. $y - x = 2$ i. $y - 2x + 1 = 0$

   j. $y = 0.5x - 1$ k. $y = -1.5x - 2$ l. $y = -3x - 3$

Review Problems

4. a. If $y = -7x + 10$, and if $x = 10$, then $y =$ ___.
   b. If $y = 13x$, and if $x = \pi$, then $y =$ ___. (give the exact answer)
   c. Describe the line $y = x$.
   d. Describe the line $y = -x$.

5. Graph: $y = -3x - 1$

6. Sketch each line and determine the only quadrant that the line does <u>not</u> pass through:

a. $y = 2x + 7$ b. $y = 3x - 9$ c. $y = -2x + 7$

d. $y = -5x + 1$ e. $y = -x - 9$ f. $y = \pi x - \sqrt{2}$

7. Determine whether or not the graph of the given equation passes through the *origin* [the point (0, 0)].

a. $y = x + 1$ b. $y = x$ c. $y = x - 5$

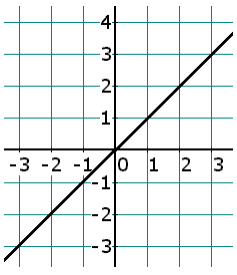
d. $y = 7x$ e. $y = x^2$ f. $y = x^2 + 1$

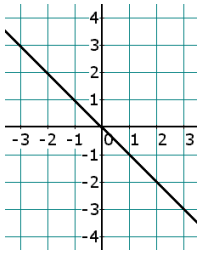
g. $y = 7 - x$ h. $y = -x$ i. $y = x^3$

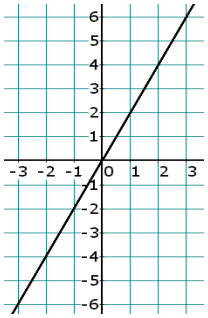
j. $y = \frac{x}{3}$ k. $y = \frac{x+1}{2}$ l. $y = 0.3x^2$

Solutions

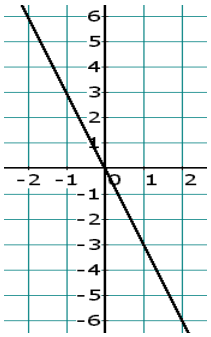
1. a. Yes b. No c. No d. Yes

2. a. 

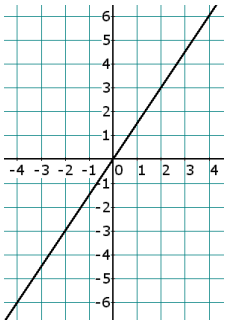
b. 

c. 

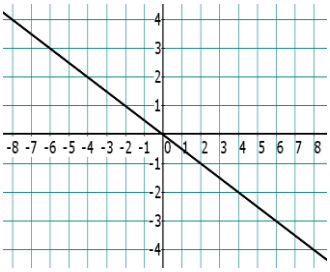
d.



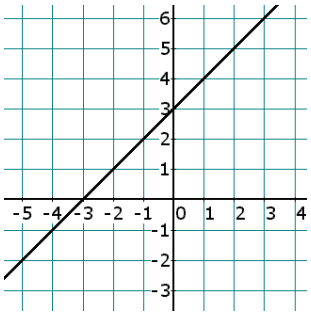
e.



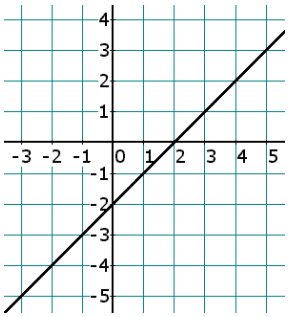
f.



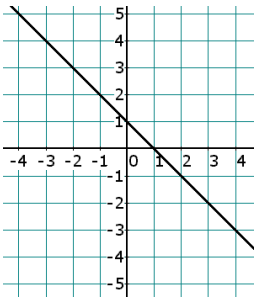
3. a.



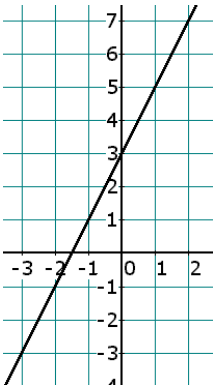
b.



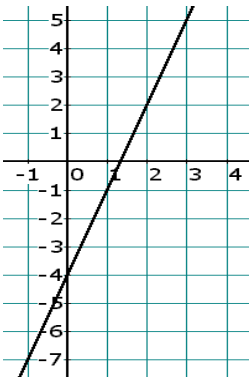
c.



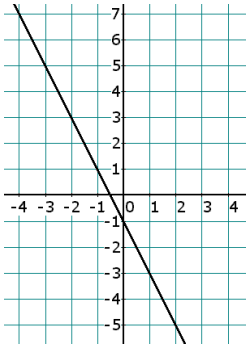
d.



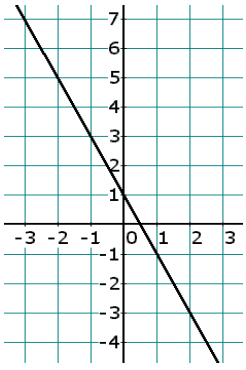
e.



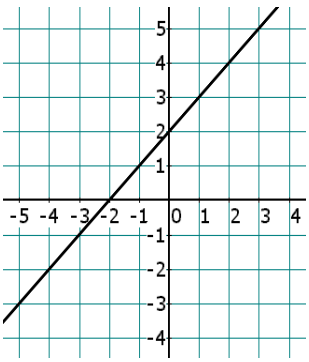
f.



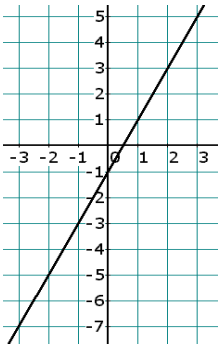
g.

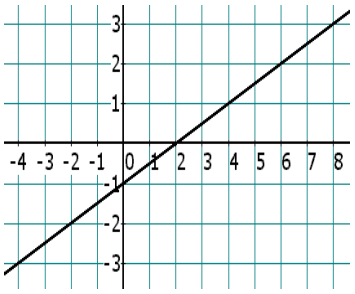


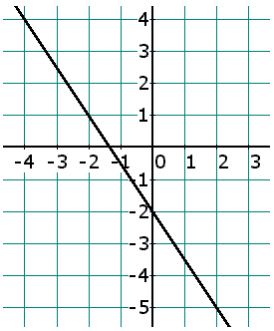
h.

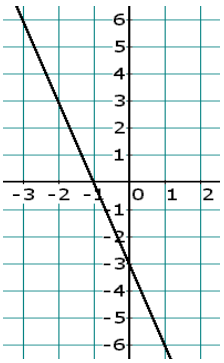


i.



j. 

k. 

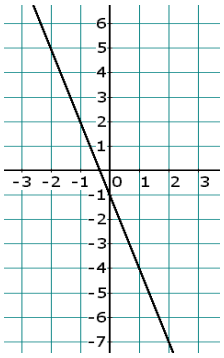
l. 

4. a. –60

b. 13π

c. diagonal, thru the origin, going thru Quadrants I and III

d. diagonal, thru the origin, going thru Quadrants II and IV

5. 

6. a. IV b. II c. III d. III e. I f. II

7. a. No b. Yes c. No d. Yes e. Yes f. No
g. No h. Yes i. Yes j. Yes k. No l. Yes

❑ *TO ∞ AND BEYOND*

A. Graph: $y = x^2$

B. Graph: $x^2 + y^2 = 25$ Think 0's & 5's; then think 3's & 4's.

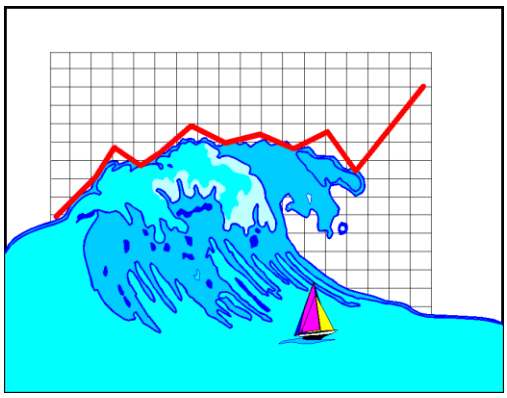
"To the world,
you might be just one person,
but to one person,
you might just be the world."

– Theodore Geisel (Dr. Seuss)



CH 15 – GRAPHING

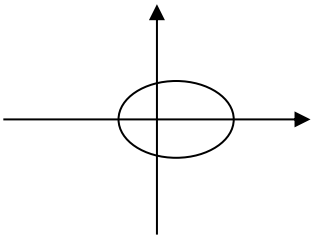
Now we tie together some of the ideas from previous chapters – symmetry, domain, intercepts – into one concept: the transformation of an equation with one or two variables into a picture in 2-dimensional space.



❑ Symmetry

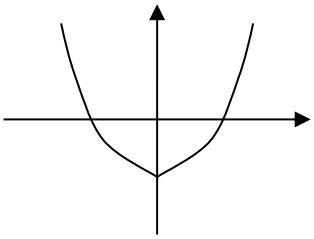
In the previous chapter, The Real Plane, we discussed what it meant for a pair of points to have some kind of symmetry. Now we extend this concept for any graph in the real plane.

_x_-axis symmetry



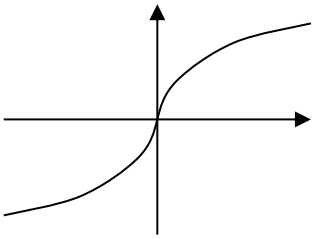
Given any point on the graph, there's an equidistant point on the other side of the _x_-axis.

_y_-axis symmetry



Given any point on the graph, there's an equidistant point on the other side of the _y_-axis.

origin symmetry



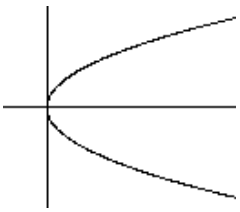
Given any point on the graph, there's an equidistant point on the other side of the origin.

The Litmus Tests for Symmetry

Now that we can visualize the three types of symmetry, we need a way to test a given equation for symmetry without having to graph it.

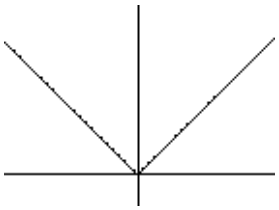
x-axis Replace y with $-y$ everywhere in the given equation. If the new equation is equivalent to the old one, then the graph of the equation has x-axis symmetry.

For example, consider the sideways-opening parabola $x = y^2$. Replacing y with $-y$ gives the equation $x = (-y)^2$, which is the same as $x = y^2$. Since the replacement didn't really change the equation, we have x-axis symmetry.



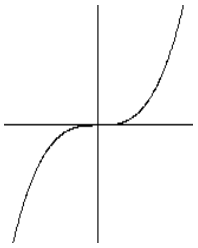
y-axis Replace x with $-x$ everywhere in the given equation. If the new equation is equivalent to the old one, then the graph of the equation has y-axis symmetry.

For instance, look at the absolute value function $y = |x|$. Replacing x with $-x$ gives the equation $y = |-x|$, which is actually the same as $y = |x|$ (since the absolute values of a number and its opposite are the same). Thus, the graph has y-axis symmetry.



origin Replace x with $-x$ and replace y with $-y$ everywhere in the equation. If the new equation is equivalent to the old one, then the graph of the equation has origin symmetry.

Consider the cubic function $y = x^3$. Replacing x with $-x$ and y with $-y$ gives the equation $-y = (-x)^3$, which implies that $-y = -x^3$, which implies that $y = x^3$, so there was no net change in the equation. Therefore, the graph has origin symmetry.



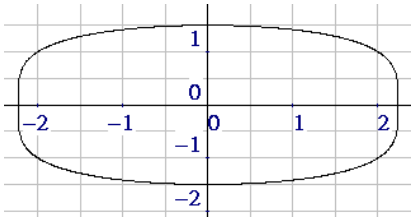
EXAMPLE 1: **Find all the symmetries of $x^2 + y^4 = 5$.**

Solution:

x-axis Replace y with $-y$: $x^2 + (-y)^4 = 5 \Rightarrow$ $x^2 + y^4 = 5$, the same as the original equation; therefore the graph has x-axis symmetry.

y-axis Replacing x with $-x$ will again result in the same equation; thus the graph has y-axis symmetry.

origin Replace both variables with their negatives: $(-x)^2 + (-y)^4 = 5 \Rightarrow x^2 + y^4 = 5$, the original equation; we conclude that the graph has origin symmetry as well.



EXAMPLE 2: **Find all the symmetries of $y^2 = x^2 + y^3$.**

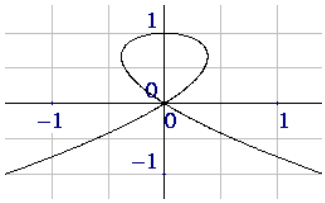
Solution:

x-axis Replace y with $-y$: $(-y)^2 = x^2 + (-y)^3$
$\Rightarrow y^2 = x^2 - y^3$, which is not the same as the original equation, nor can it be made to be the same. Thus, the graph does not have x-axis symmetry.

y-axis Replace x with $-x$: $y^2 = (-x)^2 + y^3$
$\Rightarrow y^2 = x^2 + y^3$, which is the same as the original equation. Therefore, the graph has y-axis symmetry.

origin Replace x with $-x$ and y with $-y$: $(-y)^2 = (-x)^2 + (-y)^3 \Rightarrow$ $y^2 = x^2 - y^3$, which is not the same as the original equation -- and will never be -- no matter what we do to each side of the

equation. We conclude that the graph does not have origin symmetry.



In summary, the graph has y-axis symmetry only.

Homework

1. Sketch a line with x-axis symmetry, a line with y-axis symmetry, and a line with origin symmetry.

2. a. Sketch a parabola with y-axis symmetry.
   b. Sketch a parabola with x-axis symmetry.

3. Sketch a circle which has x-axis symmetry, but neither of the other two.

4. Sketch a circle which has y-axis symmetry, but neither of the other two.

5. Sketch a circle which has origin symmetry, but neither of the other two.

6. A graph with origin symmetry has some points in the 2nd quadrant. Which other quadrant must contain some points of the graph?

7. Use the litmus tests to find the symmetries of each graph:

   a. $x^2 - y^2 = 1$
   b. $y = x^2 + x - 1$
   c. $y = x^2 - 8$
   d. $x = |y|$
   e. $y = \frac{1}{x}$
   f. $y = x^6 - x^4 + x^2$

8. If a graph has x-axis symmetry and y-axis symmetry, does that imply that it has origin symmetry? What about the converse?

❑ Absolute Values and Square Roots

EXAMPLE 3: **Graph: $y = |x - 3| + 2$**

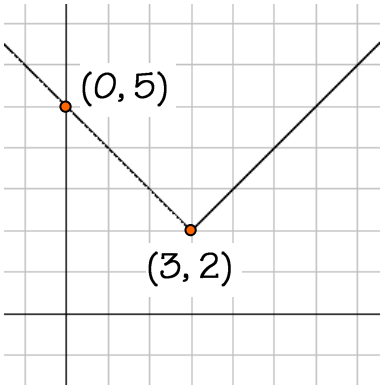
Solution: First, we'll analyze the **domain**, so that we'll know what values of x we'll be allowed to use. Could anything go wrong with the absolute value formula? No matter what real number x is, you can always subtract 3 from it, take the absolute value, and then add 2. In other words, the domain is all real numbers, $\oplus$.

Second, it appears that we have a **function**, since a given x will produce just one y-value.

Third, we seek the intercepts. Setting x to 0 gives $y = |0 - 3| + 2 = 5$. The y-intercept is therefore (0, 5). Now we set $y = 0$, which gives us $0 = |x - 3| + 2 \Rightarrow |x - 3| = -2$, which is impossible. Thus, there are no x-intercepts.

Now we'll plot lots of points, and see what we end up with.

x	y
–1	6
0	5
1	4
2	3
3	2
4	3
5	4
6	5
7	6



$y = |x - 3| + 2$

Notice that the graph is in the shape of a "V." It is sharp at its bottom point (3, 2), not smooth and curvy like a parabola. It

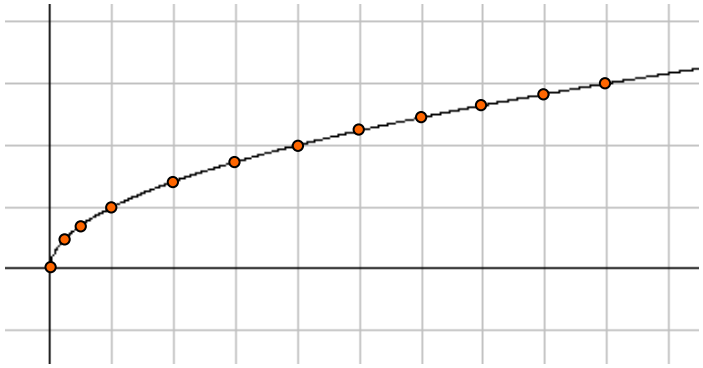
appears from the graph that it's a **function**. Note also that the graph appears to have no ***x*-intercept**, as we showed even before we plotted points.

EXAMPLE 4: **Graph:** $f(x) = \sqrt{x}$

Solution: First we note that f is really a **function**. Since the radical sign, $\sqrt{\ }$, represents the non-negative square root only, there can be only one y-value for a given x in the domain.

We now determine the **domain** of this function. Then we'll know which real numbers we can use for x and which we can't. The square root of a quantity is a real number only when the quantity is greater than or equal to zero. That is, the domain is $\{x \in \oplus \mid x \geq 0\}$, or $[0, \infty)$. So we can use 0 or any positive number for x, but we can't use any negative values for x.

x	y
0	0
0.25	0.5
0.5	0.707
1	1
2	1.41
3	1.73
4	2
5	2.24
6	2.45
7	2.65
8	2.83
9	3



$y = \sqrt{x}$

It may be hard to tell, but the graph we've just drawn is actually the top half of a sideways parabola. The origin is the only **intercept** of this graph; it's both an x-intercept and a y-intercept.

Homework

9. Graph $f(x) = |x|$, $g(x) = |x - 3|$, and $h(x) = |x + 2|$ on the same grid. Explain how the –3 and the 2 affect the graph of f.

10. Graph $f(x) = |x|$, $g(x) = |x| + 4$, and $h(x) = |x| - 2$ on the same grid. Explain how the 4 and the –2 affect the graph of f.

11. Graph $f(x) = |x|$, $g(x) = 3|x|$, and $h(x) = \frac{1}{2}|x|$ on the same grid. Explain how the 3 and the $\frac{1}{2}$ affect the graph of f.

12. Graph $f(x) = -|x|$ and compare it to the graph of $|x|$.

13. Graph $f(x) = |x + 2| - 3$.

14. Graph $g(x) = -|1 - x| + 2$.

15. Graph $h(x) = \sqrt{x+4}$.

16. Graph $y = \sqrt{x+2} - 3$.

17. Graph $y = \sqrt{-x}$.

18. Graph $y = -\sqrt{3-x}$.

19. We've seen the graph that is the top half of a sideways parabola. Can you find an equation whose graph will be the bottom half of the sideways parabola? What about an equation that will represent the entire sideways parabola?

❑ Cube Roots and Other Graphs

The graph of the square root function, $y = \sqrt{x}$, is limited by the fact that the domain of this function is [0, ∞), since $\sqrt{x}$ results in a real number only if $x \geq 0$. The cube root doesn't have this restriction, so it makes for an interesting graph.

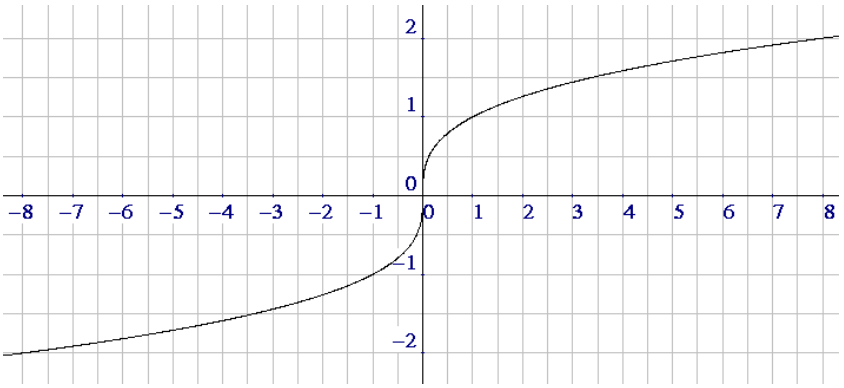
EXAMPLE 5: **Analyze and graph $y = \sqrt[3]{x}$.**

Solution: First we look at the **domain**. The cube root of a positive number is positive (e.g., $\sqrt[3]{8} = 2$), the cube root of 0 is 0, and the cube root of a negative number is negative (e.g., $\sqrt[3]{-1} = -1$). There doesn't seem to be any number whose cube root we couldn't legally take. It appears that the domain of this function is ⊕.

As for **intercepts**, letting $x = 0$ produces 0 for y, and vice versa. Thus, the only intercept is (0, 0), the origin.

Here are a few **ordered pairs** for this function. Plotting them will give the graph that follows.

(27, 3) (8, 2) (4, 1.59) (1, 1) (0, 0) (–1, –1)
(–6, –1.82) (–8, –2) (–27, – 3) (1/8, 1/2) (–1/8, –1/2)



Notice that at the origin the graph seems to be practically vertical. We could say that the tangent line at the origin has infinite slope.

This graph has origin **symmetry**. First of all, it looks like it does. Second, changing x to $-x$ and y to $-y$ produces the equation

$$-y = \sqrt[3]{-x} \quad \Rightarrow \quad -y = -\sqrt[3]{x} \quad \Rightarrow \quad y = \sqrt[3]{x}\,.$$

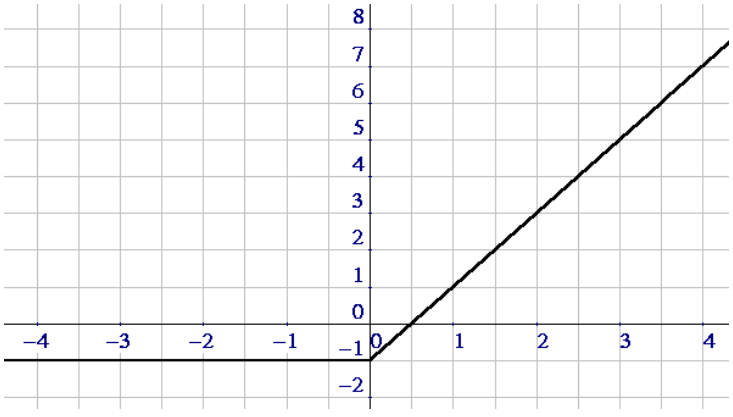
Third, notice that for each point (x, y) on the graph, the point $(-x, -y)$ is also on the graph. Anyway you look at it, the graph possesses origin symmetry.

EXAMPLE 6: **Graph $y = x + |x| - 1$.**

Solution: Noting that the domain of the function is $\oplus$, let's plot a lot of points and see what we get.

x	–4	–3	–2	–1	0	1/2	1	2	3	4
y	–1	–1	–1	–1	–1	0	1	3	5	7

Before we graph the function, notice that if $x \le 0$, y has the constant value –1. Thus, part of the graph will be horizontal. We can also see that there's an x-intercept at (1/2, 0) and a y-intercept at (0, –1). Let's put it all together:



Notice the sharp turn at the y-intercept (0, –1).

Homework

Graph each function:

20. $y = \sqrt[3]{x+2}$ 21. $y = \sqrt[3]{x-3}$ 22. $y = \sqrt[3]{x}+3$

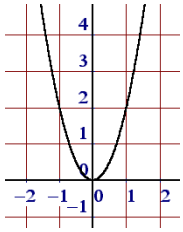
23. $y = \sqrt[3]{x}-4$ 24. $y = \sqrt[3]{x+1}+2$ 25. $y = x + |x|$

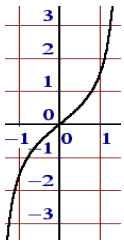
26. $y = x - |x|$ 27. $y = 2x - |x|$ 28. $y = x^2 - |4x|$

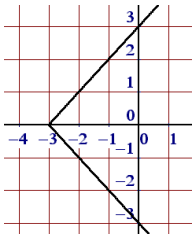
29.** Graph the function $y = \dfrac{x^2-4}{x-2}$.

Practice Problems

30. Find all the symmetries:

a. 

b. 

c. 

d. $x = |y| + 5$ e. $y = x^8 + x^4 - x^2$ f. $y = \dfrac{3}{x}$

31. Find all the symmetries of $x^2 + y^2 = 99$.

32. Find all the symmetries of $x - y^2 = 1$.

33. Graph $y = \sqrt{x-4}+3$ and state its domain.

34. Graph $y = |x+2|-1$ and state its domain.

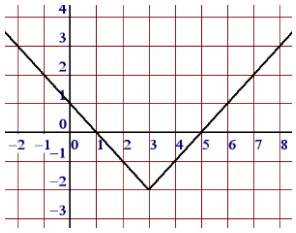
35. Graph $y = \sqrt[3]{x+3}$ and state its domain.

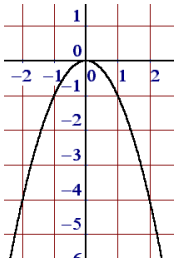
36. Graph $y = -\sqrt[3]{2-x}$ and state its domain.

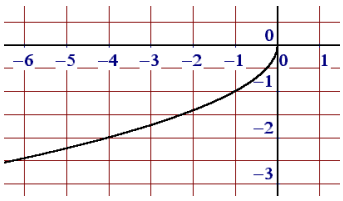
37. Graph $y = -\sqrt{1-x}+2$ and state its domain.

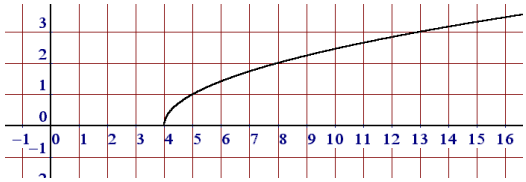
38. Graph $y = \dfrac{|x|}{x}$ and state its domain.

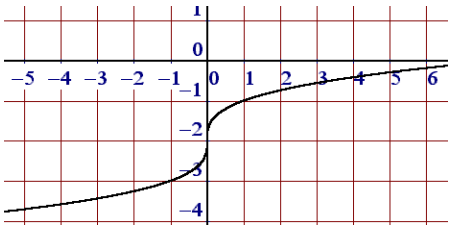
39. Find the equation of each function:

a. 

b. 

c. 

d. 

e. 

40. True/False:

a. The graph of $y = x^4 - 7x^2 + 1$ has y-axis symmetry.

b. The graph of $x^2 + y^2 = 2$ has all three symmetries.

c. The graph of $y^2 - y^3 = x^2$ has origin symmetry.

d. A circle may have x-axis symmetry only.

e. The graph of $y = \dfrac{1}{x^5}$ has origin symmetry.

f. The "turning point" of the graph of $y = |x - 3| + 2$ is (2, 3).

g. The graph of $g(x) = \sqrt{x}$ is the top-half of a sideways parabola.

h. The graph of $y = \sqrt{-x}$ is just the origin.

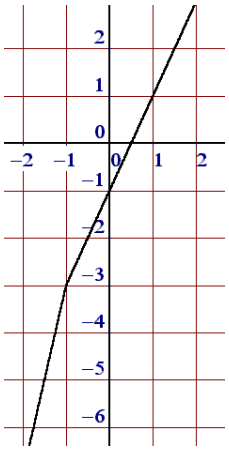
i. The graph of $y = \frac{1}{3}|3x - 10|$ is "skinnier" than the graph of $y = |3x - 10|$.

j. The graphs of $f(x) = |x - 2|$ and $g(x) = |2 - x|$ are identical.

k. The graph of $y = \sqrt[3]{x}$ has an infinite slope at the origin.

l. The graph of $y = \sqrt[3]{x + 2}$ passes through all the quadrants except the second.

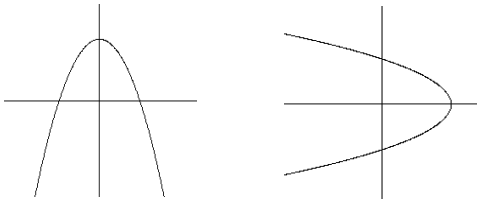
m. The graph of the function $y = 3x - |x + 1|$ is



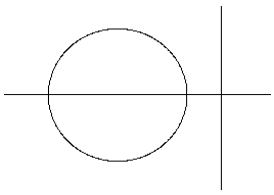
Solutions

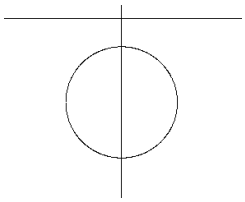
1. any vertical line; any horizontal line; any line through the origin

2.



3.



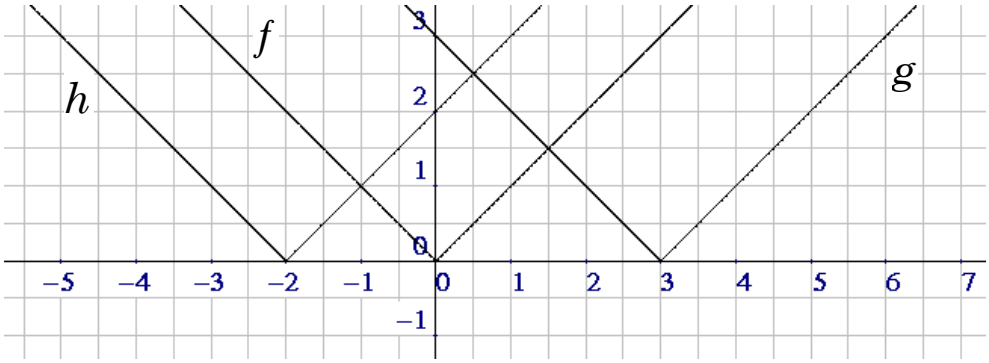
4. 

5. Impossible **6.** Quadrant IV

7. a. all b. none c. y-axis d. x-axis e. origin f. y-axis

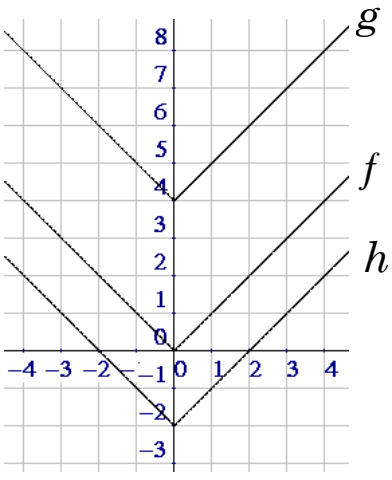
8. Yes; but the converse is false -- consider $y = 2x$. It has origin symmetry but neither of the other two.

9.



g is f shifted 3 units to the right. h is f shifted 2 units to the left. Generalization: The graph of $|x + a|$ is the graph of $|x|$ shifted a units to the right if $a < 0$ and to the left if $a > 0$.

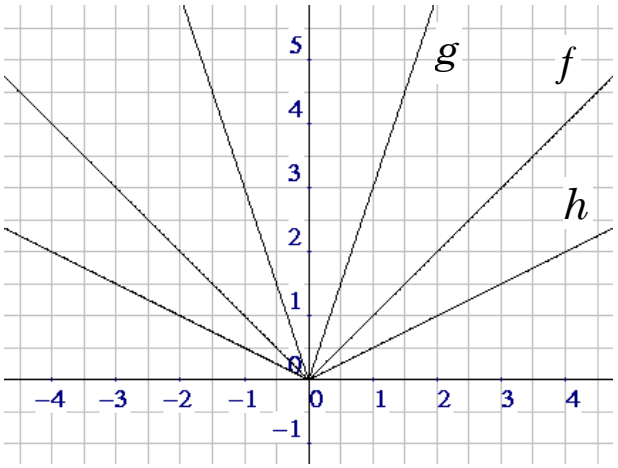
10.



g is shifted up 4 units; h is shifted down 2 units.

Generalization: The graph of $|x| + a$ is the graph of $|x|$ shifted a units up if $a > 0$ and a units down if $a < 0$.

11.

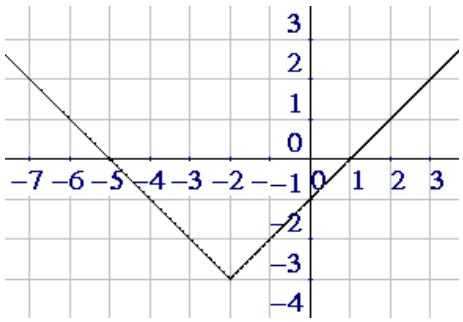


The 3 makes the graph skinnier;

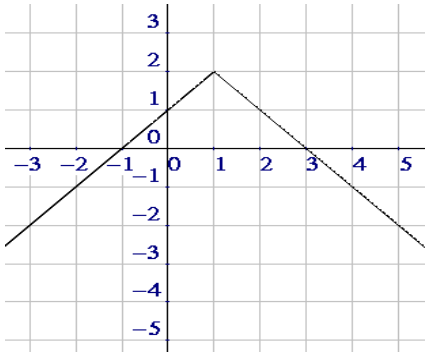
the $\frac{1}{2}$ makes the graph wider.

12. The graph of f is the graph of $|x|$ but flipped over the x-axis.

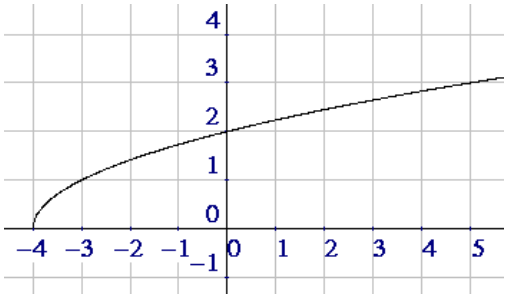
13.



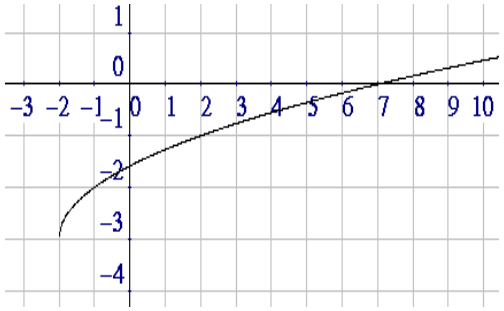
14.



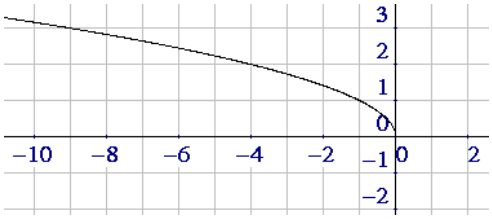
15.



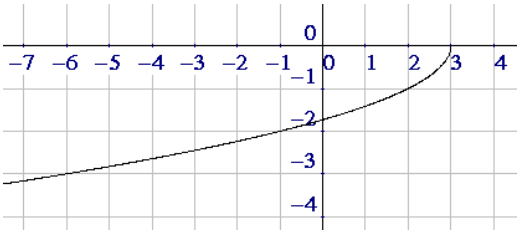
16.



17.

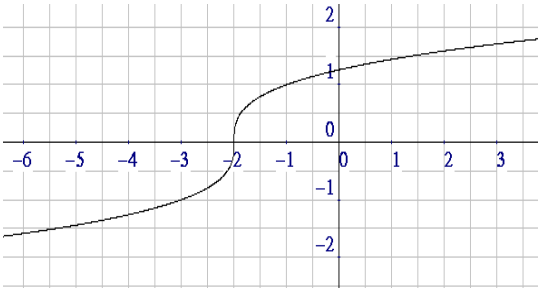


18.

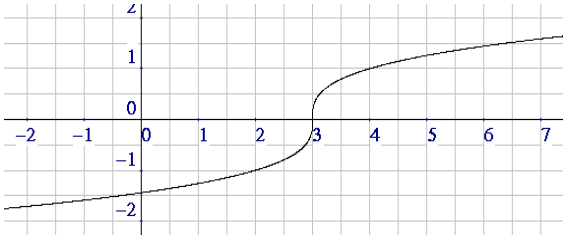


19. What equations did you come up with?

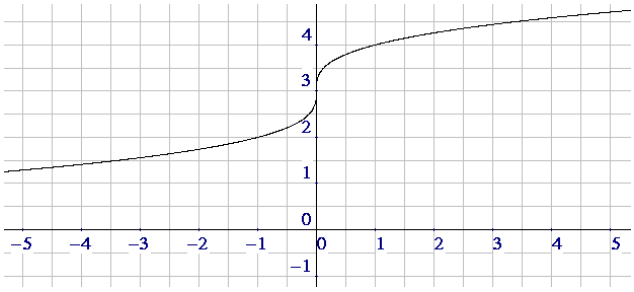
20.



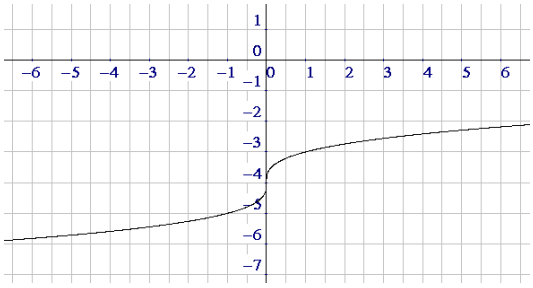
21.



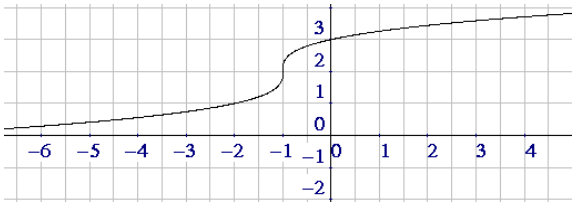
22.



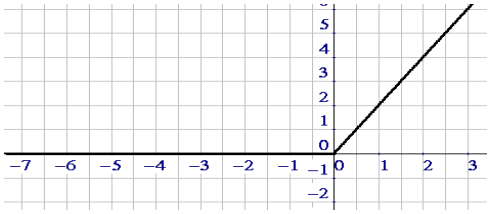
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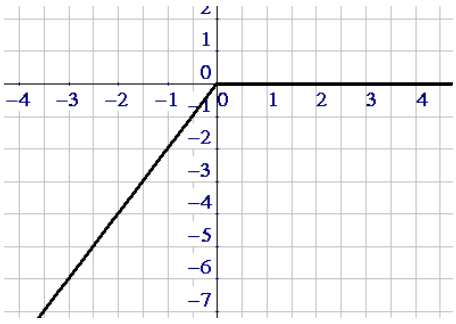
24.



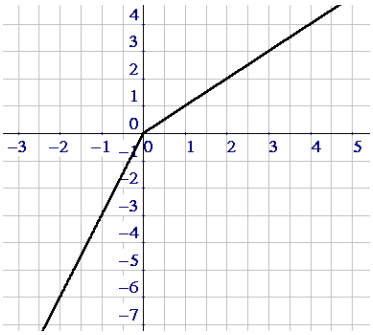
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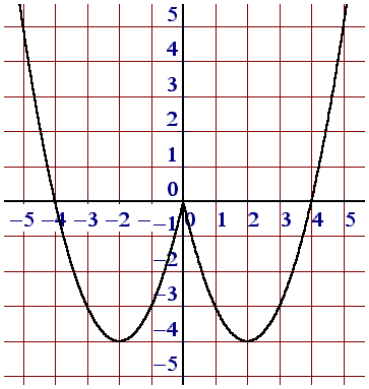
26.



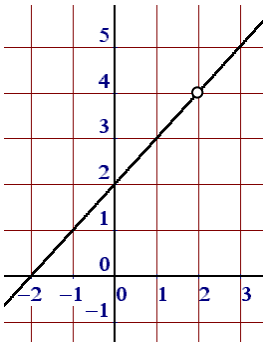
27.



28.



29. The graph is the line $y = x + 2$ with a hole at (2, 4).

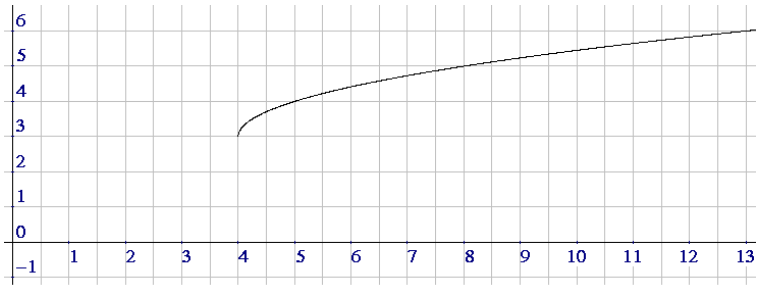


30. a. y-axis b. origin c. x-axis d. x-axis e. y-axis f. origin

31. x-axis, y-axis, and origin

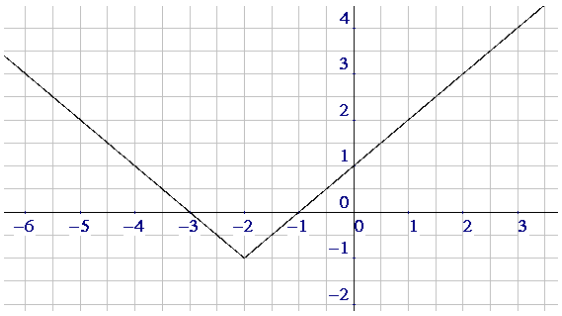
32. x-axis

33.



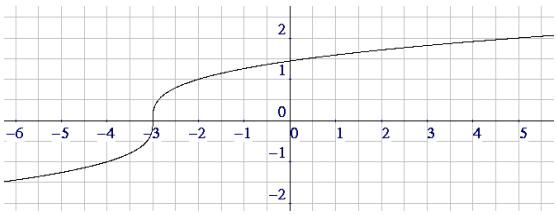
Domain = $[4, \infty)$

34.



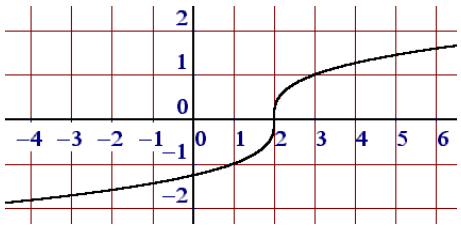
Domain = ⊕

35.



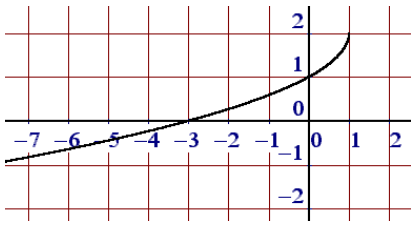
Domain = ⊕

36.



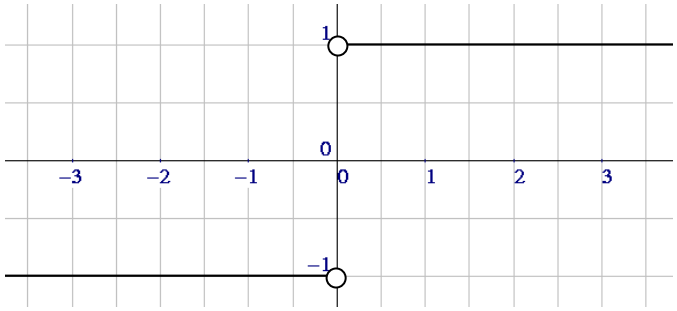
Domain = ⊕

37.



Domain = $(-\infty, 1]$

38.



Domain = $\oplus - \{0\}$

39. a. $y = |x-3| - 2$ b. $y = -x^2$ c. $y = -\sqrt{-x}$

d. $y = \sqrt{x-4}$ e. $y = \sqrt[3]{x} - 2$

40. a. T b. T c. F d. T e. T f. F g. T h. F i. T
j. T k. T l. F m. T

I have no particular talent.

I am merely inquisitive.

Albert Einstein

The Growth and Decay Formula

Many things grow and shrink (decay) at a rate based on how much stuff there is at the moment.

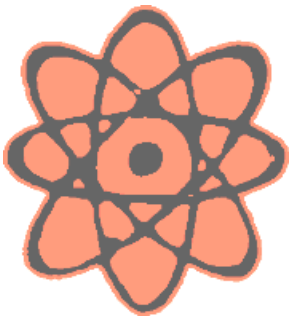




For instance, the number of births in a **population** is based on the number of people living at the time (since the more people there are, the more <u>new</u> people there will be).

Another example is **compound interest**: As money accumulates in the account (due to earned interest), the more interest the account will earn; that is, the interest will earn interest.





In science, we can observe the **radioactive decay** of an element like uranium. As the uranium decays, less of it will decay because as time goes on, there's less of it left to decay.

❑ THE GROWTH AND DECAY FORMULA

Here's a formula to help you predict how much of something there will be in the future. Among hundreds of applications, it works for business depreciation, the spread of disease, and the half-life of a drug:

Let

A_0 = starting amount (read: "A sub zero" or "A naught")
A = ending amount
e = a constant whose value is **approximately 2.718**.
k = growth or decay rate (expressed as a decimal)
t = time

Then

$$A = A_0 e^{kt}$$

This formula assumes **continuous** growth or decay, which means at "every moment in time." Don't be dismayed if that is hard to fathom.

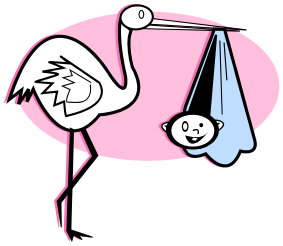
Get your calculator out. Here's a quick review of the "exponent" button. To calculate $(3.2)^{4.15}$, try either

3.2 $\boxed{y^x}$ 4.15 $\boxed{=}$ OR 3.2 $\boxed{\wedge}$ 4.15 $\boxed{=}$

The answer is about 124.845. [Your calculator may use $\boxed{\text{enter}}$ instead of the equals sign.]

The number ***e*** used in the Growth Formula is one of the most important numbers in math, statistics, science, engineering, and business. Like the number π, this real number contains an infinite number of digits which never form a repeating pattern (*e* and π and $\sqrt{2}$ are called *irrational* numbers).

EXAMPLE 1: **Assuming an initial population of 1,506 and a growth rate of 25% per year compounded continuously, predict the population in 10 years.**



Solution: Let's start by writing the formula that will solve this problem:

$$A = A_0 e^{kt}$$

The initial population is 1,506; so $A_0 = 1{,}506$.

The growth rate is 25%; thus $k = 0.25$.

We're talking about a period of 10 years; therefore $t = 10$.

Plug all these values into our formula (using 2.718 for e):

$$A = 1{,}506(2.718)^{(0.25)(10)}$$

$$\Rightarrow \quad A = 1{,}506(2.718)^{2.5}$$

$$\Rightarrow \quad A = 1{,}506(12.18) \qquad \text{(we'll round this result to 2 digits)}$$

$$\Rightarrow \quad \boxed{A = 18{,}343}$$

EXAMPLE 2: **Assuming a starting investment of $2,401 and an annual interest rate of 13% compounded continuously, predict the value of the investment in 11 years.**



Solution: The formula we used for population growth works equally well for compound interest.

$$A = A_0 e^{kt}$$

$$\Rightarrow \quad A = 2{,}401(2.718)^{(0.13)(11)}$$

$$\Rightarrow \quad A = 2{,}401(2.718)^{1.43}$$

$\Rightarrow \quad A = 2{,}401(4.18)$ (we'll round this result to 2 digits)

$\Rightarrow \quad \boxed{A = \$10{,}036}$

EXAMPLE 3: **Starting with 624 grams of uranium and assuming an annual decay rate of 5%, compute the number of grams of uranium remaining after 9 years.**



Solution: We use the same growth/decay formula; but since the amount of uranium is shrinking (decaying) rather than growing, we will use a decay rate of **–5%** (that's negative 5 percent) in our formula:

$A = A_0 e^{kt}$

$\Rightarrow \quad A = 624(2.718)^{(-0.05)(9)}$

$\Rightarrow \quad A = 624(2.718)^{-0.45}$

$\Rightarrow \quad A = 624(0.64)$ (we'll round this result to 2 digits)

$\Rightarrow \quad \boxed{A = 399 \text{ grams}}$

Homework

Note: The result of raising 2.718 to the exponent is assumed to be rounded to 2 digits, as in the examples.

1. Starting with 178 grams of uranium and assuming an annual decay rate of 17%, predict the number of grams remaining after 18 years.

2. Assuming an initial population of 3,902 and a growth rate of 14% per year, determine the population in 10 years.

3. Assuming an initial investment of $1,299 and an annual interest rate of 14% compounded continuously, predict the value of the investment in 4 years.

4. Assuming an initial investment of $9,574 and an annual interest rate of 11% compounded continuously, compute the value of the investment in 2 years.

5. Starting with 416 grams of thorium and assuming an annual decay rate of 3%, determine the number of grams remaining after 26 years.

6. Assuming an initial population of 2,586 and a growth rate of 16% per year, predict the population in 4 years.

7. Assuming an initial population of 1,422 and a growth rate of 12% per year, compute the population in 2 years.

8. Assuming an initial investment of $6,897 and an annual interest rate of 14% compounded continuously, compute the value of the investment in 6 years.

9. Starting with 215 grams of plutonium and assuming an annual decay rate of 15%, calculate the number of grams remaining after 26 years.

10. Assuming an initial population of 2,159 and a growth rate of 6% per year, calculate the population in 11 years.

[As mentioned on the second page of this Chapter, technically speaking, the growth/decay formula we've been using applies only when the growth or decay is *continuous*, which means the growth or decay occurs at every single moment. This may not strictly be the case in all situations (for example, in the births of people), but let's not worry about it in this chapter. Let's just use the formula to solve the problems.]

Solutions

1.	9 g	**2**.	15,803	**3**.	$2,273	**4**.	$11,968
5.	191 g	**6**.	4,913	**7**.	1,806	**8**.	$16,001
9.	4 g	**10**.	4,167				

"Education is not the
filling of a pail,
but the lighting of a fire."

— *William Butler Yeats*

THE HYPERBOLA



The hyperbola is the basis for a navigation system called LORAN, (Long Range Navigation.) The hyperbola is the shape of the orbits of certain comets, and is also used to design telescopes and the cooling towers at nuclear power plants.



❑ *THE MAIN EXAMPLE*

Graph the hyperbola $x^2 - y^2 = 9$.

First, let's find the **domain**. We solve for y, and then analyze all the legal values of x:

$$x^2 - y^2 = 9 \;\Rightarrow\; -y^2 = -x^2 + 9 \;\Rightarrow\; y^2 = x^2 - 9 \;\Rightarrow\; y = \pm\sqrt{x^2 - 9}$$

Because of the square root, y is a real number only when the radicand, $x^2 - 9$, is greater than or equal to zero: $x^2 - 9 \geq 0$. You can solve this inequality by the Boundary Point Method. When you do, you will conclude that the domain of the graph is $\mathbf{(-\infty, -3]} \cup \mathbf{[3, \infty)}$. By the way, can you explain why the hyperbola is <u>not</u> a function?

Second, we'll check out **intercepts**. Since 0 is not in the domain, there cannot possibly be any y-intercepts. [Indeed, letting $x = 0$ gives $0^2 - y^2 = 9 \;\Rightarrow\; y^2 = -9 \;\Rightarrow\; y = \pm\sqrt{-9}$, not members of $\mathbb{R}$.] Thus, there are no y-intercepts. As for x-intercepts, let $y = 0$; this gives $x^2 - 0^2 = 9 \;\Rightarrow\; x^2 = 9 \;\Rightarrow\; x = \pm 3$. Therefore, there are two x-intercepts: **(3, 0)** and **(–3, 0)**.

Third, let's tackle **symmetry**. Replace x with $-x$: $(-x)^2 - y^2 = 9$, which is equivalent to $x^2 - y^2 = 9$, the original equation. Thus, the graph has y-axis symmetry. Similarly, you can show that the graph has x-axis symmetry. In fact, the graph of the hyperbola also has origin symmetry.

Fourth, we analyze **asymptotes**. Since there are no fractions in the hyperbola formula, it appears that the graph has no vertical asymptotes. To find any other possible asymptotes, we'll let $x \to \infty$ and see if y approaches anything interesting.

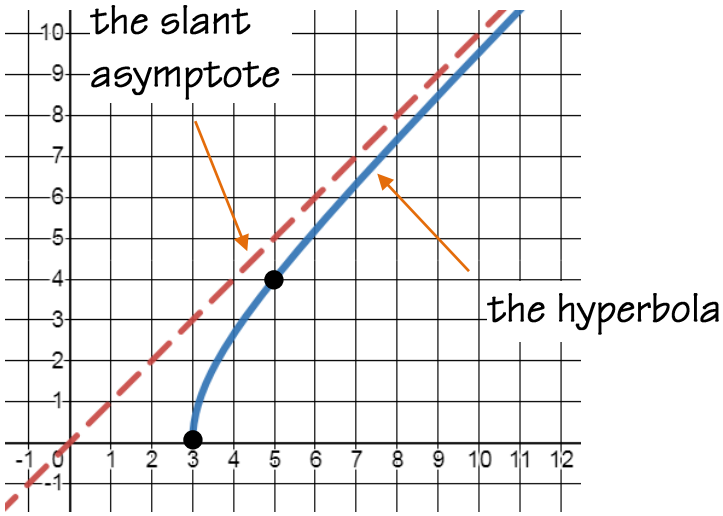
Let's construct a little x-y chart, and then choose really big values for x. Use your calculator to confirm these results.

x	y
5	± 4
10	± 9.5394
100	± 99.5499
1,000	± 999.9955
10,000	± 9,999.9996

What's going on here? Let's look just at the positive y-values. On the surface, it certainly appears that as x grows larger and larger, the y-values grow larger and larger. But look more carefully: The y-value is actually approaching the x-value. That is, as $\boldsymbol{x \to \infty, y \to x}$. We can write this limit as

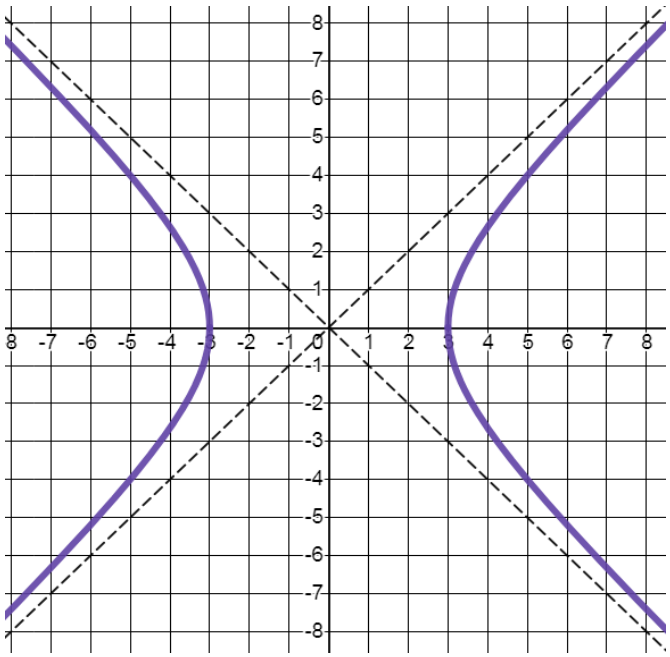
$$\lim_{x \to \infty} \sqrt{x^2 - 9} = x$$

This means that as x grows larger and larger, the curve is approaching the line $y = x$ (a 45° line passing through the origin). Since this line is neither vertical nor horizontal, we need a new adjective. We call this kind of asymptote a ***slant***, or ***oblique***, asymptote.



The analysis just completed gives a picture of the hyperbola in the first quadrant. It has an x-intercept at (3, 0), and a slant asymptote of $y = x$. Also, from the chart above, the hyperbola passes through the point (5, 4).

Since the hyperbola has y-axis symmetry, another piece of the hyperbola lies in the second quadrant. But the pieces of the hyperbola in Quadrants I and II have symmetric pieces in Quadrants III and IV, due to the x-axis symmetry of the hyperbola.



We're now ready to graph the entire hyperbola. Note that the dashed lines $y = x$ and $y = -x$ are the slant asymptotes of the hyperbola, and are graphed just for reference — they are <u>not</u> part of the hyperbola. Finally, the graph should convince you of the fact deduced earlier that the domain is $(-\infty, -3] \cup [3, \infty)$. In addition, the graph shows us that the **range** of the hyperbola is $\mathbb{R}$.

Homework

1. Regarding the main example above, $x^2 - y^2 = 9$, explain, using the formula, why it isn't a line, a parabola, a circle, or an ellipse.

2. Find the slant asymptotes for each hyperbola:

   a. $x^2 - y^2 = 25$ b. $x^2 - 9y^2 = 9$

   c. $36x^2 - y^2 = 36$ d. $9x^2 - 4y^2 = 36$

3. Find the domain and range of each hyperbola in the previous problem.

4. Perform a complete analysis (including a graph) on the hyperbola $x^2 - y^2 = 49$.

❑ *ANOTHER EXAMPLE*

Graph the hyperbola $y^2 - 4x^2 = 25$.

First we solve for y — this will allow us to find the domain and the slant asymptotes.

$$y^2 - 4x^2 = 25 \Rightarrow y^2 = 25 + 4x^2 \Rightarrow y = \pm\sqrt{25 + 4x^2}$$

Look at the radicand. No matter what kind of number x is, its square, x^2, is non-negative, which implies that $4x^2$ is non-negative, which implies that $25 + 4x^2$ is <u>at least</u> 25. So surely the radicand can <u>never</u> be negative. Thus, the **domain** of the hyperbola is $\mathbb{R}$.

Now it's time for the **intercepts**. Let $x = 0$; then $y^2 = 25$, or $y = \pm 5$. So we have a couple of y-intercepts: **(0, 5) and (0, –5)**. Now let $y = 0$; then $-4x^2 = 25 \Rightarrow x^2 = -\frac{25}{4} \Rightarrow$ no real solution for x. Therefore, there are no x-intercepts.

I'll leave it to you to verify that the hyperbola possesses all three types of **symmetry**.

To find the slant **asymptotes**, we refer to the form of the equation calculated above,

$$y = \pm\sqrt{25+4x^2}$$

Use your calculator to confirm the following ordered pairs (using just the positive square root.):

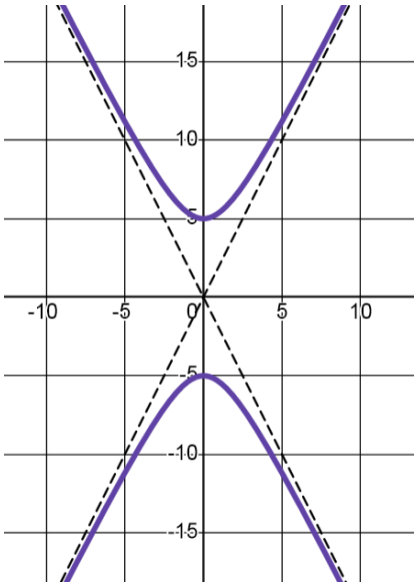
(10, 20.616) (100, 200.062) (1000, 2000.006)

As x grows larger and larger, the y-values appear to be growing closer and closer to whatever twice x is. That is, as $\boldsymbol{x \to \infty}$, $\boldsymbol{y \to 2x}$. We write

$$\lim_{x\to\infty} \sqrt{25+4x^2} = 2x$$

Thus, one of the slant asymptotes is the line $\boldsymbol{y = 2x}$.

The graph goes through the intercept (0, 5) and then approaches the slant asymptote $y = 2x$. The x-axis and y-axis symmetries then give us the rest of the hyperbola in the other three quadrants. As before, the asymptotes (dashed) are graphed for reference.



Last, we discuss the **range** of the hyperbola. The y-values appear to be numbers that are either less than or equal to –5, OR greater than or equal to 5. That is, the range of the hyperbola is $\{y \in \mathbb{R} \mid y \le -5 \text{ or } y \ge 5\}$, which is $\boldsymbol{(-\infty, -5] \cup [5, \infty)}$.

Apollonius (Greek: 262 BC – 190 BC) gave the hyperbola its present name. See how the graph of the hyperbola seems to "fly away" in all directions? Look up the word "hyperbole" in a dictionary (a big book that predates Google) and see if its meaning has any connection to the graph of a hyperbola.

Homework

5. Find the slant asymptotes for each hyperbola:

   a. $y^2 - x^2 = 144$ b. $y^2 - 16x^2 = 16$

   c. $9y^2 - x^2 = 9$ d. $25y^2 - 9x^2 = 225$

6. Find the domain and range of each hyperbola in the previous problem.

Perform a complete analysis (including a graph) on each hyperbola:

7. $y^2 - x^2 = 25$

8. $y^2 - 9x^2 = 16$

9. $16y^2 - 9x^2 = 144$

10. $\frac{x^2}{16} - \frac{y^2}{4} = 1$

11. Classify each of the following as a line, parabola, circle, ellipse, or hyperbola:

   a. $7x^2 - \pi x + 2y + 9 = 0$ b. $83x + 99y = 34$

   c. $7x^2 + 2y^2 - x = 10$ d. $y^2 + 7y - x^2 - 2x - 100 = 0$

   e. $7x + 99 = 400$ f. $32 - 9y + \pi = 0$

   g. $\sqrt{2}\,x^2 + \sqrt{2}\,y^2 - \sqrt{e}\,x + \sqrt[3]{\pi}\,y - 10^6 = 0$

Review Problems

12. Classify as parabola, circle, ellipse, hyperbola, or none of them:

a. $x^2 + y^2 = -1$ b. $y^2 - x^2 = 5$ c. $2x^2 + 4y^2 = 7$

d. $x^2 - y = 3$ e. $2x^2 - \pi y^2 = \sqrt{2}$ f. $3y^2 = x^2 + 5$

13. a. Find the intercepts of $3x^2 - 2y^2 = 2$.

b. Find the intercepts of $10y^2 - 30x^2 = 3$.

c. Find the domain of $x^2 - 14y^2 = 144$.

d. Find the range of $9x^2 - 36y^2 = 41$.

e. Find the asymptotes of $49x^2 - y^2 = 10$.

f. Find the asymptotes of $36y^2 - 4x^2 = 1$.

g. Find all symmetries of $4x^2 - 9y^2 = 36$.

14. Find the intercepts, domain, range, asymptotes, and symmetry of the hyperbola $x^2 - y^2 = 4$. Graph the hyperbola.

15. Find the asymptotes and the range of $y^2 - 9x^2 = 16$. Sketch the graph.

16. True/False:

a. Some of the hyperbolas in this chapter are functions.

b. The domain of the hyperbola is $x^2 - y^2 = 9$ is $[-3, 3]$.

c. Hyperbolas in this chapter always possess all three symmetries.

d. The slant asymptotes for $16x^2 - 9y^2 = 1$ are $y = \pm\frac{4}{3}x$.

e. There are no slant asymptotes for $25x^2 + 49y^2 = 1$.

f. The range of a hyperbola may consist of the union of two intervals.

g. Some of the hyperbolas in this chapter pass through the origin.

❑ *TO ∞ AND BEYOND*

Find the **center** of the hyperbola

$$25x^2 - 9y^2 + 50x + 90y - 425 = 0$$

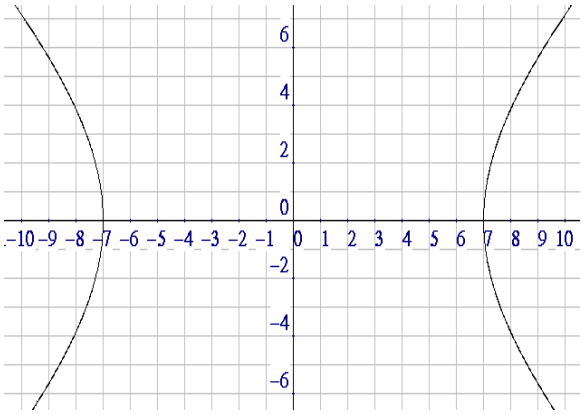
Hint: We note that the *origin* is the center of each hyperbola in this chapter.

Solutions

1. Any letter squared $\Rightarrow$ it's not a line
Both letters squared $\Rightarrow$ not a parabola
Different coefficients on the squares $\Rightarrow$ not a circle
Opposite signs on the coefficients on the squares $\Rightarrow$ not an ellipse

2. a. $y = \pm x$ b. $y = \pm\frac{1}{3}x$ c. $y = \pm 6x$ d. $y = \pm\frac{3}{2}x$

3. a. $(-\infty, -5] \cup [5, \infty)$; $\mathbb{R}$ b. $(-\infty, -3] \cup [3, \infty)$; $\mathbb{R}$
c. $(-\infty, -1] \cup [1, \infty)$; $\mathbb{R}$ d. $(-\infty, -2] \cup [2, \infty)$; $\mathbb{R}$

4. 

Domain = $(-\infty, -7] \cup [7, \infty)$

Range = $\mathbb{R}$ Not a function

No y-intercepts

x-intercepts: $(-7, 0)$ and $(7, 0)$

Symmetry: all three

$\lim_{x \to \infty} \sqrt{x^2 - 49} = x$

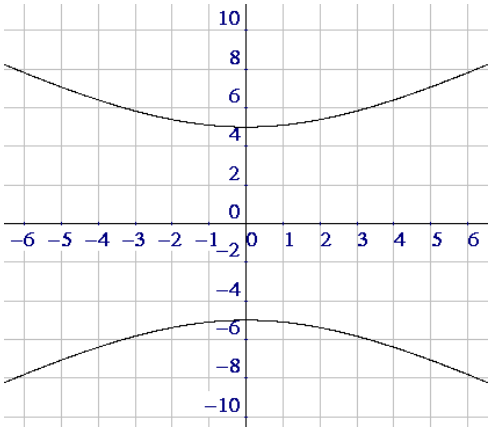
Slant asymptotes: $y = x$ and $y = -x$

5. a. $y = \pm x$ b. $y = \pm 4x$ c. $y = \pm \frac{1}{3}x$ d. $y = \pm \frac{3}{5}x$

6. a. $\mathbb{R}$; $(-\infty, -12] \cup [12, \infty)$ b. $\mathbb{R}$; $(-\infty, -4] \cup [4, \infty)$

c. $\mathbb{R}$; $(-\infty, -1] \cup [1, \infty)$ d. $\mathbb{R}$; $(-\infty, -3] \cup [3, \infty)$

7.

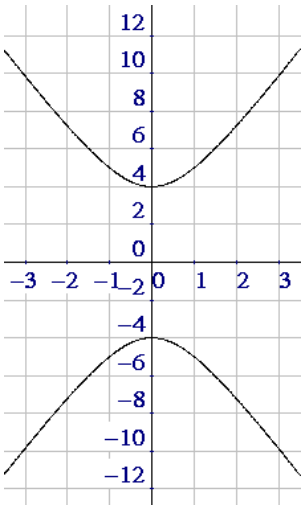


Domain = $\mathbb{R}$ Range = $(-\infty -5] \cup [5, \infty)$ Not a function

No x-intercept; y-intercepts: $(0, 5)$ and $(0, -5)$

Symmetry: all three Asymptotes: $y = x$ and $y = -x$

8.



Domain = $\mathbb{R}$

Range = $(-\infty, -4] \cup [4, \infty)$

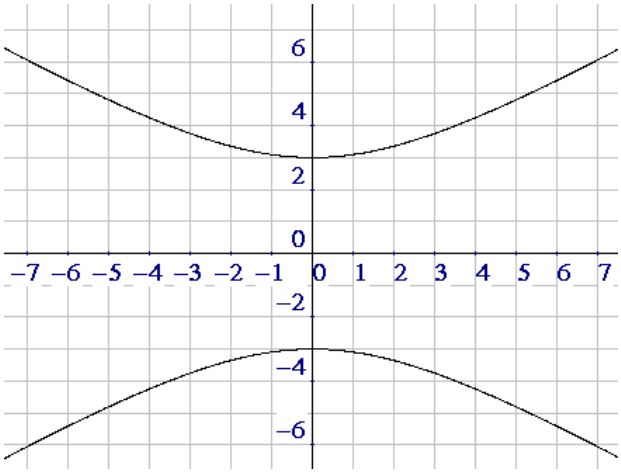
Not a function

No x-intercepts; y-intercepts: (0, 4) and (0, −4)

Symmetry: all three

Asymptotes: $y = 3x$ and $y = -3x$

9.



Domain = $\mathbb{R}$

Range = $(-\infty -3] \cup [3, \infty)$

Not a function

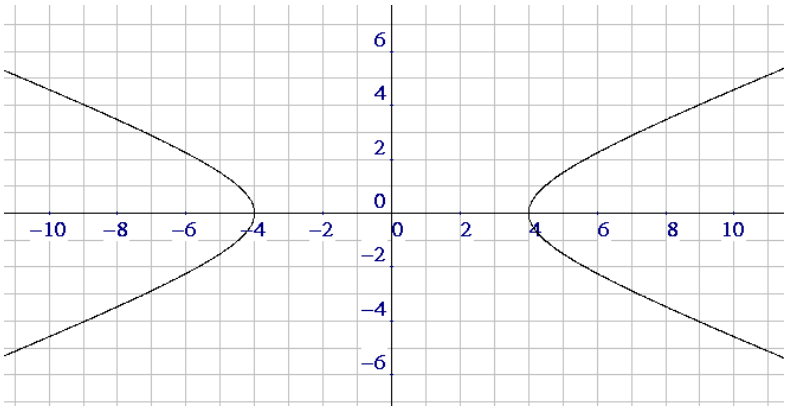
No x-intercepts

y-intercepts: (0, 3) and (0, −3)

Symmetry: all three

Asymptotes: $y = \pm\frac{3}{4}x$

10.



Domain = $(-\infty, -4] \cup [4, \infty)$ Range = $\mathbb{R}$ Not a function

Intercepts: $(-4, 0)$ and $(4, 0)$ Symmetry: all three

Slant asymptotes: $y = \pm\frac{1}{2}x$

11. a. parabola b. line c. ellipse d. hyperbola

e. (vertical) line f. (horizontal) line g. circle

12. a. none b. hyperbola c. ellipse

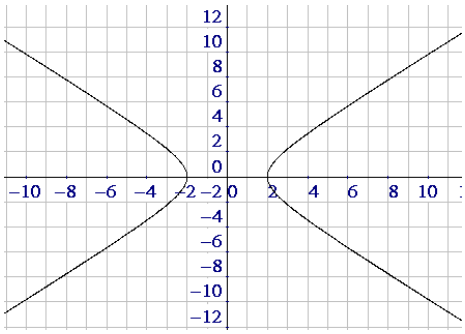
d. parabola e. hyperbola f. hyperbola

13. a. $\left(\pm\sqrt{\frac{2}{3}}, 0\right)$ b. $\left(0, \pm\sqrt{\frac{3}{10}}\right)$ c. $(-\infty, -12] \cup [12, \infty)$ d. $\mathbb{R}$

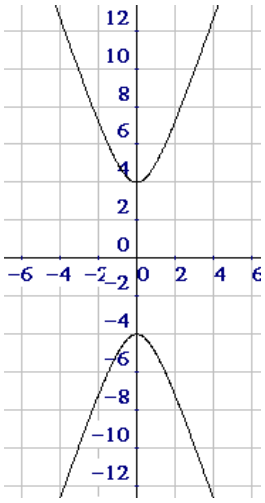
e. $y = \pm 7x$ f. $y = \pm\frac{1}{3}x$ g. x-axis, y-axis, and origin

14. $(\pm 2, 0)$ Dom = $(-\infty, -2] \cup [2, \infty)$ Rg = $\mathbb{R}$ asym: $y = \pm x$

all 3 symmetries



15. $y = \pm 3x$; $(-\infty, -4] \cup [4, \infty)$



16. a. F b. F c. T d. T e. T f. T g. F

"Effort only fully releases its reward after a person refuses to quit."

Napoleon Hill

GRAPHING INEQUALITIES IN THE PLANE

❑ *INEQUALITIES IN TWO VARIABLES*

Consider the following inequality in two variables:

$$y \leq 2x - 3$$

In English, it says that the value of y must be less than or equal to the quantity *3 less than twice the value of x*. To get a feel for this, let's test some x-y pairs. Then we'll discuss how to make a graph of the inequality.

Let's try the ordered pair (5, –1); that is, $x = 5$ and $y = -1$:

$$y \leq 2x - 3$$
$$\mathbf{-1} \leq 2(\mathbf{5}) - 3$$
$$-1 \leq 10 - 3$$
$$-1 \leq 7, \text{ which is } \underline{\text{true}}$$

Thus, the point **(5, –1)** WILL be on the graph of $y \leq 2x - 3$.

Now let's test the point (0, 4):

$$y \leq 2x - 3$$
$$\mathbf{4} \leq 2(\mathbf{0}) - 3$$
$$4 \leq 0 - 3$$
$$4 \leq -3, \text{ which is } \underline{\text{false}}$$

We conclude that **(0, 4)** will NOT be on the graph of $y \leq 2x - 3$.

Last, we we test the point (7, 11):

$$y \leq 2x - 3$$

$$\mathbf{11} \leq 2(\mathbf{7}) - 3$$

$$11 \leq 14 - 3$$

$$11 \leq 11, \text{ which is } \underline{\text{true}}$$

Thus, the point **(7, 11)** WILL be on the graph of $y \leq 2x - 3$.

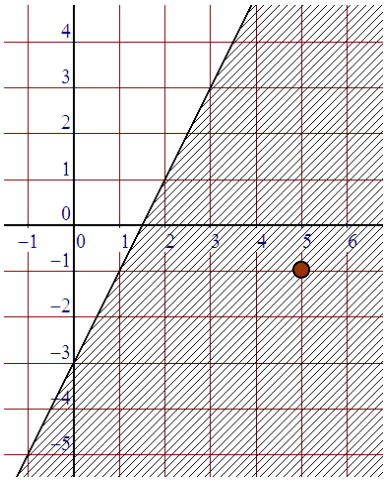
> **NOTE:** The point (7, 11) is actually ON the line $y = 2x - 3$. This is why the inequality $11 \leq 11$ occurred at the last step.

❑ *GRAPHING INEQUALITIES IN THE PLANE*

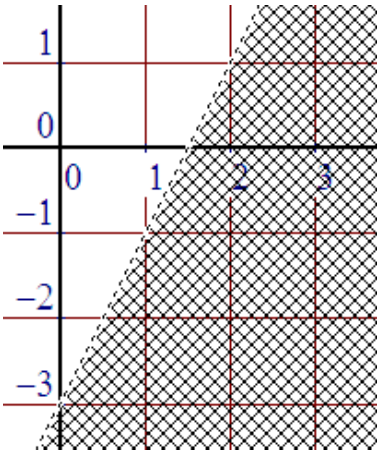
It's now time to graph the inequality $y \leq 2x - 3$.

Here's the process in a nutshell: Graph the <u>line</u> $y = 2x - 3$ (that is, ignore the inequality symbol for now). Then choose any point in the plane (that's not on the line) and test it against the inequality. If that point works, the graph of the inequality is the intirety of the side of the line containing the point; if the point does not work, the graph is the <u>other</u> side of the line.

Using the point (5, –1) that we showed before <u>is</u> a solution of the inequality, we shade the portion of the plane below the line. Notice that the line itself is graphed solid — this is because of the symbol "≤."



If the inequality had been $y < 2x - 3$, then the graph would be the same except that we would *dash* the line and it would look something like this.



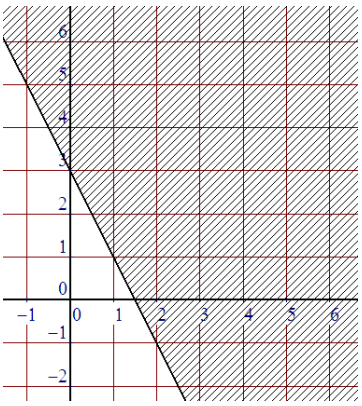
Homework

Graph each inequality:

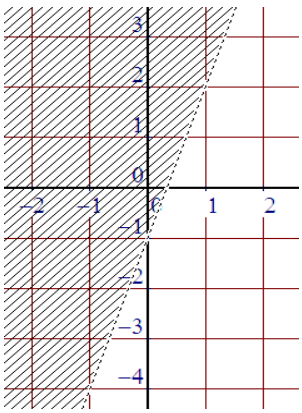
1. $y \geq -2x + 3$
2. $y > 3x - 1$
3. $y \leq x$
4. $y < -x - 2$
5. $y \leq -3x + 2$
6. $-x + y \geq 4$

Solutions

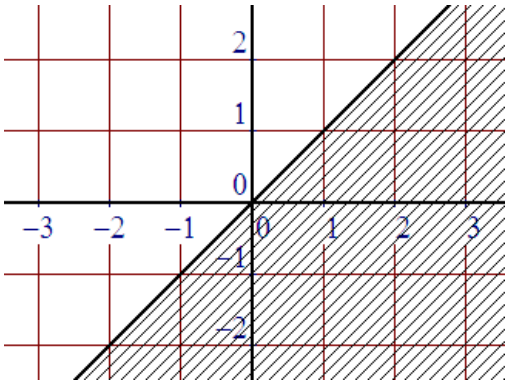
1.



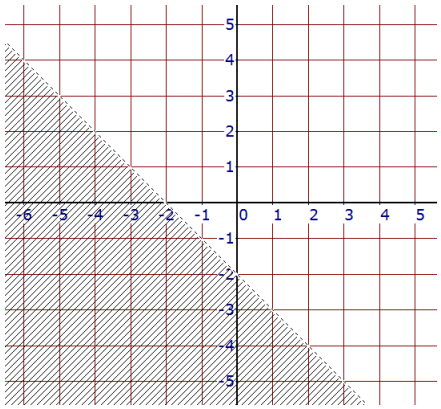
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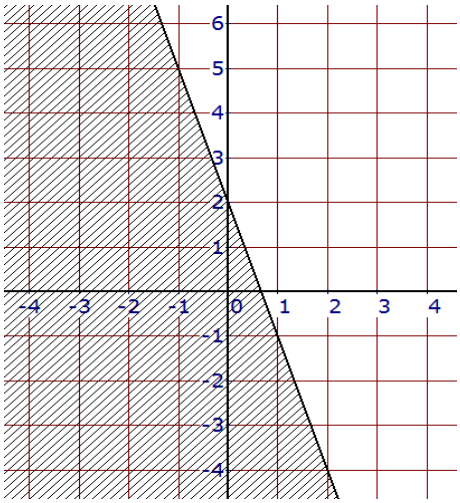
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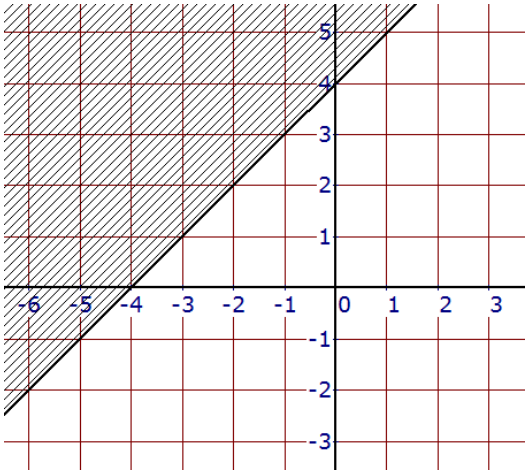
4.



5.



6.



Edith Hamilton:

"To be able to be caught up into the world of thought – that is educated."

INEQUALITIES

You must score between 80% and 89% to get a B in your math class.

You must be at least 18 years of age to vote.

You can be no taller than 48 inches to play in the park.



These are all examples of quantities being greater than something or less than something (or between two somethings). Since they are not equalities, they are called **inequalities**.

❑ *NOTATION*

We know that 5 is bigger than 3, which we can write as "**5 > 3**." The symbol ">" can also be read as "is larger than" or "is greater than."

But, of course, the fact that 5 is larger than 3 is the same as the fact that 3 is smaller than 5. This is written "**3 < 5**."

> means "is greater than"

< means "is less than"

≥ means "is greater than or equal to"

≤ means "is less than or equal to"

The symbol "≥" can read "is greater than or equal to." For example, **9 ≥ 7** because 9 is indeed greater than or equal to 7. (Actually, it's greater than 7, but that doesn't change the fact that it's greater than or equal to 7.)

Believe it or not, **12 ≥ 12** is also a true statement — after all, since 12 equals 12, it's certainly the case that 12 is greater than or equal to 12.

The symbol "≤" is read "is less than or equal to." A couple of examples are $6 \le 10$ and $8 \le 8$.



Homework

1. True/False:

   a. $7 > 3$ b. $-2 < 1$ c. $13 \ge 13$ d. $-9 \le -9$

   e. $12 \ge 9$ f. $-18 \le -20$ g. $\pi > 0$ h. $-\sqrt{2} \le 0$

2. Express each statement as an inequality:

   a. Your age, a, must be at least 18 years.

   b. Your height, h, can be no taller than 48 inches.

   c. Your years of experience, y, must exceed 10 years.

   d. The number of driving tickets, t, must be fewer than 5.

   e. The mean, μ (Greek letter mu), must be at least 75.

   f. The standard deviation, σ (Greek letter sigma), must be no more than 10.

   g. The energy, E, must be more than 100.

   h. The mass, m, must be less than 3.7.

❑ *THE BASIC PRINCIPLE OF SOLVING INEQUALITIES*

To solve an inequality such as $-2x + 7 \ge -4$, we have to perform the operations necessary to isolate the x. We will accomplish this goal by using the standard *do the same thing to each side* rule, but since we're

talking about an inequality — not an equation — we have to be very careful. The following experiment should illustrate the potential problems, and give us the techniques to overcome these problems.

Let's perform six experiments on a given inequality. Consider the true statement

$$4 < 6$$

i. Add 10 to each side: $4 + \mathbf{10} < 6 + \mathbf{10}$

$\Rightarrow 14 < 16$ ✓

ii. Subtract 3 from each side: $4 - \mathbf{3} < 6 - \mathbf{3}$

$\Rightarrow 1 < 3$ ✓

iii. Multiply each side by 5: $4(\mathbf{5}) < 6(\mathbf{5})$

$\Rightarrow 20 < 30$ ✓

iv. Divide each side by 2: $\frac{4}{2} < \frac{6}{2}$

$\Rightarrow 2 < 3$ ✓

v. Multiply each side by –3: $4(-3) < 6(-3)$

$\Rightarrow -12 < -18$ ✘

vi. Divide each side by –2: $\frac{4}{-2} < \frac{6}{-2}$

$\Rightarrow -2 < -3$ ✘

What can we deduce (conclude) from these six calculations? The first two show that adding the same number to each side of an inequality, or subtracting the same number from each side of an inequality, are both things we can do without any issues; they lead to a true inequality.

Calculations iii. and iv. indicate that multiplying or dividing each side of an inequality by a positive number leads to an inequality which is just as valid as the original one.

Unfortunately, the last two cases — multiplying or dividing by a negative number — have led to false statements. So in these two scenarios, we must reverse the order of the inequality sign (flip it around) in order to maintain a true statement. Problem solved!

These numerical experiments, by no means a complete proof of the principle we are about to state, are convincing enough for me.

The Basic Principle of Solving Inequalities

Only if you multiply or divide each side of an inequality by a *negative* number, must you reverse the inequality symbol.

EXAMPLES: **Solve each inequality:**

A. Solve the inequality: $\boldsymbol{x + 3 > 4}$

Subtract 3 from each side $\boldsymbol{x > 1}$

Note: The inequality symbol was not reversed.

B. Solve the inequality: $\boldsymbol{-2n - 9 \leq 13}$

Add 9 to each side: $-2n \leq 22$
(The inequality symbol was not reversed.)

Divide each side by –2: $\boldsymbol{n \geq -11}$

This time the inequality symbol was reversed, since we divided each side of the inequality by a negative number.

C. Solve the inequality: $\frac{u}{5} + 3 < -4$

Subtract 3 from each side: $\frac{u}{5} < -7$

Multiply each side by 5: $u < -35$

Neither operation required reversing the inequality.

Homework

3. Solve each inequality:

   a. $x + 7 > -10$ b. $x - 3 \leq 5$ c. $2x \geq 14$

   d. $-3x < -42$ e. $\frac{x}{8} > -3$ f. $\frac{x}{-5} \geq 10$

4. Solve for y: $6(y - 5) - (y + 1) \leq 12y + 21$

5. Solve for u: $-2(7 + 3u) - (1 - u) > 2u - 10$

6. a. Solve for x: $\frac{-3x + 5}{-2} + 9 > 16$ Note: The process is more important than the answer. Do you know what this means?

   b. Solve for a: $\frac{7a - 5}{-3} - 12 < 12$

   c. Solve for y: $\frac{9 - y}{-2} \geq -2$

   d. Solve for n: $\frac{-14 - 2n}{12} \leq -1$

❑ *UNDERSTANDING DOUBLE INEQUALITIES*

Suppose Edwin wants to take part in a political poll. The pollster only cares about voters whose age, A, is between 25 and 40 (including the 25 and the 40). Here's one way to write this requirement using algebra:

$A \geq 25$ **and** $A \leq 40$ (a double inequality using **and**)

By the way, the word **and** between the two inequalities is crucial; Edwin must be at least 25 **and** at most 40. Do you see that replacing the **and** with **or** would change the meaning entirely?

Here's the other way to write that double inequality — we sandwich the variable A between the lower limit (25) and the upper limit (40):

$$\mathbf{25 \leq A \leq 40}$$

This can be read in two equivalent ways:

1. $A \geq 25$ **and** $A \leq 40$ [Note that $25 \leq A$ is the same as $A \geq 25$.]
2. A is ***between*** 25 and 40, including the 25 and including the 40.

For a second example, the phrase "*x must be between 0 and 10, but not equal to either 0 or 10,"* is written

$$\mathbf{0 < x < 10}$$

Homework

7. True/False:
   a. 7 satisfies the double inequality: $0 < x < 9$.
   b. 0 satisfies the double inequality: $0 < x < 9$.
   c. 10 satisfies the double inequality: $-5 \leq x \leq 10$.
   d. 0 satisfies the double inequality: $0 \leq x < \pi$.

e. 2π satisfies the double inequality: $0 < x < 7$.

f. 100 satisfies the double inequality: $100 \le x < 101$.

8. a. Name the only solution of the double inequality

$$23 \le x \le 23$$

b. Find all values of x which satisfy the double inequality

$$50 < x < 50$$

9. Express each situation as a *double inequality*:

a. The weight, w, must be between 20 lb and 50 lb, including the 20 and the 50.

b. The score, s, must be between 100 pts and 300 pts, excluding the 100 and the 300.

c. The commission, C, must be between \$2000 and \$5750, inclusive of the endpoints.

d. The distance, d, must be between 123 m and 250 m, exclusive of the endpoints.

❑ *TO ∞ AND BEYOND*

A. Determine the values of a which satisfy the compound inequality:

$$a \ge 25 \text{ **or** } a \le 40$$

B. Marty was trying to solve the inequality

$$ax + b > c$$

for x, and wrote the following:

$$ax + b > c$$

$$\Rightarrow \quad ax > c - b$$

$$\Rightarrow \quad \boxed{x > \frac{c - b}{a}}$$

Explain the fallacy (faulty logic) in Marty's reasoning.

Solutions

1. a. T b. T c. T d. T e. T f. F
g. T h. T

2. a. $a \geq 18$ b. $h \leq 48$ c. $y > 10$ d. $t < 5$
e. $\mu \geq 75$ f. $\sigma \leq 10$ g. $E > 100$ h. $m < 3.7$

3. a. $x > -17$ b. $x \leq 8$ c. $x \geq 7$
d. $x > 14$ e. $x > -24$ f. $x \leq -50$

4. $y \geq -\frac{52}{7}$

5. $u < -\frac{5}{7}$

6. a. $x > \frac{19}{3}$ b. $a > -\frac{67}{7}$

c. $y \geq 5$ d. $n \geq -1$

7. a. True b. False c. True
d. True e. True f. True

8. I'd like to know what you think.

9. a. $20 \leq w \leq 50$ b. $100 < s < 300$

c. $2000 \leq C \leq 5750$ d. $123 < d < 250$

"Do not judge me by my successes. Judge me by how many times I fell down and got back up again."

— Nelson Mandela



INTERVAL NOTATION



Consider all the real numbers greater than 3. One simple way to express this set of numbers is the following ***inequality***:

$$x > 3$$ [Do you see that this inequality can also be written "3 < x"?]

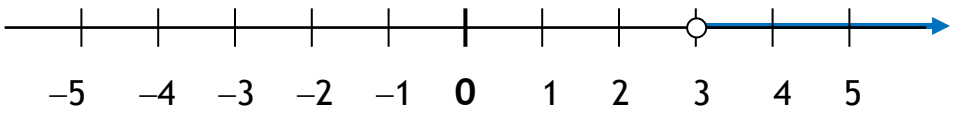
But there are other ways to represent the inequality $x > 3$, ways that might be easier to visualize.

❑ *EXAMPLES*

First Example: Let's continue with the inequality stated in the introduction:

$$x > 3$$

A great way to visualize this inequality is to ***graph*** this set of numbers on a number line:



Notice that we put an "open dot" at $x = 3$ to show that the 3 is not part of the set of numbers. But the arrow goes infinitely to the right because $x > 3$ is the set of numbers greater than 3.

Note that the numbers 3.01, π, 17, and 200 are part of the set; but the numbers –5, 0, 2.5, and 3 are not part of the set.

And a third way to denote this interval is called ***interval notation***, and for this inequality, we write:

$$(3, \infty)$$

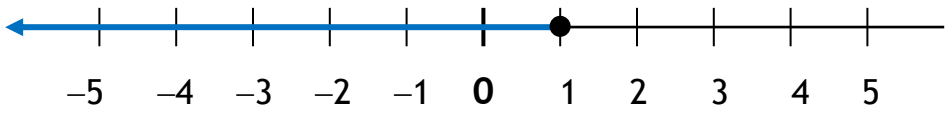
The parenthesis next to the 3 is analogous to the "open dot" on the graph — it means exclude the endpoint. And you always use a parenthesis at the ∞ or –∞ end of an interval because ∞ is not really a number, so you can't possibly include it.

Second Example: Now we consider all the real numbers less than or equal to 1. This set of numbers can be written as an inequality like this:

$$x \leq 1$$

Note that this inequality can also be expressed as: $1 \geq x$

As a graph on a number line, we write:



The "solid dot" is used to show that $x = 1$ is part of the set of numbers. And since x must be less than or equal to 1, the arrow goes infinitely to the left. Either as an inequality or a graph, you should see that the numbers –3, –1.1, $\frac{7}{8}$, and 1 are part of the set, while the numbers 1.001 and $\sqrt{2}$ are not part of the set.

As for interval notation, we write

$$\mathbf{(-\infty, 1]}$$

The (square) bracket next to the 1 is analogous to the "solid dot" on the graph — it means include the endpoint. And, as mentioned before, always use a parenthesis with ± ∞, since you can't ever "get" to ∞.

Third Example: Now it's time for an interval that represents all the numbers between two numbers. Consider the double inequality

$$-2 \leq x < 5$$

The inequality can also be read as "all real numbers that are both greater than or equal to 2 AND also less than 5.

This can be read as "all real numbers between –2 and 5, including the –2, but excluding the 5."

As for interval notation and a graph, here they are:

$[-2, 5)$



Note that some numbers in the interval are –2, 0, $\sqrt{24}$, and 4.9999. But the numbers –2.1, 5, and 2π are not in the interval.

Fourth Example: Sometimes answers to an inequality problem end up looking something like this:

$$x < 3 \text{ OR } x \geq 5$$

This means pretty much what it says: x can be less than 3, or it can be greater than or equal to 5. As long as x satisfies (at least) one of the two conditions, it's part of the answer. So some x's that satisfy the inequality are 0, 2.9, 5, and 3π. On the other hand, some numbers that do NOT satisfy it are 3, π, and 4.99.

How do we express this in interval notation? Like this, using the **union** symbol:

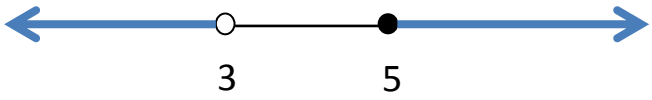
$$(-\infty, 3) \cup [5, \infty)$$

The *union* symbol, $\cup$, just means putting all the elements of the two sets into one big set; for example,

$\{a, b\} \cup \{a, c, d\} = \{a, b, c, d\}$

Note that the "overlap" of 'a' is listed only once in the final set.

And as a graph, we write:



Can you see that the interval $(-\infty, \infty)$ is the same as the entire set of Real numbers?

$$(-\infty, \infty) = \mathbb{R}$$

In fact, using the *union* notation, we can also write

$$\mathbb{R} = \text{Rationals} \cup \text{Irrationals}$$

where the Rational numbers are all decimals that do repeat, while the Irrational numbers are the decimals that do not repeat. So $\mathbb{R}$ is simply the set of ALL decimals.

❑ *Summary of Inequalities and Intervals*

Inequality	Interval
$x > b$	(b, ∞)
$x \geq b$	$[b, \infty)$
$x < a$	$(-\infty, a)$
$x \leq a$	$(-\infty, a]$
$a \leq x \leq b$ which means $x \geq a$ AND $x \leq b$, that is, x is *between* a and b, including both the a and the b.	$[a, b]$
$x < a$ OR $x > b$	$(-\infty, a) \cup (b, \infty)$

Homework

1. Convert each inequality to *interval notation*:

   a. $x > 2$ b. $x \le 5$ c. $-1 \le x < 6$ d. $x < -3$ OR $x > 0$

2. Convert each interval to an *inequality*:

   a. $[3, \infty)$ b. $(-\infty, -5)$ c. $(-1, 8]$ d. $(-\infty, -2] \cup (7, \infty)$

3. Prove that the statement

   $$x < 3 \text{ AND } x \ge 5$$

   consists of absolutely nothing, the null set, $\varnothing$.

4. What is the relationship between the notation $\mathbb{R}$ and the notation $\mathbf{(-\infty, \infty)}$?

Solutions

1. a. $(2, \infty)$ b. $(-\infty, 5]$ c. $[-1, 6)$ d. $(-\infty, -3) \cup (0, \infty)$

2. a. $x \ge 3$ b. $x < -5$ c. $-1 < x \le 8$ d. $x \le -2$ OR $x > 7$

3. There is <u>no</u> number that is both less than 3 <u>and</u> greater than or equal to 5. The two sets of numbers are disjoint, and so their intersection is $\varnothing$.

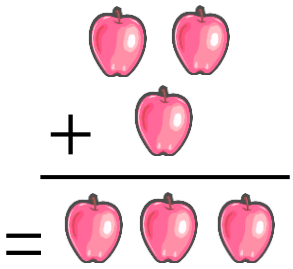
4. They both mean exactly the same thing: the set of all **real numbers**.

"Develop a passion for learning. If you do, you will never cease to grow."

Anthony J. D'Angelo

COMBINING LIKE TERMS

You've probably solved some simple equations in your course, equations like $x + 5 = 10$ or $-3n = 20$. These equations have the variable in just one place. What would we do if we came across an equation like $7x - 2x - 9 = 12x + 1$, with that darned variable all over the place? I suppose you can guess that this chapter will give us the tool we need to solve that kind of equation.



❑ THE COMMON-SENSE APPROACH TO COMBINING LIKE TERMS

If we add 5 inches and 6 inches, we get a sum of 11 inches:

5 inches + 6 inches = **11 inches**

If we subtract \$14 from \$50, we get a difference of \$36:

\$50 – \$14 = **\$36**

If 7 widgets were sold last week and 3 widgets were sold this week, then the two-week total is 10 widgets. We can write

7 widgets + 3 widgets = **10 widgets**

And if we let w represent a widget, we can write this fact as

$7w + 3w = \mathbf{10}\boldsymbol{w}$

and we say that $7w$ and $3w$ are ***like terms***, because each term represents some number of the exact same item.

Similarly, if 500 widgets were stolen from a warehouse that originally stored 2,000 widgets, we can write the amount remaining as 2,000 widgets – 500 widgets = 1,500 widgets, or

$$2{,}000w - 500w = \mathbf{1{,}500w}$$

We can also do this simplifying with more than two terms. For example, if we add 7 apples and 9 apples, then subtract 6 apples, and lastly add 10 additional apples, we can find the final number of apples like this:

$$7A + 9A - 6A + 10A = \mathbf{20A}$$

Homework

1. Work each problem, using the idea of *combining like terms*:

a.	9 feet + 12 feet	b.	12 lbs – 3 lbs
c.	14 apples + 14 apples	d.	\$123 + \$234
e.	100 meters – 98 meters	f.	17 widgets + 3 widgets
g.	93 cm – 90 cm	h.	3 iPads + 7 iPads
i.	€17 – €7 (€ = euro)	j.	34 m^2 + 16 m^2 (m^2 = square meter)

❑ *SOME TERMS CANNOT BE COMBINED*

We cannot do this adding (or subtracting) unless the items we're adding or subtracting are exactly the same. This is the old "*you can't add apples and oranges*" idea. Consider trying to analyze the sum "4 apples + 1 orange." Although you may "add" these together and say that the total is 5 pieces of fruit, you must agree that the sum is not 5 apples and it's not 5 oranges.



We say that "4 apples" and "1 orange" are ***unlike terms***; they cannot be added together. The terms $3x$ and $2w$ are also unlike terms and so the sum $3x + 2w$ cannot be simplified. Similarly, $7n$ and 5 are unlike terms and cannot be added together to make a single term.

This "simplification" process (assuming we have like terms) is called ***combining like terms*** and is used constantly in algebra. It basically says that regardless of what a variable might mean, we can combine two or more terms with the same variable into a single term by merely doing the arithmetic on the numbers connected to the variables; these numbers are called ***coefficients***. For example, when we write

$$5u + u = 6u$$

we say that the first term has a *coefficient* of 5, the second term has a *coefficient* of 1 (understood), and the sum (the answer) has a *coefficient* of 6.

❑ *EXAMPLES*

A. $10c + 20c = 30c$

B. $9x + x = 9x + 1x = 10x$

C. $-7y + 4y = -3y$

D. $\frac{2}{3}t - \frac{2}{3}t = 0t = 0$

E. $18m - 20m = -2m$

F. $-10h - 3h = -13h$

G. $25a - 13a - a = 11a$

H. $7x + 9y$ cannot be simplified (unlike terms)

I. $13N - 2$ cannot be simplified (unlike terms)

J. $7x - 7 - 9x + 12 = 7x - 9x - 7 + 12 = -2x + 5$

Homework

2. Your study partner is resisting the idea of like terms, and insists that $3x + 7$ ought to equal $10x$. By letting $x = 5$, prove your partner wrong.

3. Simplify each expression by combining like terms:

a. $10n + 7n$	b. $8x - 8x$	c. $14z + 14z$
d. $19a + a$	e. $32r - 20r$	f. $8x + 7y$
g. $77n + 77$	h. $24z - 24z$	i. $12Q - 11Q$
j. $-8x + 5x$	k. $-9t - t$	l. $12a - 20a$
m. $-9 + 9L$	n. $-7e - 7e$	o. $-12k - 13k$

4. Simplify each expression by combining like terms:

a. $3x + 7x - 8$	b. $9 + 2g - g$	c. $7x - 3x + 10$
d. $7 + 5x - 3$	e. $12d - 13d + d$	f. $4 - 7x + 9x$
g. $7y + 3y - 10y$	h. $-x - 2x - 3x$	i. $7w + 2w - 20w$
j. $-6r + 3r - r$	k. $30 - 3q + 3q$	l. $9u - 12 - 12u$

5. Simplify each expression by combining like terms:

a. $7x - 3 + 8x - 5$	b. $17 - n - n - 1$
c. $20q - 8q + 8 - 8$	d. $-1 - x - 1 - x$
e. $40i - 8 - 39i - 8$	f. $4v + 7 - 7v - 34$
g. $33w + 99 - 99 - 33w$	h. $-7x + 6x + 2x - 8 + 8$

Review Problems

6. Simplify by combining like terms:

$$7n + 2n - 8 - 1 + 5n - n - n + 9 - 13n + 20n$$

7. Now your study partner is insisting that adding $3n$ and $2n$ should come out $5n^2$. That is, he claims that $3n + 2n = 5n^2$. Prove him wrong again.

Solutions

1. a. 21 feet b. 9 lbs c. 28 apples d. \$357
e. 2 meters f. 20 widgets g. 3 cm h. 10 iPads
i. €10 j. 50 m^2

2. We will compare the expression $3x + 7$ with the expression $10x$ specifically when $x = 5$:

$3x + 7 = 3(\mathbf{5}) + 7 = 15 + 7 = 22$

$10x = 10(\mathbf{5}) = 50$; not the same

Therefore, $3x + 7 \neq 10x$. By the way, $3x + 7x$ would equal $10x$.

3. a. $17n$ b. 0 c. $28z$ d. $20a$ e. $12r$ f. As is
g. As is h. 0 i. Q j. $-3x$ k. $-10t$ l. $-8a$
m. As is n. $-14e$ o. $-25k$

4. a. $10x - 8$ b. $9 + g$ c. $4x + 10$ d. $5x + 4$ e. 0
f. $4 + 2x$ g. 0 h. $-6x$ i. $-11w$ j. $-4r$
k. 30 l. $-3u - 12$

5. a. $15x - 8$ b. $16 - 2n$ c. $12q$ d. $-2 - 2x$
e. $i - 16$ f. $-3v - 27$ g. 0 h. x

6. $19n$

7. We'll show him the error of his ways with a numerical counterexample; we'll let $n = 10$. On the one hand,

$$3n + 2n = 3(\mathbf{10}) + 2(\mathbf{10}) = 30 + 20 = 50.$$

On the other hand,

$$5n^2 = 5(\mathbf{10}^2) = 5(100) = 500.$$

Hence, $3n + 2n \neq 5n^2$.

P.S. Get a new study partner!

"In the middle of difficulty lies opportunity."

--Albert Einstein (1879-1955)

SOLVING LINEAR EQUATIONS USING LIKE TERMS

We now have the tool – ***combining like terms*** – that we need to solve equations like

$$7x - 2x - 9 = 12x + 1$$

This chapter will start with solving equations and then end with an application analyzing cell phone plans.

❑ *SOLVING EQUATIONS BY COMBINING LIKE TERMS*

EXAMPLE 1: **Solve each equation:**

A. $7x + 3x = 20$

$10x = 20$ (combine like terms)

$\frac{10x}{\mathbf{10}} = \frac{20}{\mathbf{10}}$ (divide each side by 10)

$\boldsymbol{x = 2}$ (simplify)

B. $4y - 2y = 15$

$2y = 15$ (combine like terms)

$\frac{2y}{\mathbf{2}} = \frac{15}{\mathbf{2}}$ (divide each side by 2)

$\boldsymbol{y = \frac{15}{2}}$ (simplify)

C. $23a = 3a - 80$

This is a little different from the equations we've seen so far in this course. Why? Because the variable sits on <u>both</u> sides of the equation — it's hard to isolate something when pieces of it lie all over the place. Our plan is as follows: Get all the variables to one side of the equation. You have the option of which side, but I prefer the left side.

So, let's begin by writing the given equation:

$$23a = 3a - 80$$

I hope it's clear to you that if we're going to get all the a's to the left side of the equation, we must <u>remove</u> all the a's on the right side. So our first step is to subtract $3a$ from <u>each</u> side of the equation:

$$\begin{array}{rcl} 23a & = & 3a - 80 \\ \mathbf{-\,3a} & & \mathbf{-\,3a} \end{array}$$

Now we combine the like terms on both sides of the equation:

$$20a = -80$$

The rest is old hat; divide each side by 20:

$$\frac{20a}{\mathbf{20}} = \frac{-80}{\mathbf{20}}$$

And simplify:

$$\boldsymbol{a = -4}$$

CHECK:

$23a$	$3a - 80$
$23(\mathbf{-4})$	$3(\mathbf{-4}) - 80$
$\underline{-92}$	$-12 - 80$
	$\underline{-92}$

✓

D. $10n - 3 = 9 - 6n$

$10n \mathbf{+ 6n} - 3 = 9 - 6n \mathbf{+ 6n}$ (add $6n$ to each side)

$16n - 3 = 9$ (combine like terms)

$16n - 3 \mathbf{+ 3} = 9 \mathbf{+ 3}$ (add 3 to each side)

$16n = 12$ (simplify)

$\frac{16n}{\mathbf{16}} = \frac{12}{\mathbf{16}}$ (divide each side by 4)

$\boldsymbol{n} = \frac{\mathbf{3}}{\mathbf{4}}$ (simplify)

EXAMPLE 2: **Solve for u: $3u - 7 + 8u - 1 = 9 - 2u + 12 - 6u$**

Solution: This is one long equation! But we have all the skills required to solve it. The secret is to temporarily ignore the equation aspect of the problem, and start by simplifying each side of the equation separately — by combining like terms. Then the equation will look similar to part D of the previous example, and we'll be in good shape.

$3u - 7 + 8u - 1 = 9 - 2u + 12 - 6u$ (the original equation)

$\Rightarrow \quad 3u + 8u - 7 - 1 = -2u - 6u + 9 + 12$ (rearrange the terms)

$\Rightarrow \quad 11u - 8 = -8u + 21$ (combine like terms)

$\Rightarrow \quad 11u \mathbf{+ 8u} - 8 = -8u \mathbf{+ 8u} + 21$ (add $8u$ to each side)

$\Rightarrow \quad 19u - 8 = 21$ (combine like terms)

$\Rightarrow \quad 19u - 8 \mathbf{+ 8} = 21 \mathbf{+ 8}$ (add 8 to each side)

$\Rightarrow \quad 19u = 29$ (do the arithmetic)

$\Rightarrow \quad \frac{19u}{\mathbf{19}} = \frac{29}{\mathbf{19}}$ (divide each side by 19)

$\Rightarrow \quad \boxed{u = \frac{29}{19}}$ (simplify)

Homework

1. Solve each equation:

a. $2x + 1 = 10$ b. $-3n - 7 = -1$ c. $7z - 1 = -5$

d. $7x - 2x = 50$ e. $-6y + y = 12$ f. $30 = 7n - n$

g. $7a - 7 = 2a$ h. $8 - 2x = 10x$ i. $8m - 5 = m$

j. $4 - q = 10$ k. $20 - 7u = 7u$ l. $10 - 7x = 17x$

2. Solve each equation:

a. $4x - 7 = 2x + 1$ b. $20n - 20 = 18n - 1$

c. $12 - n = 14 - 2n$ d. $6v - 2v = 10 + 20$

e. $2h + 7 = -3h + 7$ f. $12p + 13 = 11p + 14$

g. $-7x - 8 = -9x - 8$ h. $9b + 7 = 8 + 12b$

i. $17a - 15 = -9a - 5$ j. $23c - 80 = -17c - 90$

3. Solve each equation:

a. $8x - 3x + 7 = 14x - 8$

b. $9e + 7 - 3e - 8 = 12 - 5e + 6 - 4e$

c. $3u - 7 + 9u - 1 = 19 - 2u + 13 - 5u$

d. $3x + 2x + 3 + 2 = 19x - 17x + 8 - 3$

e. $23a - 17 - 18 - 24a = 100 + a - 44a - 105$

f. $-5h + 2 - 3h - 1 = -5h + 7 - 4h + 11$

g. $-10a + 8 + 5a + 8 = -2a + 4 + 4a + 12$

h. $-2j - 6 + 6j - 12 = 9j + 2 - 7j - 6$

❑ *CELL PHONE APPLICATION*



EXAMPLE 3: **Sprint offers you a monthly phone plan that has a sign-up fee of $20, plus $0.25/min. Verizon's plan has no sign-up fee, but charges $0.65/min.**

a. Assuming an average of 40 minutes per month, which is the better plan?

b. Assuming an average of 70 minutes per month, which is the better plan?

c. How many minutes of phone use would result in an equal charge from each company?

Solution: Let's write a formula for each company's cell phone plan. If we let m represent the number of minutes of use, we get the following two formulas:

$S = 20 + 0.25m$ (for Sprint)

$V = 0.65m$ (for Verizon)

a. Calculate each plan's cost for 40 minutes of use:

$S = 20 + 0.25m = 20 + 0.25(\mathbf{40}) = 20 + 10 = 30$

$V = 0.65m = 0.65(\mathbf{40}) = 26$

For 40 minutes of use, Sprint charges $30.00, while Verizon charges $26.00. Thus, in this scenario, Verizon has the better plan.

b. Now we'll assume 70 minutes of use:

$$S = 20 + 0.25m = 20 + 0.25(\mathbf{70}) = 20 + 17.5 = 37.5$$

$$V = 0.65m = 0.65(\mathbf{70}) = 44.5$$

For 70 minutes, Sprint charges $37.50 and Verizon charges $44.50. So this time Sprint has the better plan.

c. To determine the *break-even point* — the number of minutes where each plan costs the same — all we have to do is set Sprint's charges equal to Verizon's and solve the resulting equation:

$$S = V$$

Write each company's formula for their charges:

$$20 + 0.25m = 0.65m$$

To get all the variables on the left side, subtract $0.65m$
And simplify by combining like terms:

$$20 - 0.4m = 0$$

Subtract the constant from each side of the equation:

$$20 - \mathbf{20} - 0.4\text{m} = 0 - \mathbf{20}$$

And simplify:

$$-0.4m = -20$$

Last step — divide each side of the equation by –0.4:

$$\frac{-0.4m}{\mathbf{-0.4}} = \frac{-20}{\mathbf{-0.4}}$$

And we're done:

$$m = 50$$

We conclude that if we talk 50 minutes in a month, each plan will cost the same. From parts a. and b. of this problem, it should also be clear that Verizon is cheaper if you use less than 50 minutes of talk time each month. However, Sprint's plan is better if you usually exceed 50 minutes per month.

Homework

4. For each problem, I give you two cell phone plans. You need to determine the number of minutes that would result in the *same charge* for each plan:

a.	Plan #1	Sign-up fee: $40	Per minute: $0.10
	Plan #2	Sign-up fee: $0	Per minute: $0.20
b.	Plan #1	Sign-up fee: $25	Per minute: $0.20
	Plan #2	Sign-up fee: $50	Per minute: $0.15
c.	Plan #1	Sign-up fee: $15	Per minute: $0.25
	Plan #2	Sign-up fee: $38	Per minute: $0.05



Review Problems

5. Solve each equation:

   a. $w + 3 - 8w + 8 = 7w + 5 - 9w - 10$

   b. $3b + 8 - 2b + 6 = -2b + 8 - 6b - 1$

   c. $-10g + 6 + 5g + 2 = 10g + 9 - 3g + 6$

   d. $-7t - 2 + 9t - 4 = 9t - 2 + 5t + 10$

6. Here are two cell phone plans:

Plan #1	Sign-up fee: \$19.00	Per minute: \$0.65
Plan #2	Sign-up fee: \$6.00	Per minute: \$0.90

How many minutes of talk time would result in the same charge for each plan?

Solutions

1. a. $x = \frac{9}{2}$ b. $n = -2$ c. $z = -\frac{4}{7}$ d. $x = 10$

e. $y = -\frac{12}{5}$ f. $n = 5$ g. $a = \frac{7}{5}$ h. $x = \frac{2}{3}$

i. $m = \frac{5}{7}$ j. $q = -6$ k. $u = \frac{10}{7}$ l. $x = \frac{5}{12}$

2. a. $x = 4$ b. $n = \frac{19}{2}$ c. $n = 2$ d. $v = \frac{15}{2}$

e. $h = 0$ f. $p = 1$ g. $x = 0$ h. $b = -\frac{1}{3}$

i. $a = \frac{5}{13}$ j. $c = -\frac{1}{4}$

3. a. $x = \frac{5}{3}$ b. $e = \frac{19}{15}$ c. $u = \frac{40}{19}$ d. $x = 0$

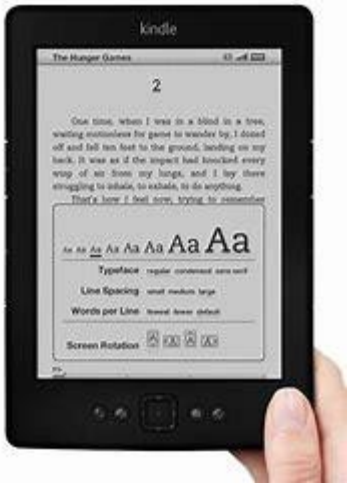
e. $a = \frac{5}{7}$ f. $h = 17$ g. $a = 0$ h. $j = 7$

4. a. 400 minutes b. 500 minutes c. 115 minutes

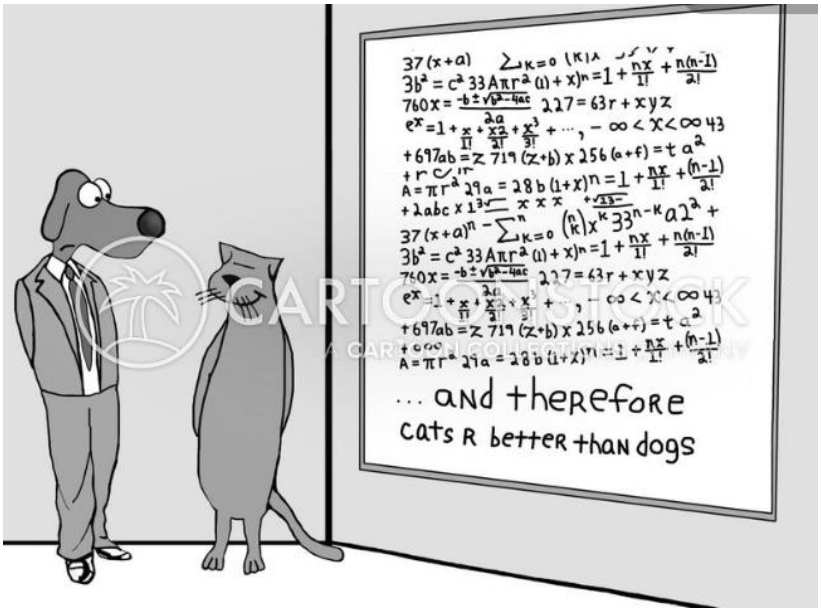
5. a. $w = \frac{16}{5}$ b. $b = -\frac{7}{9}$ c. $g = -\frac{7}{12}$ d. $t = -\frac{7}{6}$

6. 52 minutes

❑ *TO ∞ AND BEYOND!*



Max wants to save up for a Kindle costing $200. Max's dad said that if he gets a job, whatever he earns, dad will give him twice that amount toward the purchase of the Kindle. In addition, Aunt Edna has promised to contribute $20 to the cause. By letting *M* represent the amount of money Max earns at his job, write an equation representing this situation. Solve the equation to see how much Max must earn himself to reach his $200 goal.



"When I get a little money,
I buy books;
and if any is left,
I buy food and clothes."

Erasmus, early 1500s



CREATING THE SLOPE-INTERCEPT ($y = mx + b$) FORM OF A LINE



Some chapters back we played around with straight lines. We graphed a few, and we learned how to find their *intercepts* and *slopes*. Now we're ready to formalize the whole concept into a single idea — the kind of thing you'll need for your next algebra course, statistics, chemistry, business, economics, and many other disciplines.

Before we tackle these lines, let's review the algebra skills and straight-line concepts we'll need for this chapter.

Homework

1. Solve each formula for y:

   a. $-4x + y - 1 = 0$ b. $-3x - y = 7$

   c. $-2x + y + 1 = 0$ d. $3x + 2y = 10$

   e. $2x - 7y = 14$ f. $-8x - 2y + 7 = 0$

2. Find all the intercepts of the line $3x - 7y = 42$.

3. Find the slope of the line connecting the points $(2, -3)$ and $(-8, 9)$.

❑ *Calculating the Slope & y-intercept of a Given Line*

Consider the line

$$y = 2x + 3$$

Let's see what we can discover about this line, using just the ideas from the previous chapters.

First, we'll determine the **slope** of the line. To do this, we need a pair of points on the line, which are simply solutions of the line equation. To get points on the line, we'll choose a couple of x-values off the top of our head, and then calculate the corresponding y-values.

Suppose $x = 5$; then $y = 2(\mathbf{5}) + 3 = 10 + 3 = 13$.
Thus, **(5, 13)** is a point on the line.

Now let $x = -1$; and so $y = 2(-\mathbf{1}) + 3 = -2 + 3 = 1$.
Therefore, **(–1, 1)** is a point on the line.

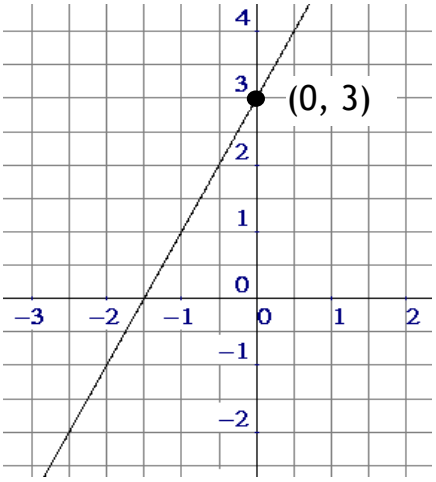
We can now compute the slope using the points we just calculated, (5, 13) and (–1, 1):

$$m = \frac{\Delta y}{\Delta x} = \frac{13-1}{5-(-1)} = \frac{13-1}{5+1} = \frac{12}{6} = \mathbf{2}$$

Second, we'll calculate the ***y*-intercept** of the line. Recall that the y-intercept of any graph is found by setting x to 0. Here's what we get:

$$y = 2(\mathbf{0}) + 3 = 0 + 3 = \mathbf{3}$$

The y-intercept is therefore **(0, 3)**. In summary,



The line

$$\boldsymbol{y = 2x + 3}$$

has a slope of 2 and a *y*-intercept of (0, 3).

Homework

4. Consider the line $y = -3x + 7$.

   a. Find two points on the line. For instance, let $x = 1$ and then let $x = 3$ (or choose your own x's).

   b. Find the slope of the line using the two points you found in part a. by applying the definition of slope, $m = \frac{\Delta y}{\Delta x}$.

   c. Find the y-intercept of the line (by letting $x = 0$, of course).

5. Consider the line $y = 5x - 14$.

   a. Find two points on the line. For instance, let $x = -2$ and then let $x = 4$.

   b. Find the slope of the line using the two points you found in part a. by applying the definition of slope, $m = \frac{\Delta y}{\Delta x}$.

   c. Find the y-intercept of the line by letting $x = 0$.

6. Consider the line $y = -x - 17$.

   a. Find two points on the line.

   b. Find the slope of the line using the two points you found in part a.

   c. Find the y-intercept of the line.

7. Consider the line $y = x + 44$.

   a. Find two points on the line.

   b. Find the slope of the line using the two points you found in part a.

   c. Find the y-intercept of the line.

❑ THE $y = mx + b$ FORM OF A LINE

Here's a summary of the four homework problems you just completed (You did them, right?):

$y = -3x + 7$	$m = -3$	y-int = $(0, 7)$
$y = 5x - 14$	$m = 5$	y-int = $(0, -14)$
$y = -x - 17$	$m = -1$	y-int = $(0, -17)$
$y = x + 44$	$m = 1$	y-int = $(0, 44)$



See a pattern here? The slope of each line is simply the number in front of the x, the *coefficient* of the first term. For example, the slope of the line $y = 5x - 14$ is 5.



Important: Note that the slope of the line is **5**, NOT $5x$.

Also, the y-intercept is essentially the number hanging off the end of the equation. ["Essentially" means that the y-intercept is the *point* (0, *something*), and that *something* is the number at the end of the line equation.] For example, for the line $y = x + 44$, the y-intercept is (0, 44). (Remember that some teachers and books will allow you to write the y-intercept simply as 44.)

> Therefore, if you're asked for the slope and the y-intercept of the line $y = -14x + 17$, for example, you should be able to immediately reply that the slope is -14 and the y-intercept is (0, 17).

Conversely, if you're asked to come up with the equation of the line whose slope is 23 and whose y-intercept is $(0, \pi)$, be sure you understand that no calculations are required to come up with the equation $\boldsymbol{y = 23x + \pi}$.

Let's generalize the four examples above. We will, as usual, let m represent the slope of the line, and let b (for some odd reason) stand for the y-coordinate of the y-intercept. Here's what it all boils down to:

Creating the Slope-Intercept ($y = mx + b$) Form of a Line

$$y = mx + b$$

SLOPE

y-INTERCEPT

To be precise, *b* is not the *y*-intercept; *b* is the *y*-coordinate of the *y*-intercept. The *y*-intercept is properly written (0, *b*).

EXAMPLE 1:

A. Find the slope and the *y*-intercept of the line $y = -\frac{2}{3}x - 5$.

Answer: The slope is $-\frac{2}{3}$ and the *y*-intercept is (0, –5).

B. Find the equation of the line whose slope is –6 and whose *y*-intercept is $\left(0, \frac{4}{5}\right)$. Answer: $y = -6x + \frac{4}{5}$.

EXAMPLE 2: **Find the slope and the *y*-intercept of the line $3x - 5y + 15 = 0$.**

Solution: The given equation, $3x - 5y + 15 = 0$, doesn't fit the slope-intercept form, $y = mx + b$, of a line. But we can make it fit; we can solve the equation $3x - 5y + 15 = 0$ for *y*:

	$3x - 5y + 15 = 0$	(the original line)
$\Rightarrow$	$3x - 5y = -15$	(subtract 15 from each side)
$\Rightarrow$	$-5y = -3x - 15$	(subtract $3x$ from each side)
$\Rightarrow$	$\frac{-5y}{-5} = \frac{-3x-15}{-5}$	(divide each side by –5)

$\Rightarrow \quad y = \frac{3x}{5} + \frac{15}{5}$ (split the right-hand fraction)

$\Rightarrow \quad y = \frac{3}{5}x + 3$ (rewrite the first fraction)

Now that the line is in the $y = mx + b$ form, we conclude that

The slope is $\frac{3}{5}$ and the y-intercept is (0, 3).

Homework

8. a. Find the slope and y-intercept of the line $y = -17x + 13$.

   b. Find the equation of the line whose slope is 99 and whose y-intercept is $(0, -101)$

   c. Find the equation of the line whose slope is $\frac{2}{3}$ and whose y-intercept is $(0, -\pi)$.

   d. Find the equation of the line whose slope is $-\frac{5}{4}$ and whose y-intercept is $(0, 2\pi)$.

9. Find the slope and y-intercept of each line by converting the line to $y = mx + b$ form, if necessary:

   a. $y = 132x - 1000$
   b. $y = -\frac{8}{7}x - \frac{13}{9}$
   c. $7x - 9y = 10$
   d. $-3x - 5y + 1 = 0$
   e. $2x + 7y = 13$
   f. $-5x + 2y + 3 = 0$
   g. $y = \frac{9x - 5}{2}$
   h. $-17x - y = 4$
   i. $2x - 6y = 8$
   j. $-x + 4y - 2 = 0$
   k. $7y - 2x = 0$
   l. $7x + 4y + 5 = 0$

❑ *A Proof of the Slope–Intercept Form of a Line*

We've learned the following:

The line with slope m and y-intercept $(0, b)$ has the equation $\boldsymbol{y = mx + b}$.

We did this by viewing some homework results, but examples prove nothing. So let's do it the right way, by proving that $y = mx + b$ truly has the properties we've been claiming it has.

Claim #1: **The line $y = mx + b$ has a y-intercept of $(0, b)$.**

Proof: To find the y-intercept(s) of any graph, we set x to 0 and solve for y:

$$y = m(\mathbf{0}) + b \;\Rightarrow\; y = 0 + b \;\Rightarrow\; y = b.$$

In other words, when $x = 0$, $y = b$, which means that $(0, b)$ is on the line, and thus $(0, b)$ is precisely the y-intercept.

Claim #2: **The line $y = mx + b$ has a slope of m.**

Proof: Our definition of slope, $m = \dfrac{\Delta y}{\Delta x}$, will be used to calculate the slope of the line $y = mx + b$. To apply this definition, we need two points on the line. One of them might as well be the y-intercept calculated above: **$(0, b)$**. For the other point, pick $x = 1$. This yields a y-value of $y = m(\mathbf{1}) + b = m + b$. Therefore, **$(1, m + b)$** is a second point on the line. Now we can find the slope, using the two points $(0, b)$ and $(1, m + b)$:

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{(m+b)-b}{1-0} = \frac{m+b-b}{1} = \frac{m}{1} = \boldsymbol{m}$$

and thus the slope of the line $y = mx + b$ is indeed m.
Our two claims have verified our assertion, and the proof is complete.

Review Problems

10. Consider the line $y = 10x - 13$.

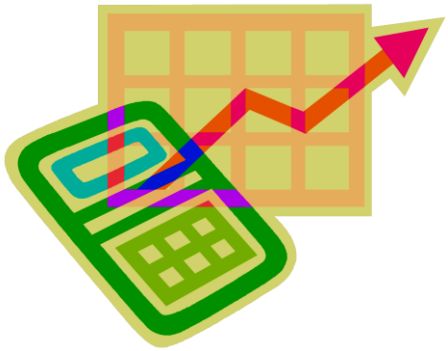
    a. Find two points on the line. For instance, let $x = 1$ and then let $x = -2$.

    b. Find the slope of the line using the two points you found in part a. by applying the definition of slope, $m = \frac{\Delta y}{\Delta x}$.

    c. Find the y-intercept of the line by letting $x = 0$.

11. Find the equation of the line whose slope is -17 and whose y-intercept is $(0, 99)$.

12. Find the slope and the y-intercept of the line $-28x + 4y = 16$ by converting the line to $y = mx + b$ form.



Solutions

1. a. $y = 4x + 1$

b. $-3x - y = 7$

$\Rightarrow \quad -y = 3x + 7$

$\Rightarrow \quad \frac{-y}{-1} = \frac{3x + 7}{-1}$ (you could also multiply each side by -1)

$\Rightarrow \quad y = -3x - 7$

c. $y = 2x - 1$

d. $3x + 2y = 10 \Rightarrow 2y = -3x + 10$
$\Rightarrow y = \frac{-3x+10}{2}$
$\Rightarrow y = -\frac{3}{2}x + 5$

e. $y = \frac{2}{7}x - 2$

f. $y = -4x + \frac{7}{2}$

2. x-int: (14, 0); y-int: (0, –6)

3. $m = -\frac{6}{5}$

4. a. (1, 4) and (3, –2) b. $m = \frac{\Delta y}{\Delta x} = \frac{4-(-2)}{1-3} = \frac{6}{-2} = -3$

c. $y = -3(\mathbf{0}) + 7 = 7$; so the y-intercept is (0, 7).

5. a. (–2, –24) and (4, 6) b. $m = \frac{\Delta y}{\Delta x} = \frac{-24-6}{-2-4} = \frac{-30}{-6} = 5$

c. $y = 5(\mathbf{0}) - 14 = -14$; so the y-intercept is (0, –14).

6. a. You choose the two points. b. $m = -1$ c. y-int = (0, –17)

7. a. You choose the two points. b. $m = 1$ c. y-int = (0, 44)

8. a. $m = -17$ y-int = (0, 13) b. $y = 99x - 101$

c. $y = \frac{2}{3}x - \pi$ d. $y = -\frac{5}{4}x + 2\pi$

9. a. $m = 132$ y-int = (0, –1000) b. $m = -\frac{8}{7}$ y-int = $(0, -\frac{13}{9})$

c. $m = \frac{7}{9}$ y-int = $(0, -\frac{10}{9})$ d. $m = -\frac{3}{5}$ y-int = $(0, \frac{1}{5})$

e. $m = -\frac{2}{7}$ y-int = $(0, \frac{13}{7})$ f. $m = \frac{5}{2}$ y-int = $(0, -\frac{3}{2})$

g. $m = \frac{9}{2}$ y-int = $(0, -\frac{5}{2})$ h. $m = -17$ y-int = (0, –4)

i. $m = \frac{1}{3}$ y-int = $(0, -\frac{4}{3})$ j. $m = \frac{1}{4}$ y-int = $(0, \frac{1}{2})$

k. $m = \frac{2}{7}$ y-int = (0, 0) l. $m = -\frac{7}{4}$ y-int = $(0, -\frac{5}{4})$

10. a. (1, –3) and (–2, –33)

b. $m = \frac{\Delta y}{\Delta x} = \frac{-3-(-33)}{1-(-2)} = \frac{-3+33}{1+2} = \frac{30}{3} = 10$

c. $y = 10(\mathbf{0}) - 13 = 0 - 13 = -13$; the y-intercept is (0, –13).

11. $y = -17x + 99$

12. $m = 7$; y-int: (0, 4)

❑ *TO ∞ AND BEYOND*

Consider the infinite sequence of numbers:

8, 10, 12, 14, 16, . . .

If 8 is the 1st term, and 10 is the 2nd term, etc., what is the 1,000th term?

"Without education, we are in a horrible and deadly danger of taking educated people seriously."

– G.K. Chesterton

THE EQUATION OF A LINE – $y = mx + b$

Given an equation like $y = 7x + 3$, we've learned that its graph is a line with a slope of 7 and a y-intercept of (0, 3), and we certainly could find other points on the line. In this chapter we turn the tables: Given some information about the line, we try to seek the equation of the line. For you CSI fans, it's like using blood and hair samples to determine the identity of the killer.



❑ *FINDING THE LINE GIVEN SLOPE AND Y-INTERCEPT*

EXAMPLE 1: **Find the equation of the line which has a slope of –3 and whose y-intercept is the point (0, 9).**

Solution: This is the easiest possible problem asking us to find the equation of a line. Here's why: The slope-intercept form of a line we've learned is

$$y = mx + b$$

where m is the slope and $(0, b)$ is the y-intercept. To "fill in the blanks" of this equation, we need m and b. Now look at what's given to us in the problem: the slope and the y-intercept. That is, we're told that $m = -3$ and $b = 9$. We're done!

$$\boxed{y = -3x + 9}$$

❑ *FINDING THE LINE GIVEN THE SLOPE AND A POINT*

EXAMPLE 2: **Find the equation of the line which has a slope of 7 and which passes through the point (–5, 3).**

Solution: The line equation we are using is $y = mx + b$, where m is the slope and $(0, b)$ is the y-intercept. In this example, we are given the slope of 7. That's good.

And so the line equation $y = mx + b$

becomes $y = 7x + b$ (putting the 7 in for slope)

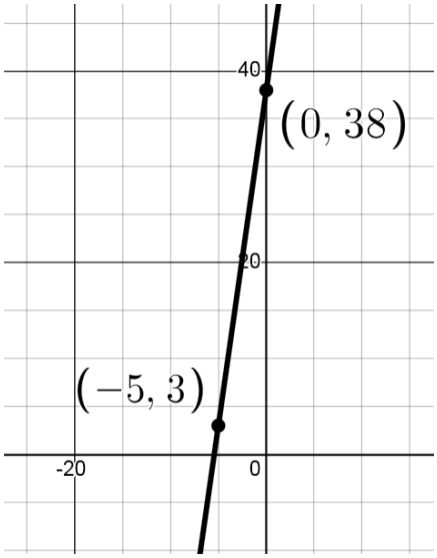
But did the problem give us the y-intercept? No; instead, we are given a point on the line, but I know that it's not the y-intercept. (How can I be so sure?) So now our goal is to find the value of b.

Consider that the problem tells us that the point $(-5, 3)$ is on the line. Therefore, the point must work in the equation we just wrote, $y = 7x + b$. So we plug –5 in for x and plug 3 in for y:

	$y = 7x + b$	(our line with the slope plugged in)
$\Rightarrow$	$\mathbf{3} = 7(\mathbf{-5}) + b$	(since (–5, 3) lies on the line)
$\Rightarrow$	$3 = -35 + b$	(multiply)
$\Rightarrow$	$b = 38$	(solve for b)

Now we put it all together. The value of m is 7 (given in the problem) and the value of b is 38 (we just calculated it). We get our final answer:

$$\boxed{y = 7x + 38}$$



Homework

1. Find the equation of the line with the given slope and passing through the given point:

a. $m = -3$; (8, –16)

b. $m = 4$; (–1, –11)

c. $m = 2$; (0, –10)

d. $m = -3$; (3, –2)

e. $m = 9$; (2, 18)

f. $m = -1$; (–3, 11)

g. $m = 1$; (5, 5)

h. $m = 7$; (0, 10)

i. $m = -8$; (–2, 29)

j. $m = 1$; (10, –89)

❑ *FINDING THE LINE GIVEN TWO POINTS*

EXAMPLE 3: **Find the equation of the line passing through the points (–1, 3) and (8, –15).**

Solution: This problem will really test our deductive skills. Let's begin with the slope-intercept form of a line:

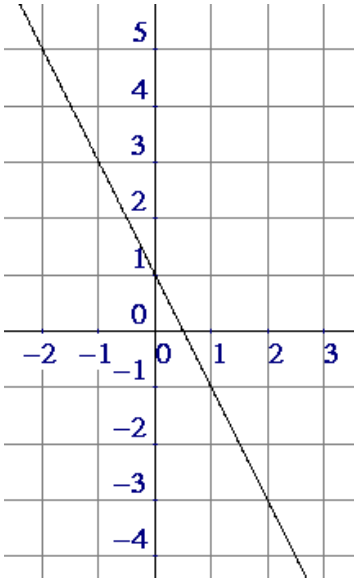
$$y = mx + b$$ [we need the slope and the *y*-intercept]

Did the problem tell us what the slope is? No. Did the problem give us the *y*-intercept? No. This is not good — how can we possibly solve this problem? Well, even though the slope was not handed to us on a silver platter, we can use the two given points on the line to calculate the slope, using our $m = \frac{\Delta y}{\Delta x}$ formula. So, using the given points (–1, 3) and (8, –15), we find the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{3-(-15)}{-1-8} = \frac{3+15}{-1-8} = \frac{18}{-9} = -2$$

We can now write our line as

$$y = -2x + b$$



How do we find b? The same way we did in the previous example: Plug the coordinates of one of the original points in for x and y — either point will do the job, since each of them lies on the line. Let's use the point (–1, 3).

$$y = -2x + b$$
$$\Rightarrow \quad \mathbf{3} = -2(\mathbf{-1}) + b$$
$$\Rightarrow \quad 3 = 2 + b$$
$$\Rightarrow \quad b = 1$$

Putting the values of m and b into the formula $y = mx + b$ gives us the line which passes through the two points:

$$\boxed{y = -2x + 1}$$

✓ **Check:** Let's check our final answer. If $y = -2x + 1$ is really the line passing through the two given points, then obviously each point should lie on the line. That is, each point should satisfy the equation of the line.

$$y = -2x + 1 \qquad \text{Is this the right equation?}$$

(–1, 3): $\mathbf{3} = -2(\mathbf{-1}) + 1 \Rightarrow 3 = 2 + 1 \Rightarrow 3 = 3$ ✓

(8, –15): $\mathbf{-15} = -2(\mathbf{8}) + 1 \Rightarrow -15 = -16 + 1 \Rightarrow -15 = -15$ ✓

Homework

2. Find the equation of the line passing through the two given points. Also be sure you know how to check your answer:

a. (3, 13) and (–1, 5)

b. (1, –9) and (–5, 39)

c. (0, –1) and (2, 9)

d. (1, –11) and (6, –1)

e. (1, –8) and (–2, 31)

f. (0, 0) and (–5, 35)

g. (7, 7) and (–3, –3)

h. (0, 17) and (–17, 0)

i. (–5, –4) and (2, –11)

j. (1, –3) and (–4, –38)

k. (2, –30) and (–1, –3)

l. (–4, –23) and (–1, 7)

m. (25, –43) and (–10, 27)

n. (1, –13) and (–13, –97)

EXAMPLE 4: **Find the equation of the line passing through the points (2, 5) and (–1, –3).**

Solution: This is the same type of question as in the previous example, except that fractions will be involved. In order to utilize the slope-intercept form of a line, $y = mx + b$, we need to know the slope and the y-intercept. Since neither one of these was explicitly given in the problem, we'll have to calculate them ourselves.

First, we find the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{5-(-3)}{2-(-1)} = \frac{5+3}{2+1} = \frac{8}{3}$$

Our line can now be written as $y = \frac{8}{3}x + b$.

Second, we find the value of b. To accomplish this, we can place either point from the problem into the equation we just wrote — let's choose (2, 5):

$$5 = \frac{8}{3}(2) + b \quad \Rightarrow \quad 5 = \frac{16}{3} + b \quad \Rightarrow \quad \frac{15}{3} - \frac{16}{3} = b \quad \Rightarrow \quad b = -\frac{1}{3}$$

Putting it all together, the equation of the line is

$$\boxed{y = \frac{8}{3}x - \frac{1}{3}}$$

Homework

3. Find the equation of the line passing through the two given points:

   a. (10, 9) and (7, –7)

   b. (–4, –5) and (–8, 0)

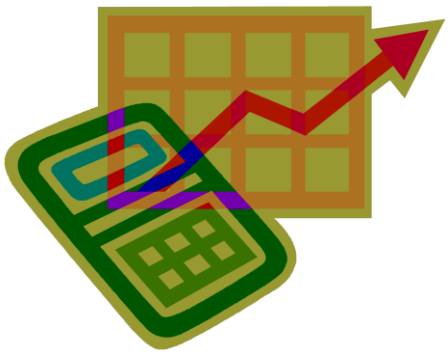
   c. (–2, 4) and (1, –6)

   d. (–2, 10) and (–7, –7)

   e. (7, 5) and (–4, –3)

   f. (–8, 4) and (–2, –4)

Review Problems



4. Find the slope and y-intercept of the line $-28x + 4y = 16$ by converting the line to $y = mx + b$ form.

5. Find the equation of the line which has slope 5 and which passes through the point (6, 30).

6. Find the equation of the line which passes through the points (2, 15) and (–2, –9).

7. Find the slope and y-intercept of the line $-45x - 5y = -15$ by converting the line to $y = mx + b$ form.

8. Find the equation of the line which has slope –4 and which passes through the point (–5, 22).

9. Find the equation of the line which passes through the points (–2, –6) and (7, 3).

Solutions

1. a. $y = -3x + 8$ b. $y = 4x - 7$ c. $y = 2x - 10$
d. $y = -3x + 7$ e. $y = 9x$ f. $y = -x + 8$
g. $y = x$ h. $y = 7x + 10$ i. $y = -8x + 13$
j. $y = x - 99$

2. a. $y = 2x + 7$ b. $y = -8x - 1$ c. $y = 5x - 1$
d. $y = 2x - 13$ e. $y = -13x + 5$ f. $y = -7x$
g. $y = x$ h. $y = x + 17$ i. $y = -x - 9$
j. $y = 7x - 10$ k. $y = -9x - 12$ l. $y = 10x + 17$
m. $y = -2x + 7$ n. $y = 6x - 19$

3. a. $y = \frac{16}{3}x - \frac{133}{3}$ b. $y = -\frac{5}{4}x - 10$
c. $y = -\frac{10}{3}x - \frac{8}{3}$ d. $y = \frac{17}{5}x + \frac{84}{5}$
e. $y = \frac{8}{11}x - \frac{1}{11}$ f. $y = -\frac{4}{3}x - \frac{20}{3}$

4. $m = 7$; y-int: $(0, 4)$

5. $y = 5x$

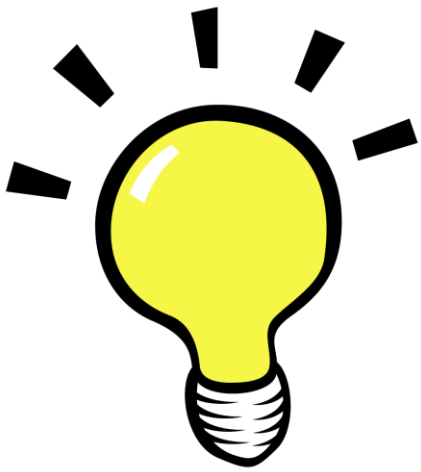
6. $y = 6x + 3$

7. $m = -9$; y-int: $(0, 3)$

8. $y = -4x + 2$

9. $y = x - 4$

"I haven't failed, I've found 1,000 ways that don't work."

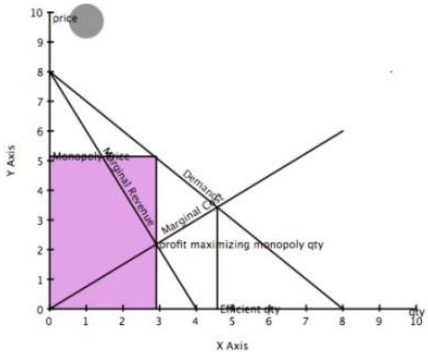




Thomas Edison

INTERCEPTS

You probably studied the chapter that taught you how to graph a straight line by plotting points. This chapter focuses on two special points that are easy to calculate and very important for future chapters in math and for business, statistics, and science courses.



❑ *CALCULATING INTERCEPTS*

Consider the line $2x - 3y = 12$. We can find one point on the line very easily, by letting $x = 0$. This produces

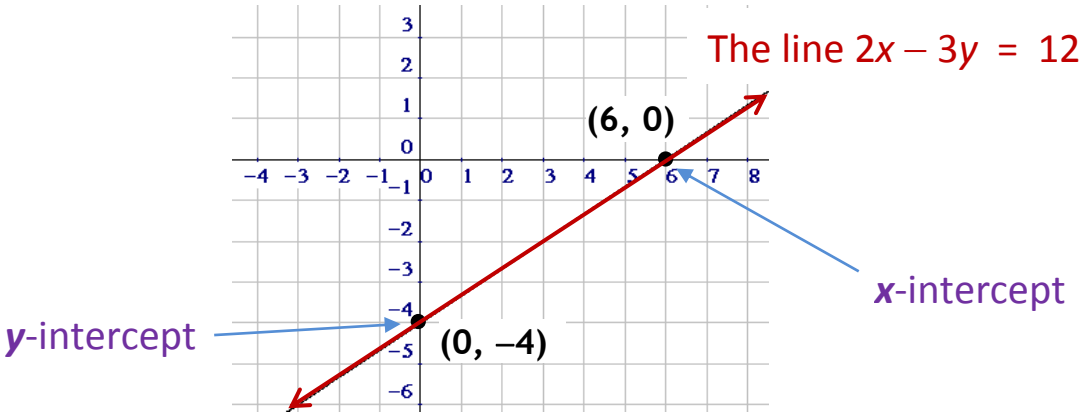
$$2(\mathbf{0}) - 3y = 12$$
$$\Rightarrow \quad 0 - 3y = 12$$
$$\Rightarrow \quad -3y = 12$$
$$\Rightarrow \quad \underline{y = -4} \qquad \text{(divide each side of the equation by } -3\text{)}$$

Since we let $x = 0$, and got the result $y = -4$, this shows us that <u>the point **(0, –4)** is on the line</u>.

Now let's set y to 0. We obtain

$$2x - 3(\mathbf{0}) = 12 \;\Rightarrow\; 2x - 0 = 12 \;\Rightarrow\; 2x = 12 \;\Rightarrow\; \underline{x = 6}$$

We conclude that <u>the point **(6, 0)** is also on the line</u>. Since two points suffice to construct a line (although plotting more than two points is an excellent idea!), we'll graph our line right now using the points **(0, –4)** and **(6, 0)**:



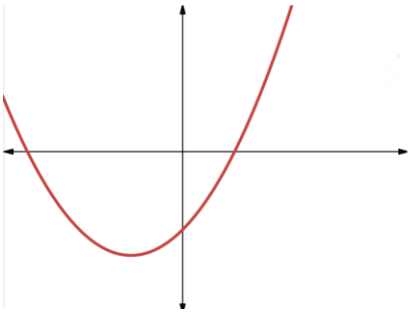
Notice that the point (6, 0), although certainly on the line $2x - 3y = 12$, is special because it lies on the x-axis. We call the point (6, 0) in this example the ***x-intercept*** of the line. Similarly, we call the point (0, –4) the ***y-intercept*** of the line.

Looking back at the calculations, we see that $x = 6$ (which gave us the x-intercept) was found by setting y to 0, and $y = -4$ (which yielded the y-intercept) was found by setting x to 0. Here's a summary of this easy way to find the intercepts of a line (and other graphs, too):

To find ***x***-intercepts, set ***y*** = 0.

To find ***y***-intercepts, set ***x*** = 0.

Of course, graphs such as circles and parabolas (which you might see later in this course) may have more than one x-intercept or more than one y-intercept (or even no intercepts of any kind). And when we get to horizontal and vertical lines pretty soon, you'll see that they may have only one kind of intercept, or they may even have an infinite number of intercepts!



This parabola has 3 intercepts

EXAMPLE 1: **Find the *x*-intercept and the *y*-intercept of the line $3x - 7y = 42$.**

Solution: To find the *x*-intercepts of this line (of any graph, in fact) we set $y = 0$ and solve for *x*:

$3x - 7y = 42$ (the line)

$\Rightarrow \quad 3x - 7(0) = 42$ (set $y = 0$)

$\Rightarrow \quad 3x - 0 = 42$ (anything times 0 is 0)

$\Rightarrow \quad 3x = 42$ (simplify)

$\Rightarrow \quad \frac{3x}{3} = \frac{42}{3}$ (divide each side by 3)

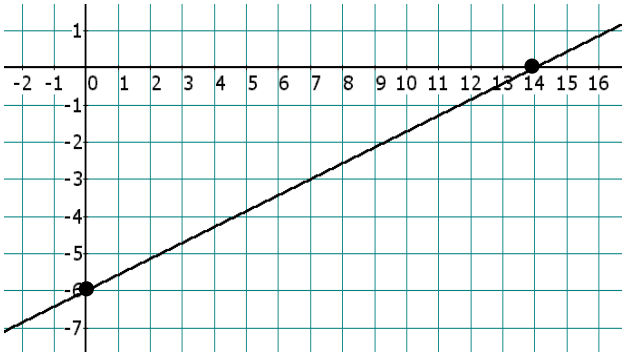
$\Rightarrow \quad x = 14$

The *x*-intercept is therefore $\boxed{(14, 0)}$

Setting $x = 0$ to find the *y*-intercept gives

$3(0) - 7y = 42 \quad \Rightarrow \quad -7y = 42 \quad \Rightarrow \quad y = -6$

and so the *y*-intercept is $\boxed{(0, -6)}$



The *x*-intercept is **(14, 0)**, the point where the line intersects the *x*-axis.

The *y*-intercept is **(0, –6)**, the point where the line intersects the *y*-axis.

Note: Some teachers will allow you to say that the *x*-intercept is 14, and some will require you to say that it's (14, 0), which is the notation promoted by this book. Ask your teacher what will be allowed.

Homework

1. Each of the following points (except one) is an intercept. Is it an *x*-intercept or a *y*-intercept?

   a. $(0, \pi)$ b. $(-99, 0)$ c. $(0, 0)$

   d. $(3\pi, 0)$ e. $(0, -3.7)$ f. $(7, 7)$

2. Find the *x*-intercept and the *y*-intercept of each line — be sure that every intercept you write consists of an ordered pair (that is, two coordinates, one of which must be 0):

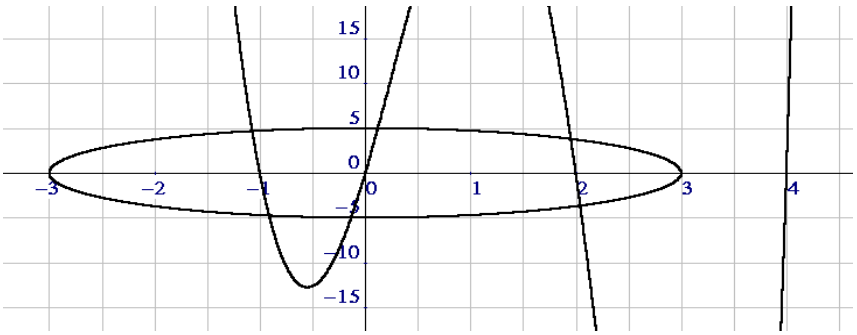
   a. $2x + y = 12$ b. $3x - 4y = 24$ c. $-4x + 7y = 28$

   d. $x - 7y = 7$ e. $y - 3x = 12$ f. $6x + 5y = 60$

   g. $4x - 3y = 2$ h. $8y + 3x = 1$ i. $3x - 9y = 0$

   j. $y = 7x - 3$ k. $y = -9x + 2$ l. $y = -x - 5$

3. Find <u>all</u> the intercepts of the following graph:



4. Sketch a graph that has NO intercepts of either kind.

Review Problems

5. Find all the intercepts of each line:

a. $y = 7x - 3$ b. $y = -9x + 8$ c. $y = x + 1$

d. $y = -x - 1$ e. $y = \frac{2}{3}x + 5$ f. $y = -\frac{1}{2}x - \frac{4}{5}$

g. $-2x - 7y = 0$ h. $4x + 8y = 6$ i. $18x - 17y = 2$

❑ *TO ∞ AND BEYOND*

A. Find all the intercepts of the graph of $x^2 + y^2 = 25$.

B. Find all the intercepts of the graph of $y = x^2 - 9$.

Solutions

1. a. y-intercept b. x-intercept c. both intercepts
d. x-intercept e. y-intercept f. neither

2. a. (6, 0) (0, 12) b. (8, 0) (0, –6) c. (–7, 0) (0, 4)
d. (7, 0) (0, –1) e. (–4, 0) (0, 12) f. (10, 0) (0, 12)
g. $(\frac{1}{2}, 0)$ $(0, -\frac{2}{3})$ h. $(\frac{1}{3}, 0)$ $(0, \frac{1}{8})$ i. (0, 0)
j. $(\frac{3}{7}, 0)$ (0, –3) k. $(\frac{2}{9}, 0)$ (0, 2) l. (–5, 0) (0, –5)

3. (–3, 0) (–1, 0) (0, 0) (2, 0) (3, 0) (4, 0) (0, –5) (0, 5)

4. A circle contained completely within Quadrant I would do the trick.

5. a. $(0, -3)$ $(\frac{3}{7}, 0)$ b. $(0, 8)$ $(\frac{8}{9}, 0)$ c. $(0, 1)$ $(-1, 0)$

d. $(0, -1)$ $(-1, 0)$ e. $(0, 5)$ $(-\frac{15}{2}, 0)$ f. $(0, -\frac{4}{5})$ $(-\frac{8}{5}, 0)$

g. $(0, 0)$ h. $(0, \frac{3}{4})$ $(\frac{3}{2}, 0)$ i. $(0, -\frac{2}{17})$ $(\frac{1}{9}, 0)$

“Effort only fully releases its reward after a person refuses to quit.”

– *Napoleon Hill*, American Author

PARALLEL AND PERPENDICULAR LINES

The chapter title says it all, so let's get to work. First, let's revisit our friend, the slope-intercept equation of a line:



$$y = mx + b$$

SLOPE — *y*-INTERCEPT

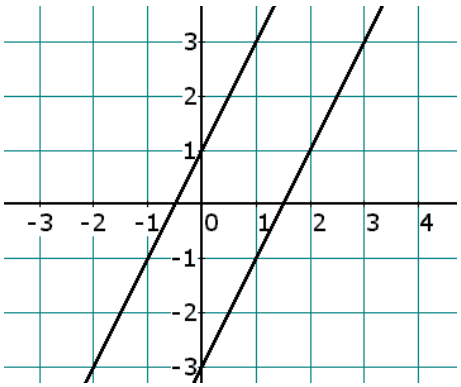
To be precise, *b* is not the *y*-intercept; *b* is the ***y*-coordinate** of the *y*-intercept. The *y*-intercept is properly written **(0, *b*)**.

❑ *PARALLEL LINES*

Let's begin by assuming that throughout this chapter we will never be referring to horizontal or vertical lines — these kinds of lines are covered in detail in the chapter *Special Lines*.

Let's graph the two lines $y = 2x + 1$ and $y = 2x - 3$ on the same grid:

[Is it clear which one is which?]



Notice that the two lines appear to be parallel (and they really are). Now, what do the equations of the two lines have in common? The formulas for the lines show that each line has a slope of **2**. Since slope is a measure of steepness, is it reasonable to claim that if two lines are parallel, then they are equally steep, and therefore they must have the same slope?

> *Parallel lines* have the **same** slope.

For a simple example: Suppose Line 1 has a slope of –9, and that Line 2 is parallel to Line 1. We can then deduce (conclude by logic) that the slope of Line 2 is also –9, without any calculating or graphing. Note that we do not consider a line to be parallel to itself.

EXAMPLE 1: **Find the slope of any line that is *parallel* to the line $7x - 5y = 2$.**

Solution: Any line that is parallel to the line $7x - 5y = 2$ must have the *same* slope as the line $7x - 5y = 2$. So, if we can compute the slope of this line, we will have the slope of any line parallel to it. The easiest way to find the slope of the line is to convert it to $y = mx + b$ form:

$$7x - 5y = 2 \;\Rightarrow\; -5y = -7x + 2 \;\Rightarrow\; y = \frac{7}{5}x - \frac{2}{5}$$

The slope of the given line is $\frac{7}{5}$, and so we conclude that any line that is parallel to the line $7x - 5y = 2$ must have a slope of

$$\boxed{\frac{7}{5}}$$

NOTE: The following examples are solved using the $\boldsymbol{y = mx + b}$ form of a line. However, if you've learned the point-slope form of a line,

$$y - y_1 = m(x - x_1),$$

you might want to use that form instead, since it actually works a little better for these problems.

EXAMPLE 2: **Find the equation of the line that is *parallel* to the line $3x + y = 5$, and that passes through the point (6, 2).**

Solution: We're looking for an unknown line

$$y = mx + b$$

The slope of our unknown line was not given to us, but we know that it's the same as the slope of the given line, since the two lines are parallel. Solving the given line for y yields the line $y = -3x + 5$, whose slope is clearly -3. So the slope of our unknown line is also -3. At this point in the problem we can write our line as

$$y = -3x + b$$

Plugging the given point (6, 2) into this equation allows us to find b:

$$\mathbf{2} = -3(\mathbf{6}) + b$$

$$\Rightarrow \quad 2 = -18 + b$$

$$\Rightarrow \quad 20 = b$$

and we're done; our line is

$$\boxed{y = -3x + 20}$$

Homework

1. a. A given line has a slope of 7. What is the slope of any line that is parallel to the given line?

   b. A given line has a slope of $-\frac{2}{3}$. What is the slope of any line that is parallel to the given line?

2. a. What is the slope of any line that is parallel to the line $y = \frac{3}{4}x - 9$?

   b. What is the slope of any line that is parallel to the line $5x + 2y = 9$?

3. a. Prove that the lines $2x - 4y = 5$ and $3x - 6y = 1$ <u>are</u> parallel.

   b. Prove that the lines $3x - y = 4$ and $5x + 2y = 10$ are <u>not</u> parallel.

4. Find the equation of the line that is *parallel* to the given line, and that passes through the given point:

   a. $y = 3x + 4$; $(3, -5)$ b. $y = \frac{1}{2}x - 5$; $(-1, 7)$

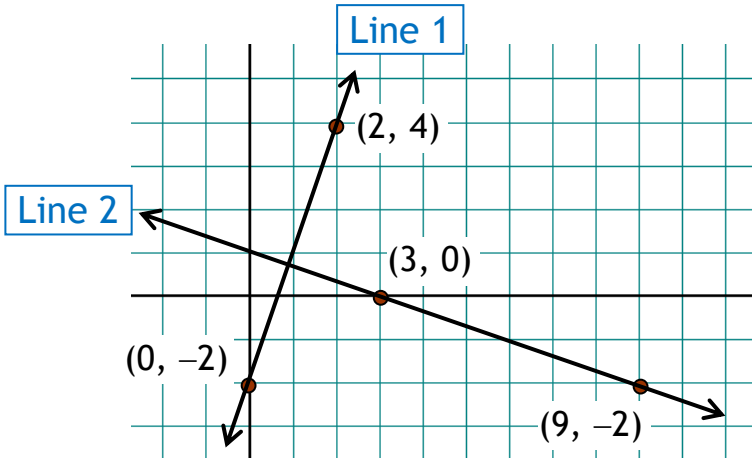
   c. $3x + y = 7$; $(1, 9)$ d. $2x - y = 8$; $(-5, -4)$

   e. $2x - 3y = 1$; $(2, -5)$ f. $3x + 4y = 10$; $(8, 0)$

❑ *PERPENDICULAR LINES*

Parallel lines have the same slope — certainly *perpendicular* lines do not. But is there some relationship, some connection, between the slopes of perpendicular lines? Let's see if we can discover one with an example. In the following grid, Line 1 is perpendicular to Line 2, and

points have been labeled so that we can easily calculate m_1 and m_2, the slopes of the two lines.



First we compute the slope of Line 1:

$$m_1 = \frac{\Delta y}{\Delta x} = \frac{4-(-2)}{2-0} = \frac{6}{2} = \mathbf{3}$$

Next let's compute the slope of Line 2:

$$m_2 = \frac{\Delta y}{\Delta x} = \frac{0-(-2)}{3-9} = \frac{2}{-6} = -\frac{\mathbf{1}}{\mathbf{3}}$$

There are two things to note regarding these two slopes of the two perpendicular lines. First, one slope is positive while the other is negative. This makes sense because as we move from left to right, Line 1 is increasing while Line 2 is decreasing.

Second, the slope of Line 1, m_1, is kind of a big number (the line's pretty steep), while the slope of Line 2, m_2, (ignoring the minus sign) is a relatively small number (the line's not very steep).

Specifically, the two slopes have opposite signs, and they are also (ignoring the minus sign) *reciprocals* of each other. In other words, when looking at the slopes of these two **perpendicular lines**, each of the slopes is the ***opposite reciprocal*** of the other.

Although we have shown this relationship (*opposite reciprocal*) for just this example, it can be proved that this relationship always works for perpendicular lines.

Perpendicular lines have slopes that are **opposite reciprocals** of each other.

For example, if a line has a slope of $\frac{7}{4}$, then any perpendicular line must have a slope of $-\frac{4}{7}$.

And consider the line $y = -5x + 1$. Since its slope is -5, it follows that the slope of any perpendicular line must be $\frac{1}{5}$.

Some books say that the slopes of two perpendicular lines are *negative reciprocals* of each other.

Alternative: We've learned that the slopes of two perpendicular lines are *opposite reciprocals* of each other. But some books say that two lines are perpendicular if their slopes have a product of -1. Do both of these statements mean the same thing?

Yes — assume that the product of their slopes is -1:

$$m_1 m_2 = -1$$

Solving for m_1 gives us the equation

$$m_1 = -\frac{1}{m_2},$$

which says that one slope is the *opposite reciprocal* of the other.

Homework

5. A given line has a slope of $-\frac{5}{3}$. What is the slope of any line that is perpendicular to the given line?

6. Prove that the lines $7x - 2y = 10$ and $4x + 14y = 23$ are perpendicular.

7. Prove that the lines $3x + 2y = 10$ and $2x + 3y = 9$ are not perpendicular.

8. Prove that the lines $5x - 3y = 10$ and $5x + 3y = 17$ are not perpendicular.

9. Find the slope of any line that is *perpendicular* to the given line:

   a. $y = -7x + 9$ b. $y = \frac{5}{4}x + 10$

   c. $2x + 7y = 10$ d. $3x - 2y = 0$

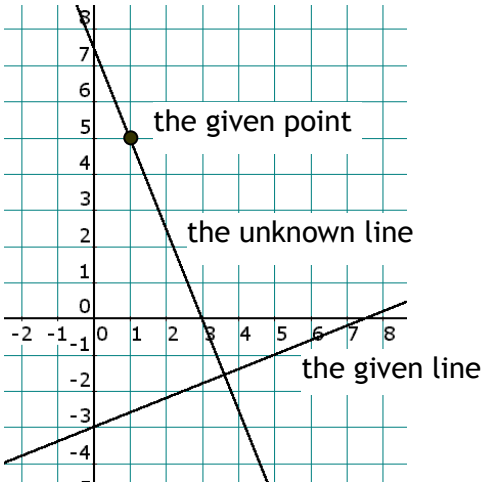
EXAMPLE 3: **Find the equation of the line that is *perpendicular* to the line $2x - 5y = 15$ and passes through the point (1, 5).**

Solution: We're looking for an unknown line

$$y = mx + b$$

The slope of our unknown line was not given to us, but we know that it's the *opposite reciprocal* of the slope of the given line, since the two lines are perpendicular. To determine the slope of the given line, we solve for y:

$$2x - 5y = 15$$



$$\Rightarrow \quad -5y = -2x + 15$$
$$\Rightarrow \quad \frac{-5y}{-5} = \frac{-2x+15}{-5}$$
$$\Rightarrow \quad y = \frac{2}{5}x - 3$$

telling us that the slope of the given line is $\frac{2}{5}$. So the slope of our unknown line is the opposite reciprocal of that, which is $\mathbf{-\frac{5}{2}}$. At this point in the problem we can write our line as

$$y = -\frac{5}{2}x + b$$

Plugging the given point (1, 5) into this equation allows us to find *b*:

$$\mathbf{5} = -\frac{5}{2}(\mathbf{1}) + b \quad \Rightarrow \quad 5 + \frac{5}{2} = b \quad \Rightarrow \quad b = \frac{15}{2}$$

and we're done; our line is

$$\boxed{y = -\frac{5}{2}x + \frac{15}{2}}$$

Homework

10. Find the equation of the line that is *perpendicular* to the given line, and that passes through the given point:

a. $y = 3x + 4$; (3, –5)

b. $y = \frac{1}{2}x - 5$; (–1, 7)

c. $3x + y = 7$; (1, 9)

d. $2x - y = 8$; (–5, –4)

e. $5x - 3y = 1$; (2, –5)

f. $3x + 4y = 10$; (8, 0)

Review Problems

11. The slopes of two parallel lines are _____.

12. The slopes of two perpendicular lines are _____.

13. The slope of a line is $\frac{5}{7}$. What is the slope of any parallel line?

14. The slope of a line is $-\frac{4}{9}$. What is the slope of any perpendicular line?

15. T/F: The lines $y = 7x - 3$ and $14x - 2y = 22$ are parallel.

16. T/F: The lines $3x - 7y = 1$ and $7x + 3y = 0$ are perpendicular.

17. Find the equation of the line that is parallel to $3x - 7y = 9$ and passes through the point $(-3, 10)$.

18. Find the equation of the line that is perpendicular to $4x - 9y = 11$ and passes through the point $(3, -13)$.

19. Find the equation of the line that is parallel to the line $3x - 4y = 1$ and that passes through the point $(-2, -7)$.

20. Find the equation of the line that is perpendicular to the line $3x - 4y = 1$ and that passes through the point $(-2, -7)$.

21. Which one of the following lines is *parallel* to the line $5x - 3y = 7$?

a. $y = \frac{3}{5}x - 1$ b. $y = -\frac{3}{5}x + 4$ c. $y = \frac{5}{3}x - 3$

d. $y = -\frac{5}{3}x + 2$ e. $y = \frac{5}{3}$

22. Which one of the following lines is *perpendicular* to the line $5x - 3y = 7$?

a. $y = \frac{3}{5}x - 1$ b. $y = -\frac{3}{5}x + 4$ c. $y = \frac{5}{3}x - 3$

d. $y = -\frac{5}{3}x + 2$ e. $y = \frac{5}{3}$

Solutions

1. a. 7 b. $-\frac{2}{3}$

2. a. $\frac{3}{4}$ b. $-\frac{5}{2}$

3. a. Each line has a slope of $\frac{1}{2}$. Same slope $\Rightarrow$ parallel lines.

b. The slopes are 3 and –5/2. Different slopes $\Rightarrow$ non-parallel lines.

4. a. $y = 3x - 14$ b. $y = \frac{1}{2}x + \frac{15}{2}$ c. $y = -3x + 12$

d. $y = 2x + 6$ e. $y = \frac{2}{3}x - \frac{19}{3}$ f. $y = -\frac{3}{4}x + 6$

5. $\frac{3}{5}$

6. The slopes are $\frac{7}{2}$ and $-\frac{2}{7}$, which are opposite reciprocals of each other.

7. The slopes are $-\frac{3}{2}$ and $-\frac{2}{3}$, which are <u>not</u> opposite reciprocals of each other. (They're reciprocals, but not opposites.)

8. The slopes are $\frac{5}{3}$ and $-\frac{5}{3}$, which are <u>not</u> opposite reciprocals of each other. (They're opposites, but not reciprocals.)

9. a. $\frac{1}{7}$ b. $-\frac{4}{5}$ c. $\frac{7}{2}$ d. $-\frac{2}{3}$

10. a. $y = -\frac{1}{3}x - 4$ b. $y = -2x + 5$ c. $y = \frac{1}{3}x + \frac{26}{3}$

d. $y = -\frac{1}{2}x - \frac{13}{2}$ e. $y = -\frac{3}{5}x - \frac{19}{5}$ f. $y = \frac{4}{3}x - \frac{32}{3}$

11. equal **12**. opposite reciprocals

13. $\frac{5}{7}$ **14**. $\frac{9}{4}$ **15**. T **16**. T

17. $y = \frac{3}{7}x + \frac{79}{7}$

18. $y = -\frac{9}{4}x - \frac{25}{4}$

19. $y = \frac{3}{4}x - \frac{11}{2}$

20. $y = -\frac{4}{3}x - \frac{29}{3}$

21. c.

22. b.

"Upon the subject of education ... I can only say that I view it as the most important subject which we as a people may be engaged in."



– *Abraham Lincoln*

THE EQUATION OF A LINE – $y - y_1 = m(x - x_1)$

The slope-intercept form of a line, $y = mx + b$, is perfect when you have the slope and the y-intercept. But the odds of this are slim. It's much more likely that you'll be working with the slope and some point on the line other than the y-intercept. So, in the following formula, m is the slope (as before), while (x_1, y_1) represents any given point on the line.



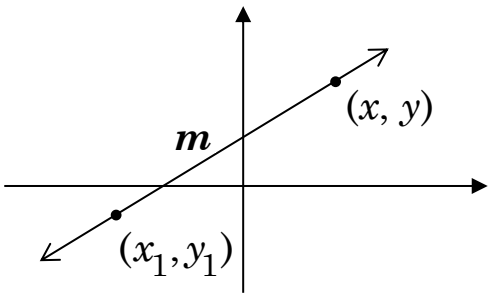
❑ THE POINT-SLOPE FORMULA

THEOREM: **The equation of the line with slope m and passing through the point (x_1, y_1) is**

$$y - y_1 = m(x - x_1)$$

The Point-Slope Form of a Line

PROOF: We begin by sketching the line and labeling the given point (x_1, y_1), the slope m, and a *generic point* (x, y):



On the one hand, the slope of the line is given by m.

On the other hand, the slope of the line can be calculated using the two points (x, y) and (x_1, y_1): $\frac{y - y_1}{x - x_1}$. And, of course, these two slopes must be the same (since there's only one line involved):

$$\frac{y - y_1}{x - x_1} = m$$

$$\Rightarrow \quad \boxed{y - y_1 = m(x - x_1)}$$

This is the form of a line that is used in Calculus.

❑ *BONUS DERIVATION*

Using the point-slope form of a line above, we can derive the slope-intercept form of a line: $y = mx + b$. Here's how:

We assume that the slope of a line is given by m, and that the y-intercept is $(0, b)$, which is simply a point on the line; so it's the point (x_1, y_1) in the formula $y - y_1 = m(x - x_1)$. Thus,

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow \quad y - b = m(x - 0)$$

$$\Rightarrow \quad y - b = mx$$

$$\Rightarrow \quad y = mx + b$$

and we're done!

❑ *EXAMPLES OF THE POINT-SLOPE FORMULA*

EXAMPLE 1: **Find the equation of the line whose slope is –3 and which passes through the point (8, –2).**

Solution: This is precisely the data we need to use the point-slope form: $y - y_1 = m(x - x_1)$. We're given the slope, so $m = -3$. We're also given a point on the line, so $(x_1, y_1) = (8, -2)$. Plugging these values into the point-slope form gives us

$$y - (-2) = -3(x - 8), \text{ or}$$

$$\boxed{y + 2 = -3(x - 8)}$$

EXAMPLE 2: **Find the equation of the line passing through the two points (3, –5) and (–2, –8).**

Solution: The point-slope form, $y - y_1 = m(x - x_1)$, requires a point (we have two of them), and the slope, which we'll have to calculate ourselves:

$$m = \frac{\Delta y}{\Delta x} = \frac{-5 - (-8)}{3 - (-2)} = \frac{-5 + 8}{3 + 2} = \frac{3}{5}$$

Now, using the point (3, –5) (either point would work), we get our equation

$$y - (-5) = \frac{3}{5}(x - 3), \text{ or } \boxed{y + 5 = \frac{3}{5}(x - 3)}$$

Homework

1. Use the point-slope formula to find the equation of the line with slope 7 and passing through the point (6, –8).

2. Use the point-slope formula to find the equation of the line with slope 0 and passing through the point (–17, 9).

3. Use the point-slope formula to find the equation of the line with slope $-\frac{4}{7}$ and passing through the point $(\frac{1}{2}, \pi)$.

4. Use the point-slope formula to find the equation of the line which passes through the points (–2, 4) and (5, –5).

5. Use the point-slope formula to find the equation of the line which passes through the points $(\pi, \sqrt{2})$ and (–3, 1).

Solutions

1. $y + 8 = 7(x - 6)$

2. $y - 9 = 0$

3. $y - \pi = -\frac{4}{7}\left(x - \frac{1}{2}\right)$

4. $y - 4 = -\frac{9}{7}(x + 2)$

5. $y - \sqrt{2} = \frac{1 - \sqrt{2}}{-3 - \pi}(x - \pi)$ The slope can also be written: $\frac{\sqrt{2} - 1}{\pi + 3}$.

An investment in knowledge always pays the best interest.

Benjamin Franklin

SPECIAL LINES



You may have noticed that most (if not all) of the lines you've seen so far in your course have had either positive or negative *slopes*. You may also have observed that every line had both of the variables *x* and *y* in its equation. But there are lines whose slopes are neither positive nor negative, and lines whose equations have only one variable in them. This chapter deals with these special lines.

❑ *GRAPHING*

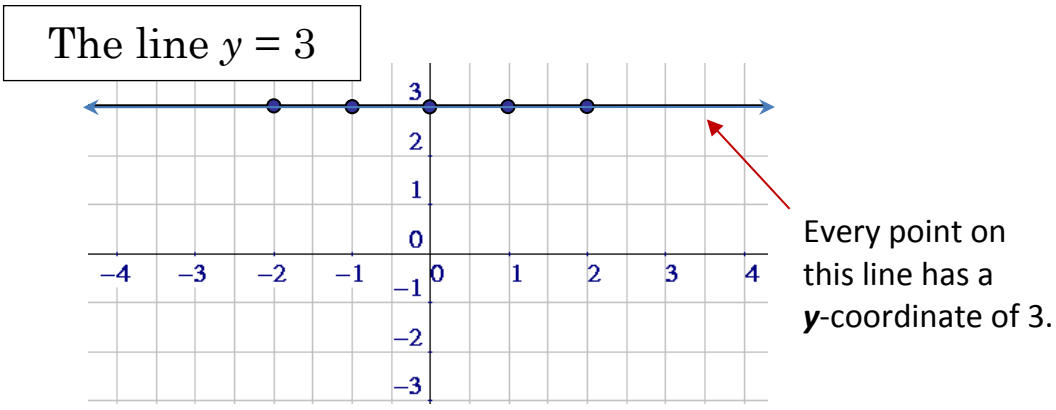
EXAMPLE 1: **Graph the line $y = 3$.**

Solution: This strange little equation doesn't even have an *x* in it. That's fine — we just think up our favorite *x*'s, and then understand that *y* is going to be 3 regardless of the *x*-value we choose. That is, *y* is a constant; it doesn't depend on *x*. Here's a possible table of values for this line. [You are more than welcome to choose *x*-values different from the ones I've chosen, but it won't make any difference in the final graph.]

x	*y*
–2	3
–1	3
0	3
1	3
2	3

We therefore have the points (–2, 3), (–1, 3), (0, 3), (1, 3), and (2, 3). Plotting these five points, and then connecting them with a straight line, produces the following **horizontal** line; notice

that the y-intercept of this line is (0, 3), and that there's no x-intercept.



EXAMPLE 2: **Graph the line $x = -2$.**

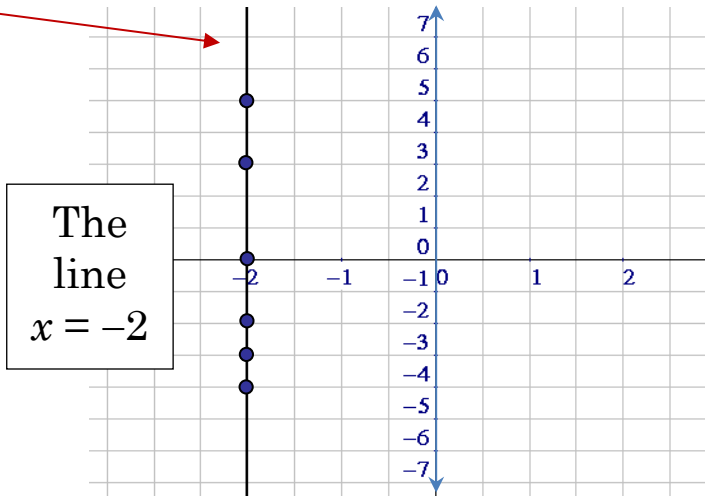
Solution: This one's as goofy as the previous one — but this time the y is missing. But more importantly, the equation clearly informs us that x must be -2. Any other choice of x would contradict this requirement. Moreover, since there's no y in the equation, we can choose any number we'd like for y. This leads to a collection of points like this:

(–2, –4) (–2, –3) (–2, –2) (–2, 0) (–2, 3) (–2, 5)

The key to every point on the line $x = -2$ is that the x-coordinate of the point must be –2, while the y-coordinate can be any number (even numbers like π and $-\sqrt{2}$).

When we plot these points and connect them with a straight line, we get the following **vertical** line; note that the x-intercept of this line is (–2, 0) and that there's no y-intercept.

Every point on this line has an ***x***-coordinate of –2.



Homework

1. Graph each line by plotting at least three points:

   a. $y = 4$ b. $y = -3$ c. $x = 5$

   d. $x = -1$ e. $y = 0$ f. $x = 0$

2. a. The horizontal line $y = 0$ is the _____.

   b. The vertical line $x = 0$ is the _____.

3. a. Is the line $x = 1{,}000{,}000$ is horizontal or vertical?

   b. Is the line $y = \sqrt{1679}/\pi$ is horizontal or vertical?

4. At what point do the lines $x = 17$ and $y = -99$ intersect?

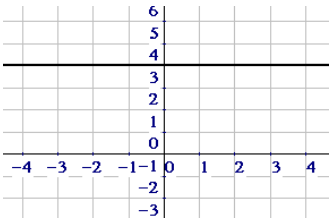
5. Find the intercepts of each line:

   a. $x = 3$ b. $y = -2$ c. $x = 0$ d. $y = 0$

   e. $y = 5$ f. $x = -\pi$ g. $x = \sqrt{2}$ h. $y = x$

❑ THE SLOPE OF A HORIZONTAL LINE

We recall (from Homework 1a) that the graph of the line with the equation $y = 4$ is a horizontal line four units above the x-axis.



Notice that the graph has a y-intercept at (0, 4) but has no x-intercepts. Other points on this horizontal line include (3, 4), (–20, 4), (π, 4), and $(-\sqrt{7}, 4)$. In other words, in the formula $y = 4$, x can be <u>any</u> number, but y <u>must</u> be 4.

Now it's time to calculate the slope of this horizontal line. We need a pair of points on this line — we'll use (3, 4) and (–20, 4).

$$m = \frac{\Delta y}{\Delta x} = \frac{4-4}{3-(-20)} = \frac{0}{23} = \mathbf{0}$$

Since all horizontal lines ought to have the same slope, we can be confident in drawing the following conclusion:

> The slope of any horizontal line is 0.

❑ THE SLOPE OF A VERTICAL LINE

Do you remember what the graph of $x = -2$ looks like? Return to Example 2 and recall that it's a vertical line with x-intercept (–2, 0).

To obtain the slope of this line, we'll use the points (–2, 0) and (–2, 5):

$$m = \frac{\Delta y}{\Delta x} = \frac{5-0}{-2-(-2)} = \frac{5}{-2+2} = \frac{5}{0} = \textbf{Undefined}$$

The conclusion that the slope is "undefined" is based on the fact that division by zero is undefined. We might also observe the "steepness" of the vertical line. It's so steep that no number could measure it, so

"undefined" is a good way to describe the slope. Since all vertical lines should have the same slope,

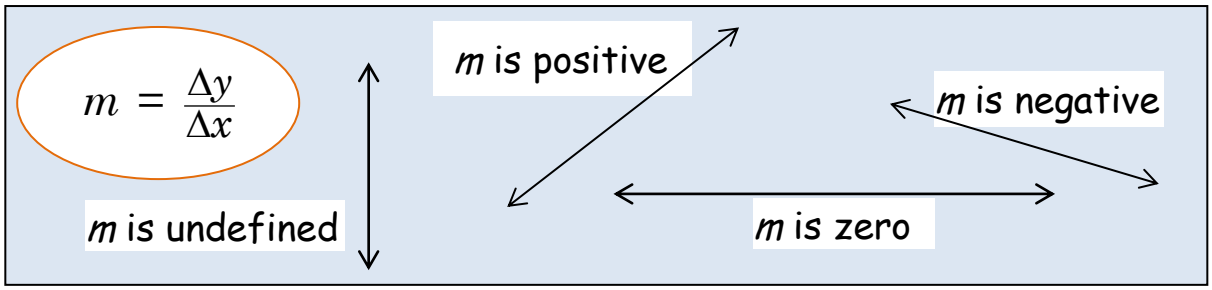
> The slope of any vertical line is *undefined*.

Homework

6. For each line, i) find two points on the line
   ii) use these points and $m = \frac{\Delta y}{\Delta x}$ to find its slope

a. $y = 3$	b. $x = 4$	c. $y = -19$
d. $x = -44$	e. $x = 0$	f. $y = 0$

The following diagram is a summary of our notion of ***slope***:



❑ *More Horizontal and Vertical Lines*

We know that a horizontal line always has an equation of the form "y = *some number*," while a vertical line always has an equation of the form "x = *some number*." We've also learned that a horizontal line has a slope of 0, while the slope of a vertical line is undefined. We put all this info into a little table to help us see all the essential facts about horizontal and vertical lines.

Equation	Type of Line	Slope
y = some number	horizontal	zero
x = some number	vertical	undefined

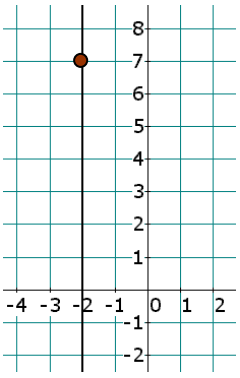
EXAMPLE 3:

A. **Find the equation of the horizontal line passing through the point (5, 3).**

Solution: A horizontal line has an equation of the form y = *some number*. Since (5, 3) is on the line, the equation of the line must be $\boldsymbol{y = 3}$.

B. **Find the equation of the vertical line passing through the point (–2, 7).**

Solution: A vertical line has an equation of the form x = *some number*. Since (–2, 7) is on the line, the line must have the equation $\boldsymbol{x = -2}$.



C. **Find the equation of the line whose slope is 0 and which passes through the point (–5, 9).**

Solution: If a line has a slope of 0, it must be a horizontal line whose equation must be of the form y = *some number*. Because (–5, 9) lies on the line, the answer is $\boldsymbol{y = 9}$.

D. **A line has an undefined slope and passes through the point (7, –12). What is the equation of the line?**

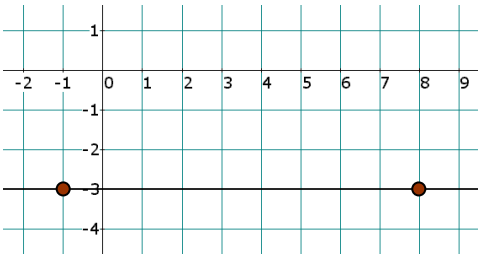
Solution: An undefined slope implies a vertical line, which means an equation like x = *some number*. Since (7, –12) is on the line, its equation must be $\boldsymbol{x = 7}$.

E. **What is the equation of the line passing through the points (9, 5) and (9, –2)?**

Solution: Plot the two points and you'll notice that (9, 5) is directly above (9, –2), yielding a vertical line. The equation must be $\boldsymbol{x = 9}$.

F. **Find the equation of the line passing through (8, –3) and (–1, –3).**

Solution: A quick sketch shows that (8, –3) lies directly to the right of (–1, –3). This creates a horizontal line whose equation is $\boldsymbol{y = -3}$.



Homework

7. Describe the line $y = 17$.
8. Describe the line $x = -99$.
9. Find the line with a slope of 0 and passing through the point (1, 0).
10. The line $y = 8$ is (horizontal, vertical) and its slope is _____.
11. What is the equation of the line passing through (3, 11) and (–3, 11)?
12. Find the equation of the horizontal line passing through the point (–13, 17).
13. Find the line with a slope of 0 and passing through the point (–1, 7).
14. The line $y = -6$ is (horizontal, vertical) and its slope is _____.
15. What is the equation of the line passing through (–17, –13) and (7, –13)?
16. Find the equation of the vertical line passing through the point (–4, –9).
17. Find the line with a slope of 0 and passing through the point (2, 1).
18. The line $x = -4$ is (horizontal, vertical) and its slope is _____.
19. What is the equation of the line passing through (–1, 9) and (–1, 11)?
20. Find the equation of the horizontal line passing through the point (–1, 0).
21. Find the line with an undefined slope and passing through the point (3, 7).
22. The line $x = -8$ is (horizontal, vertical) and its slope is _____.

23. What is the equation of the line passing through (–5, 10) and (–5, 6)?

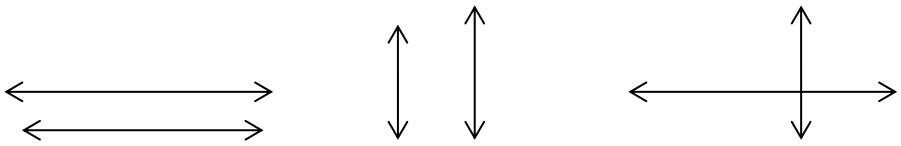
24. Find the equation of the vertical line passing through the point (–18, –11).

25. Find the line with a slope of 0 and passing through the point (–5, –6).

❑ *PARALLEL AND PERPENDICULAR LINES*

Would you agree that two different vertical lines never intersect? When two lines (in the same plane) never intersect, we say they're ***parallel***. So, for example, the lines $x = 3$ and $x = -4$ are parallel, since each is vertical. Now consider a pair of different horizontal lines. Clearly, they're parallel, too. Thus, for example, the lines $y = 2$ and $y = \pi$ are also parallel.

Now consider a vertical line and a horizontal line. They must meet at a 90° angle, and we say that the two lines are ***perpendicular*** (in the same way that the two legs of a right triangle are perpendicular to each other). We can therefore say that the lines $x = 5$ (vertical) and $y = -3$ (horizontal) are perpendicular.



Parallel lines Parallel lines Perpendicular Lines

Parallel lines and perpendicular lines can also be at an angle; you'll see how that works in the chapter *Parallel and Perpendicular Lines*.



EXAMPLE 4:

A. **Find the equation of the line which is parallel to the line $x = 7$ and which passes through the point (5, 3).**

Solution: The line $x = 7$ is vertical. Any line parallel to this line must also be vertical. What vertical line passes through the point (5, 3)? The line **$x = 5$** does.

B. **Find the equation of the line which is perpendicular to the line $x = -5$ and which passes through the point (–2, –9).**

Solution: Since the line $x = -5$ is vertical, the perpendicular line we're seeking has to be horizontal. What is the equation of the horizontal line passing through (–2, –9). The answer is **$y = -9$**.

C. **Find the equation of the line which is parallel to the line $y = 17$ and which passes through the point (–1, 3).**

Solution: This time the given line $y = 17$ is horizontal, and since we seek a parallel line, it also must be horizontal. And the horizontal line passing through the point (–1, 3) is certainly **$y = 3$**.

D. **Find the equation of the line which is perpendicular to the line $y = 11$ and which passes through the point (6, –3).**

Solution: The line $y = 11$ is horizontal, so we need a vertical line passing through (6, –3). That line is **$x = 6$**.

Homework

26. Fill in each blank with either the word 'parallel' or 'perpendicular':

    a. Two different vertical lines are _____.
    b. Two different horizontal lines are _____.
    c. A vertical line and a horizontal line are _____.

27. Fill in each blank with either the word 'vertical' or 'horizontal':

    a. A line which is parallel to a vertical line must be ___.
    b. A line which is perpendicular to a horizontal line must be ___.
    c. A line which is parallel to a horizontal line must be ___.
    d. A line which is perpendicular to a vertical line must be ___.

28. a. Are the lines $x = 9$ and $x = -1$ parallel or perpendicular?
    b. Are the lines $y = 7$ and $y = 0$ parallel or perpendicular?
    c. Are the lines $x = -9$ and $y = 7$ parallel or perpendicular?

29. a. Give an example of a line which is parallel to $x = 5$.
    b. Give an example of a line which is parallel to $y = -4$.
    c. Give an example of a line which is perpendicular to $y = -4$.
    d. Give an example of a line which is perpendicular to $x = 8$.

30. Fill in each blank with either the word 'vertical' or 'horizontal':

    a. A line which is parallel to the line $y = 7$ must be _____.
    b. A line which is perpendicular to the line $x = 3$ must be _____.
    c. A line which is parallel to the line $x = 8$ must be _____.
    d. A line which is perpendicular to the line $y = -3$ must be _____.

31. a. Find the equation of the line which is parallel to the line $x = 9$ and which passes through the point (1, 7).

b. Find the equation of the line which is perpendicular to the line $x = -3$ and which passes through the point $(-7, 0)$.

c. Find the equation of the line which is parallel to the line $y = 10$ and which passes through the point $(-5, 8)$.

d. Find the equation of the line which is perpendicular to the line $y = -9$ and which passes through the point $(7, -2)$.

Review Problems

32. a. Graph the line $x = -3$ by plotting three points.

b. Is the line horizontal or vertical?

c. Find all the intercepts of the line.

d. Use two of the points and $m = \frac{\Delta y}{\Delta x}$ to <u>calculate</u> the slope.

33. a. Graph the line $y = 2$ by plotting three points.

b. Is the line horizontal or vertical?

c. Find all the intercepts of the line.

d. Use two of the points and $m = \frac{\Delta y}{\Delta x}$ to <u>calculate</u> the slope.

34. a. What is the equation of the x-axis?

b. What is the equation of the y-axis?

35. The line $y = \sqrt{3}$ is (horizontal, vertical) and its slope is _____.

36. The line $x = \sqrt{2\pi}$ is (horizontal, vertical) and its slope is _____.

37. Graph the line $y = x$. Is it horizontal, vertical, or diagonal? What is its slope?

38. Find the equation of the horizontal line passing through the point (17, 99).

39. Find the equation of the vertical line passing through the point (34, −44).

40. Find the equation of the line with undefined slope passing through the point $(2, -\pi)$.

41. Find the equation of the line with 0 slope passing through the point $(2, -\pi)$.

42. What is the equation of the line passing through (2, 7) and (2, 1)?

43. What is the equation of the line passing through (1, 7) and (0, 7)?

44. Find the equation of the line which is parallel to the line $x = 14$ and which passes through the point (−2, −9).

45. Find the equation of the line which is perpendicular to the line $y = -23$ and which passes through the point $(\pi, 0)$.

46. True/False:

a. The line $y = \sqrt{2}$ is horizontal.

b. The line $x = 3$ has an undefined slope.

c. The line $y = 5$ has exactly one intercept.

d. The vertical line passing through (2, 7) is $x = 7$.

e. The equation of the x-axis is $y = 0$.

f. The line $x = -1$ has infinitely many intercepts.

g. The point (7, 9) lies on the line $y = 9$.

h. The line $x = -8$ has a negative slope.

i. The slope of the line $y = 3x + 4$ is 3.

j. The line passing through $(3, \pi)$ and (3, 1) is $x = 3$.

k. The line $y = x$ is horizontal.

l. A line can have two intercepts.

m. The point (−2, 5) lies on the line $x = 5$.

n. The line $y = 7$ has an undefined slope.

o. All horizontal lines have the same slope.

p. The equation of the y-axis is $y = 0$.

q. The line passing through (1, 2) and (1, 0) is $y = 1$.

r. A line can have exactly one intercept.

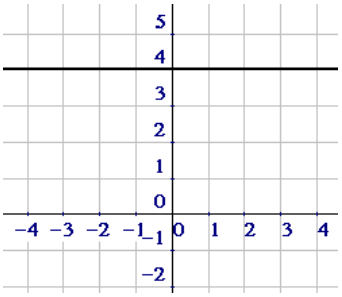
s. A line can have infinitely many intercepts.

t. The lines $x = 3$ and $y = 4$ are parallel.

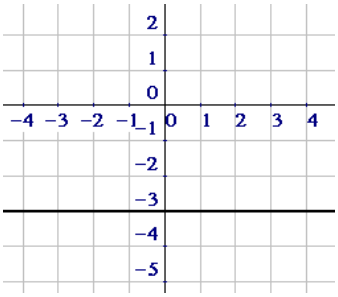
u. The slope of the line $y = x$ is 1.

Solutions

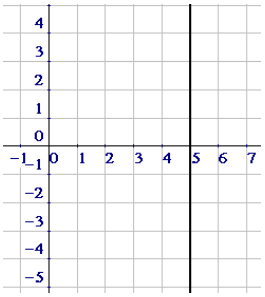
1. a.



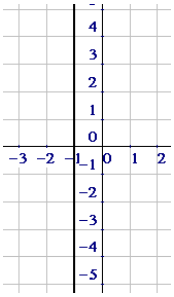
b.



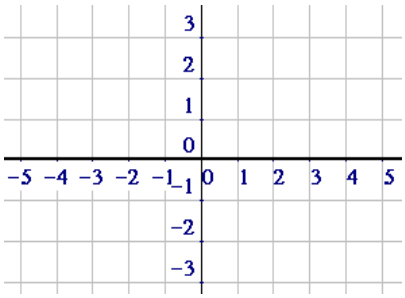
c.



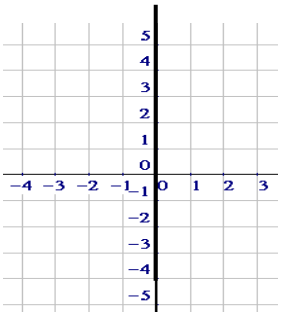
d.



e.



f.



2. a. x-axis b. y-axis **3**. a. vertical b. horizontal **4**. (17, –99)

5. a. x-intercept: (3, 0); No y-intercept

b. No x-intercept; y-intercept: (0, –2)

c. x-int: (0, 0); y-int: all the points on the y-axis are y-intercepts

d. x-int: all the points on the x-axis are x-intercepts; y-int: (0, 0)

e. No x-intercept; y-intercept: (0, 5)

f. x-intercept: $(-\pi, 0)$; No y-intercept

g. x-intercept: $(\sqrt{2}, 0)$; No y-intercept

h. x-int: (0, 0); y-int: (0, 0)

6. a. e.g., (2, 3) and (–4, 3); $m = \frac{3-3}{2-(-4)} = \frac{0}{6} = 0$

b. e.g., (4, 7) and (4, π); $m = \frac{7-\pi}{4-4} = \frac{7-\pi}{0}$ = Undefined

c. $m = 0$ d. m = Undefined

e. m = Undefined f. $m = 0$

7. $y = 17$ is a horizontal line 17 units above the x-axis. Its y-intercept is (0, 17), but it has no x-intercepts; its slope is 0.

8. $x = -99$ is a vertical line 99 units to the left of the y-axis. Its x-intercept is (–99, 0), but it has no y-intercepts; its slope is undefined.

9. $y = 0$ **10**. horizontal; 0 **11**. $y = 11$ **12**. $y = 17$ **13**. $y = 7$

14. horizontal; 0 **15**. $y = -13$ **16**. $x = -4$ **17**. $y = 1$

18. vertical; undefined **19**. $x = -1$ **20**. $y = 0$ **21**. $x = 3$

22. vertical; undefined **23**. $x = -5$ **24**. $x = -18$ **25**. $y = -6$

26. a. parallel b. parallel c. perpendicular

27. a. vertical b. vertical c. horizontal d. horizontal

28. a. parallel b. parallel c. perpendicular

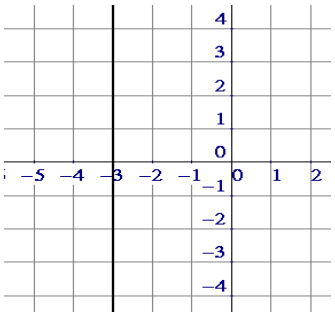
29. a. $x = 23$, for example; any line of the form $x =$ *some number* would work.
b. $y = 9$, for example; any line of the form $y =$ *some number* would work.
c. $x = -\pi$ for example; any line of the form $x =$ *some number* would work.
d. $y = -3$, for example; any line of the form $y =$ *some number* would work.

30. a. horizontal b. horizontal c. vertical d. vertical

31. a. $x = 1$ b. $y = 0$ c. $y = 8$ d. $x = 7$

32. a. For instance, (–3, 0), (–3, –2), and (–3, 4) are three points on the line.

Plotting these points and connecting them produces the graph:



b. The line is vertical.

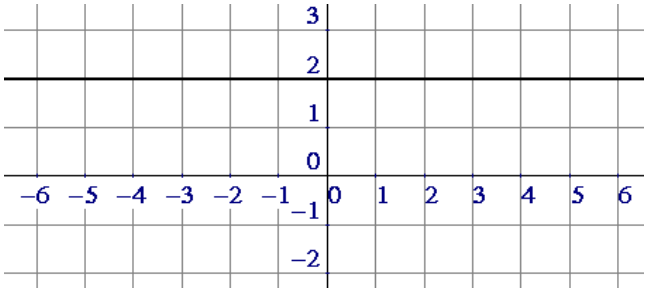
c. The only intercept of this line is (–3, 0).

d. Using the first two of the three points, we find the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{0-(-2)}{-3-(-3)} = \frac{0+2}{-3+3} = \frac{2}{0},$$

and therefore the slope is undefined.

33. a. For example, (–1, 2), (3, 2), and (4, 2) are three points on the line. Plotting these points and connecting them produces the graph:



b. The line is horizontal.

c. The only intercept of this line is (0, 2).

d. Using the first two of the three points, we find the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{2-2}{-1-3} = \frac{0}{-4} = 0$$

and therefore the slope is 0.

34. a. $y = 0$ b. $x = 0$

35. horizontal; 0 **36**. vertical; undefined **37**. diagonal; 1

38. $y = 99$ **39**. $x = 34$ **40**. $x = 2$

41. $y = -\pi$ **42**. $x = 2$ **43**. $y = 7$

44. $x = -2$ **45**. $x = \pi$

46. a. T b. T c. T d. F e. T f. F g. T h. F i. T
j. T k. F l. T m. F n. F o. T p. F q. F r. T
s. T t. F u. T

“This thing we call ‘failure’ is not
the falling down . . .
but the staying down.”

Mary Pickford

LOG EQUATIONS

Now that we have the Laws of Logs at our disposal, we can solve equations containing logs. We will also revisit the Richter scale, the pH scale, and the decibel scale.



Decibel Scale

❑ REVIEW OF THE LOG FUNCTION AND LAWS OF LOGS



pH Scale

Some log equations can be solved by inspection. For instance, the solution of the equation $\log x = 2$ is $x = 100$, since $\log 100 = 2$. But some others aren't so easy. The key to solving log equations will be the conversion from a log expression to an exponential expression; this conversion is none other than our definition of log:



Richter Scale

$$\log_b x = y \quad \text{means} \quad b^y = x$$

In addition, let's restate the 1st and 2nd laws of logs — these will also be used in this chapter:

First Law of Logs: $\log_b(xy) = \log_b x + \log_b y$

Second Law of Logs: $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

❑ *LEVEL ONE EQUATIONS*

EXAMPLE 1: **Solve for x: $\log_8 x - 2 = 0$**

Solution:

	$\log_8 x - 2 = 0$	(the original equation)
$\Rightarrow$	$\log_8 x = 2$	(begin to isolate the x)
$\Rightarrow$	$8^2 = x$	(**change to exponential form**)
$\Rightarrow$	$\boxed{x = 64}$	(calculate the value of x)

EXAMPLE 2: **Solve for y: $\log (3y + 5) = 1$**

Solution: Note: This is the common log, base 10.

	$\log (3y + 5) = 1$	(the original equation)
$\Rightarrow$	$10^1 = 3y + 5$	(**change to exponential form**)
$\Rightarrow$	$3y + 5 = 10$	(rearrange and simplify)
$\Rightarrow$	$3y = 5$	(subtract 5 from each side)
$\Rightarrow$	$\boxed{y = \frac{5}{3}}$	(divide each side by 3)

EXAMPLE 3: **Solve for x: $\ln(7 - 4x) = \frac{1}{2}$**

Solution: ***ln*** is just another log, and its base is understood to be ***e***.

	$\ln(7 - 4x) = \frac{1}{2}$	(the original equation)
$\Rightarrow$	$e^{1/2} = 7 - 4x$	(**convert to exponent form**)

$\Rightarrow \quad -4x+7 = \sqrt{e}$ (flip the equation around)

$\Rightarrow \quad -4x = \sqrt{e}-7$ (subtract 7 from each side)

$\Rightarrow \quad x = \dfrac{\sqrt{e}-7}{-4}$ (divide each side by –4)

$\Rightarrow \quad x = \dfrac{7-\sqrt{e}}{4}$ (multiply top and bottom by –1)

So the exact solution is

$$\boxed{x = \frac{7-\sqrt{e}}{4}}$$

We can use a calculator to approximate the solution as **1.338**.

Homework

1. Solve each log equation:

a. $\log_6 x = 2$

b. $\log_5 x = 3$

c. $\log_2 x = \frac{1}{2}$

d. $\log_3 x = -4$

e. $\log_7(x+2) = 1$

f. $\log(2x-1) = 2$

g. $\ln x = 3$

h. $\ln(x-1) = 1$

i. $\ln(3x-5) = 0$

j. $\ln(2x+5) = \frac{1}{4}$

❑ *LEVEL TWO EQUATIONS*

EXAMPLE 4: **Solve for x: $\log x + \log 2 = 3$**

Solution: Be honest — you don't like logs, and this equation has two of them in it! But we have a way out, using the first law of logs: $\log(ab) = \log a + \log b$

Let's begin:

	$\log x + \log 2 = 3$	(the original equation)
$\Rightarrow$	$\log (2x) = 3$	(**first law of logs**)
$\Rightarrow$	$10^3 = 2x$	(exponent form – base is 10)
$\Rightarrow$	$2x = 1000$	(rearrange and simplify)
$\Rightarrow$	$\boxed{x = 500}$	(divide each side by 2)

EXAMPLE 5: **Solve for n: $\log (n - 3) = \log 5 + 4$**

Solution: For this problem, the second law of logs, $\log \frac{a}{b} = \log a - \log b$, comes into play:

	$\log (n - 3) = \log 5 + 4$	(the given equation)
$\Rightarrow$	$\log (n - 3) - \log 5 = 4$	(bring the logs together)
$\Rightarrow$	$\log\left(\frac{n-3}{5}\right) = 4$	(**second law of logs**)
$\Rightarrow$	$10^4 = \frac{n-3}{5}$	(convert to exponent form)
$\Rightarrow$	$n - 3 = 5(10^4) = 50{,}000$	(multiply each side by 5)
$\Rightarrow$	$\boxed{n = 50{,}003}$	(solve for n)

As a quick check, note that if we plug 50,003 into the original equation, we are taking the log of a positive number – a good sign, since the log of a negative number or zero is undefined. Using a calculator, we can compute the following:

log $(n - 3)$ = log 5 + 4
log (**50,003** – 3) = log 5 + 4
log (50,000) = log 5 + 4
4.69897 = 0.69897 + 4
4.69897 = 4.69897 ✓

EXAMPLE 6: **Solve for z: $\log z + \log (z + 15) = 2$**

Solution: Let's just do it:

	$\log z + \log (z + 15) = 2$	(the given equation)
$\Rightarrow$	$\log [z(z + 15)] = 2$	(**the first law of logs**)
$\Rightarrow$	$\log (z^2 + 15z) = 2$	(distribute)
$\Rightarrow$	$10^2 = z^2 + 15z$	(change to exponent form)
$\Rightarrow$	$z^2 + 15z - 100 = 0$	(put in quadratic form)
$\Rightarrow$	$(z + 20)(z - 5) = 0$	(factor)
$\Rightarrow$	$z = -20$ OR $z = 5$	(set each factor to 0)

That yields two potential solutions, but only checking will tell which, if either, will work in the original equation:

$z = -20$: Immediate meltdown:
log z + log $(z + 15)$ = 2
log (–20) + . . . we can stop here;
this log is undefined; thus, –20 fails.

$z = 5$: It might work. Using a calculator:

$\log z + \log (z + 15) = 2$

$\log 5 + \log (5 + 15) = 2$

$\log 5 + \log 20 = 2$

$0.698970004 + 1.301029996 = 2$

$2 = 2$ ✓

Thus, the final solution to the equation is $\boxed{z = 5}$

EXAMPLE 7: **Solve for x: $\log_2(x+5) = 3 + \log_2(x-2)$**

Solution: Notice that the base of the log is 2, but the equation is really no more difficult than if it had a base of 10.

$\log_2(x+5) = 3 + \log_2(x-2)$ (the original equation)

$\Rightarrow \log_2(x+5) - \log_2(x-2) = 3$ (bring the logs together)

$\Rightarrow \log_2\left(\frac{x+5}{x-2}\right) = 3$ (the second law of logs)

$\Rightarrow 2^3 = \frac{x+5}{x-2}$ (change to exponent form)

$\Rightarrow 8x - 16 = x + 5$ (clear the fraction)

$\Rightarrow 7x = 21$ (subtract x and add 16)

$\Rightarrow \boxed{x = 3}$

To check our candidate, $x = 3$, we see that we won't be able to use our calculator, since log base 2 is not on the machine:

$\log_2(x+5) \stackrel{?}{=} 3 + \log_2(x-2)$

$\log_2(\boxed{3}+5) \stackrel{?}{=} 3 + \log_2(\boxed{3}-2)$

$\log_2 8 \stackrel{?}{=} 3 + \log_2 1$

$3 \stackrel{?}{=} 3 + 0$

$3 = 3$ ✓

EXAMPLE 8: **Solve for x: $\ln x^2 + \ln x + \ln 7 = 3$**

Solution: For this log equation, notice that the log has the understood base of e.

$\ln x^2 + \ln x + \ln 7 = 3$

$\Rightarrow \quad \ln (x^2 \cdot x \cdot 7) = 3$ (**sum of logs = log of product**)

$\Rightarrow \quad \ln (7x^3) = 3$ (simple algebra)

$\Rightarrow \quad e^3 = 7x^3$ (**change to exponent form**)

$\Rightarrow \quad x^3 = \frac{e^3}{7}$ (divide each side by 7)

$\Rightarrow \quad x = \sqrt[3]{\frac{e^3}{7}}$ (take the cube root of each side)

$\Rightarrow \quad x = \frac{\sqrt[3]{e^3}}{\sqrt[3]{7}}$ (split the radical)

$\Rightarrow \quad \boxed{x = \frac{e}{\sqrt[3]{7}}}$ (simplify the top)

An approximate answer, given by the calculator, is **1.421**.

EXAMPLE 9: **Solve for a: $\ln a = \ln (a + 5) - 4$**

Solution: $\ln a = \ln (a + 5) - 4$ (the original equation)

$\Rightarrow \quad \ln a - \ln (a + 5) = -4$ (subtract $\ln (a + 5)$)

$\Rightarrow \quad \ln\left(\frac{a}{a+5}\right) = -4$ (**second law of logs**)

$\Rightarrow \quad \frac{a}{a+5} = e^{-4}$ (**convert to exponent form**)

$\Rightarrow \quad \frac{a}{a+5} = \frac{1}{e^4}$ (convert exponent to fraction)

$\Rightarrow \quad ae^4 = a + 5$ (cross-multiply)

$\Rightarrow \quad ae^4 - a = 5$ (subtract a from each side)

$\Rightarrow \quad a(e^4 - 1) = 5$ (factor out the variable)

$\Rightarrow \quad \boxed{a = \frac{5}{e^4 - 1}}$ (divide each side by $e^4 - 1$)

Homework

Solve each log equation:

2. $\log y + \log 3 = 4$

3. $\ln z = 1 + \ln 5$

4. $\log_3(w+1) - \log_3 w = 3$

5. $\log u + \log (u + 246) = 3$

6. $\log (2x + 1) = 1 - \log (3x - 4)$

7. $\log (2x + 1) - \log 5 = \log x$

8. $\ln (x + 2) - \ln (x - 3) = 2$

9. $\ln x - \ln (x + 5) = 4$

10. $\log_2 x^2 + \log_2 x + \log_2 3 = 5$

11. Solve each log equation:

    a. $\ln x + \ln 5 = 2$

    b. $\log a = 2 + \log 3$

    c. $\log_2(y+1) - \log_2 y = 4$

    d. $\log(u+24) + \log(u+3) = 2$

    e. $\log_5(8x+1) = 2 - \log_5(x-2)$

    f. $\ln(2x) - \ln(x-1) = 4$

    g. $\log_3 n^2 + \log_3 n + \log_3 7 = 2$

    h. $\log(5-x) + \log(x+15) = 2$

12. Check your solution to problem 11h.

❑ *LOG APPLICATIONS REVISITED*

Now that we can solve equations containing logs, we will go back to our formulas for pH, decibels, and Richter numbers and solve more problems. Here are the three formulas from previous chapters:

$\text{pH} = -\log[\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration in moles/liter

$D = 10\log\left(\frac{I}{10^{-12}}\right)$ where I is the intensity in watts/m^2

$M = \frac{2}{3}\log\left(\frac{E}{2.5\times10^4}\right)$ where E is the energy in joules

EXAMPLE 10: **If the pH of an acid is 2.3, find the hydrogen ion concentration.**

Solution: To simplify the symbols, let's use just the letter H to represent the concentration.

$$\text{pH} = -\log H$$
$$\Rightarrow \quad 2.3 = -\log H$$
$$\Rightarrow \quad \log H = -2.3$$
$$\Rightarrow \quad H = 10^{-2.3}, \text{ and so the concentration is approximately}$$

0.005 moles/liter

EXAMPLE 11: **A certain sound has a decibel value of 85. Find the intensity of the sound.**

Solution: $D = 10\log\left(\frac{I}{10^{-12}}\right)$

$\Rightarrow \quad 85 = 10\log\left(\frac{I}{10^{-12}}\right)$

$\Rightarrow \quad 8.5 = \log\left(\frac{I}{10^{-12}}\right)$

$\Rightarrow \quad \frac{I}{10^{-12}} = 10^{8.5}$

$\Rightarrow \quad I = 10^{-12} \times 10^{8.5}$

$\Rightarrow \quad I = 10^{-3.5}$

$\Rightarrow \quad \boxed{I = 0.00032 \text{ W/m}^2}$

EXAMPLE 12: **An earthquake had a Richter magnitude of 6. Find the energy release in joules.**

Solution:

$M = \frac{2}{3}\log\left(\frac{E}{2.5\times10^4}\right)$

$\Rightarrow \quad 6 = \frac{2}{3}\log\left(\frac{E}{2.5\times10^4}\right)$ (we need to solve for E)

$\Rightarrow \quad 9 = \log\left(\frac{E}{2.5\times10^4}\right)$ (multiply each side by $\frac{3}{2}$)

$\Rightarrow \quad \frac{E}{2.5\times10^4} = 10^9$ (definition of log)

$\Rightarrow \quad E = 10^9 \times 2.5 \times 10^4 \Rightarrow \boxed{E = 2.5 \times 10^{13} \text{ J}}$

Homework

13. The pH of a strong base is 12.3. Find the hydrogen ion concentration.

14. Find the intensity of a sound wave whose decibel value is 50.

15. Calculate the energy released in an earthquake of magnitude 4.5 on the Richter scale.

16. The pH of a weak acid is 5.8. Find the hydrogen ion concentration.

17. Find the intensity of a sound wave whose decibel value is 100.

18. Calculate the energy released in an earthquake of magnitude 7.8 on the Richter scale.

Review Problems

19. Solve for x: $\log_3(x+4) - \log_3 x = 4$

20. Solve for x: $\ln x^3 + \ln x^2 - \ln x - 2 = 0$

21. Solve for x: $\log(x-3) = \log x - 3$

22. Solve for x: $\log_2(x+14) = 5 - \log_2 x$

23. a. Find the hydrogen ion concentration of a base with pH = 12.

b. Find the intensity of a sound whose decibel level is 40.

c. Find the energy released in an earthquake of Richter 3.

24. True/False:

a. The equation $\log_2 x = 0$ has a solution.

b. The solution of $\ln(7-4x) = \frac{1}{2}$ is $x = \frac{1}{14-8x}$.

c. The solution of $\log x + \log 5 = 4$ is $x = 2000$.

d. The solution of $\log_3(x+1) - \log_3(x-1) = 3$ is $x = 14/13$.

e. The equation $\log n = 2 - \log(n+15)$ has two solutions.

f. The solution of $\ln x^3 + \ln x^2 + \ln x + \ln 2 = 10$ is $x = \sqrt[6]{\frac{e^{10}}{2}}$.

g. If an acid has a pH of 2.5, then its hydrogen ion concentration is 0.02 moles/liter.

h. The intensity of a sound whose decibel level is 125 is $3.1623\ \text{W/m}^2$.

i. If the Richter magnitude of an earthquake is 9, then the energy release is 7.9057×10^{17} joules.

Solutions

1. a. 36 b. 125 c. $\sqrt{2}$ d. $\frac{1}{81}$ e. 5

f. $\frac{101}{2}$ g. e^3 h. $e+1$ i. 2 j. $\frac{\sqrt[4]{e}-5}{2}$

2. $\frac{10,000}{3}$ **3**. $5e$, which is ≈ 13.5914 **4**. $\frac{1}{26}$

5. –250 is an extraneous solution; the solution is {4}.

6. You should reach the quadratic equation $6x^2 - 5x - 14 = 0$, whose two solutions are 2 and $-\frac{7}{6}$. Only 2 works.

7. $\frac{1}{3}$

8. $\ln(x+2) - \ln(x-3) = 2 \Rightarrow \ln\frac{x+2}{x-3} = 2 \Rightarrow \frac{x+2}{x-3} = e^2$

$\Rightarrow x+2 = e^2x - 3e^2 \Rightarrow x - e^2x = -3e^2 - 2$

$\Rightarrow x(1-e^2) = -3e^2 - 2 \Rightarrow x = \frac{-3e^2-2}{1-e^2}$, or $\frac{3e^2+2}{e^2-1}$

9. $x = \frac{5e^4}{1-e^4}$ is the solution obtained by following the rules. The problem is that this value of x is negative. Placing this value of x into the original equation yields an error; the final solution is therefore ∅, a notation that indicates No Solution.

10. You should get the equation $3x^3 = 32$.
Solving for x gives $\sqrt[3]{\frac{32}{3}}$ or $2\sqrt[3]{\frac{4}{3}}$ or $\frac{2\sqrt[3]{36}}{3}$.

11. a. $\frac{e^2}{5}$ b. 300 c. $\frac{1}{15}$ d. 1
e. 3 f. $\frac{e^4}{e^4-2}$ g. $\sqrt[3]{\frac{9}{7}}$ h. –5

12. We suppose that $x = -5$ is a solution of $\log(5-x)+\log(x+15) = 2$.
Placing –5 in for x in the original equation, $\log(5-x)+\log(x+15) = 2$, gives

$\log(5-(-\mathbf{5}))+\log(-\mathbf{5}+15) = \log(5+5)+\log(-5+15)$
$= \log 10 + \log 10 = 1+1 = \mathbf{2}$ It works. ☺

13. 5.01×10^{-13} moles/liter

14. 10^{-7} W/m^2

15. 1.41×10^{11} joules

16. 0.000001585 moles/liter

17. 0.01 W/m^2

18. 1.253×10^{16} J

19. $x = \frac{1}{20}$

20. $x = \sqrt{e}$

21. $x = \frac{1000}{333}$

22. $x = 2$

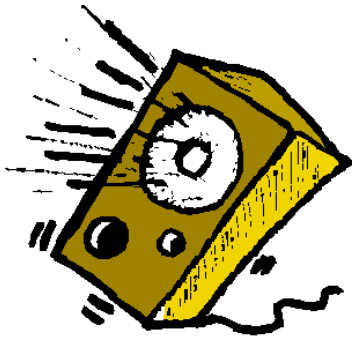
23. a. 1×10^{-12} moles/liter b. 1×10^{-8} W/m^2 c. 7.91×10^{8} joules

24. a. T b. F c. T d. T e. F f. T g. F h. T i. T

"**Education** is what survives when what has been learned has been forgotten."

BF Skinner

LOG FUNCTIONS



The **decibel scale** for measuring the loudness of sound is discussed at the end of the chapter.

To better understand logs, it would be nice to graph them; but to do that, we need to express the log concept as a *function*.

You learned how to perform operations like *squaring* back in basic algebra. But it wasn't until later in your studies that you looked at squaring as a function: a collection of inputs and outputs where the output is always the *square* of the input, and where each input has exactly one output. This new function approach led to concepts such as domain, notation like $f(x) = x^2$, and the parabolic graphs of the squaring functions.

We do the same thing now with *logs*. We've learned what the definition of log is, and we're able to calculate logs, using a variety of bases. Now it's time to formalize the concept of log in the parlance of functions.

❑ DEFINITION

Let ***b*** be a positive real number not equal to 1. Then the function

$$y = \log_b x$$

is called the ***log base b*** function.

As with exponential functions, the base, b, of a logarithmic function satisfies the property:

$$b > 0,\ b \neq 1$$

Note that another way to describe what the base ***b*** can be is to say that ***b*** must be in the set $(0, \infty) - \{1\}$, which we also write as $(0, 1) \cup (1, \infty)$.

Homework

1. In the log function $y = \log_b x$, what real numbers can ***b*** be?

2. Let $f(x) = \log_2 x$. Compute:

   a. $f(2)$ b. $f(4)$ c. $f(16)$ d. $f(128)$ e. $f(1)$

   f. $f(\frac{1}{2})$ g. $f(\frac{1}{8})$ h. $f(0)$ i. $f(-1)$ j. $f(-2)$

3. Let $g(x) = \log x$. Compute:

   a. $g(10)$ b. $g(1)$ c. $g(0)$

   d. $g(100)$ e. $g(-100)$ f. $g(\frac{1}{1000})$

4. Let $h(x) = \ln x$. Compute:

   a. $h(0)$ b. $h(1)$ c. $h(e)$ d. $h(e^2)$ e. $h(e^{10})$

   f. $h(e^x)$ g. $h(\frac{1}{e})$ h. $h(\sqrt{e})$ i. $h\left(\frac{1}{\sqrt{e}}\right)$ j. $h(-e)$

Note that there are a variety of *log* functions, one for each valid value of the base, ***b***.

❑ *DOMAIN*

From the previous Homework we extract three problems:

$\log_2\left(\frac{1}{8}\right) = -3$ $\qquad$ $\log 0$ is Undefined $\qquad$ $\ln(-e)$ is Undefined

NOTE: What these three logs, along with the rest of the Homework, should tell you is that the log function (for any legal base) is defined only for inputs that are positive numbers; that is, you can take the log of positive numbers only. Equivalently, you can never take the log of 0 or any negative number. In short,

The domain of the function $y = \log_b x$ is ***x* > 0**.

[In interval notation, the domain is **(0, ∞)**.]

EXAMPLE 1: **Find the domain of each function:**

A. $f(x) = \log_3(x+7)$

We can take the log of positive quantities only; therefore, we must solve the inequality $x + 7 > 0$, which implies that $x > -7$. So the domain is

$x > -7$ $\qquad$ Interval notation: **(–7, ∞)**

B. $g(x) = \log(x^2 - 5x - 14)$

Again, we can take the log of positive quantities only. So we must solve the inequality $x^2 - 5x - 14 > 0$. Using the *Boundary Point Method*, we solve the associated equation

and calculate the boundary points: $x^2 - 5x - 14 = 0 \Rightarrow (x+2)(x-7) = 0 \Rightarrow -2$ and 7 are the boundary points.

By choosing test points and checking them in the original inequality, together with checking the boundary points, you can verify that the domain is

$x < -2$ OR $x > 7$

Interval notation:

$(-\infty, -2) \cup (7, \infty)$

C. $h(x) = \ln(25 - x^2)$

To solve the inequality $25 - x^2 > 0$, we solve the equation to get the boundary points –5 and 5. When the test points and boundary points are tested, the domain is

$-5 < x < 5$ Interval notation: **(–5, 5)**

Homework

5. Consider the function $y = \ln(x^2 - 3x - 10)$. Without actually calculating the domain, determine which of the following are in the domain of the function:

   a. π b. -2 c. -10 d. 0 e. 5 f. e g. 12

6. Find the domain of each function:

   a. $y = \log_{23} x$ b. $y = \log(8 - 2x)$

   c. $y = \ln(x^2 + 2x - 3)$ d. $y = \log_5(x^2 - 49)$

   e. $y = \log_{12}(x^2 + 1)$ f. $y = \log_{\pi}(100 - x^2)$

❑ GRAPHING

EXAMPLE 2: **Graph: $y = \log_2 x$**

Solution:

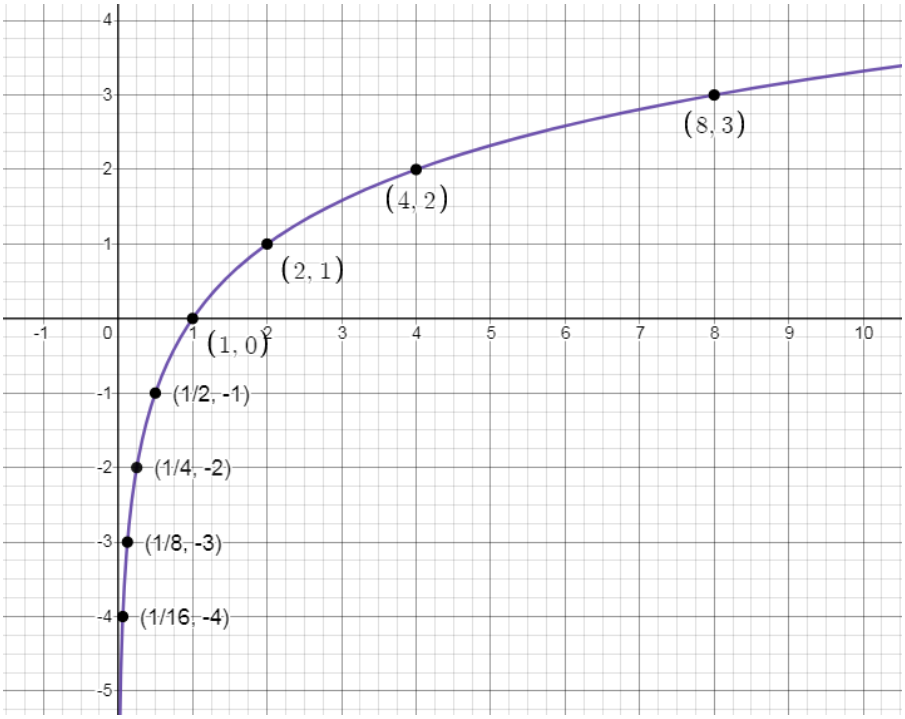
If $x = 8$, then $y = \log_2 8 = 3$ (since $2^3 = 8$).

If $x = 2$, then $y = \log_2 2 = 1$ (since $2^1 = 2$).

If $x = 1/2$, then $y = \log_2 1/2 = -1$ (since $2^{-1} = 1/2$)

Continuing in this way, and remembering that the domain of the log function is $x > 0$, we can calculate the following ordered pairs in the function. Plotting the points in the table and connecting them with a smooth curve gives us the following graph:

x	y
1	0
2	1
4	2
8	3
16	4
$\frac{1}{2}$	-1
$\frac{1}{4}$	-2
$\frac{1}{8}$	-3
$\frac{1}{16}$	-4



Let's confirm what we said earlier. The graph seems to indicate that the **domain** of the function is $\boldsymbol{x > 0}$; that is, x must be a positive real number.

As for **intercepts**, the table and the graph indicate that (1, 0) is the only x-intercept. There is no y-intercept, since x can never be 0.

Let's look at some **limits** for this function. Check out the first five points in the x-y table. As x grows larger and larger, so does y, albeit rather slowly. As slowly as the graph rises, it nevertheless rises higher and higher as x gets larger (tough to see; just trust me). Therefore,

$$\textbf{As } x \to \infty, \textbf{ } y \to \infty,$$

which can be written $\lim_{x \to \infty} \log_2 x = \infty$.

Now look at how the graph gets closer and closer to the y-axis as x gets closer and closer to 0. Remember, even though x can never be 0, that doesn't prevent us from analyzing what happens to y when x approaches 0. But, of course, looking at the graph (and the domain of the function) shows us that x cannot approach 0 from the left because there's no graph there, so we consider x approaching 0 only from the right.

Now to the question: As x approaches 0 from the right, what does y approach? It's going down and down, so it appears to be approaching $-\infty$. Here's what we're saying:

$$\textbf{As } x \to \textbf{0} \text{ (from the right)}, \textbf{ } y \to -\infty$$

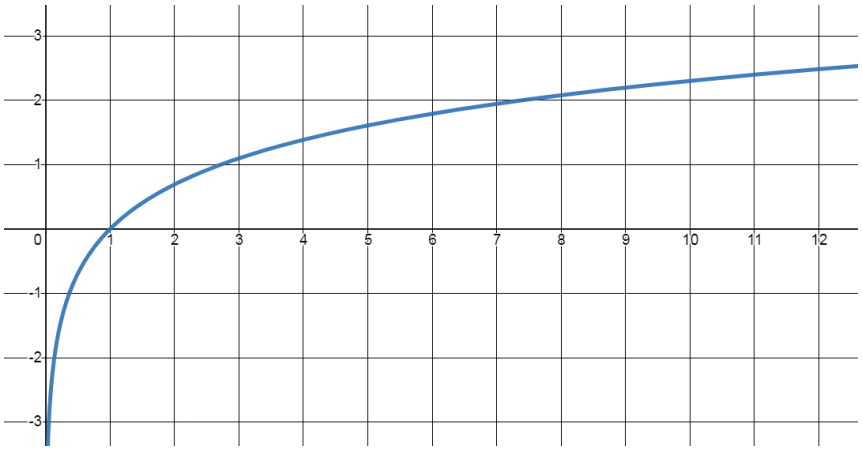
The two limits discussed seem to give evidence that the y-values take on all real numbers. In addition, the second limit indicates that the function has a **vertical asymptote** at $x = 0$ (the y-axis).

EXAMPLE 3: **Graph: $y = \ln x$**

Solution: Using a calculator when necessary, we can compute the following ordered pairs for the function:

$(\frac{1}{10}, -2.3)$	$(\frac{1}{2}, -.69)$	(1, 0)
(e, 1)	(5, 1.61)	(8, 2.08)

These points, together with the conclusions of Example 2, yield the following graph:



EXAMPLE 4: **Graph: $y = \log(x+2)$**

Solution: Note that this is a common log, base 10. We begin our analysis of this function by finding the **domain**. To do this, we create the inequality $x + 2 > 0$, which implies that $\boldsymbol{x > -2}$.

The ***x*-intercept** is found by setting y to 0. This produces the equation $\log (x + 2) = 0$. Converting this to log form gives $10^0 = x + 2$, from which it follows that $x = -1$. Thus, the x-intercept is **(–1, 0)**.

To calculate the ***y*-intercept**, we of course set x to 0. This yields the equation $y = \log (0 + 2)$, which produces a y-value of about 0.3. Therefore, the y-intercept is **(0, 0.3)**.

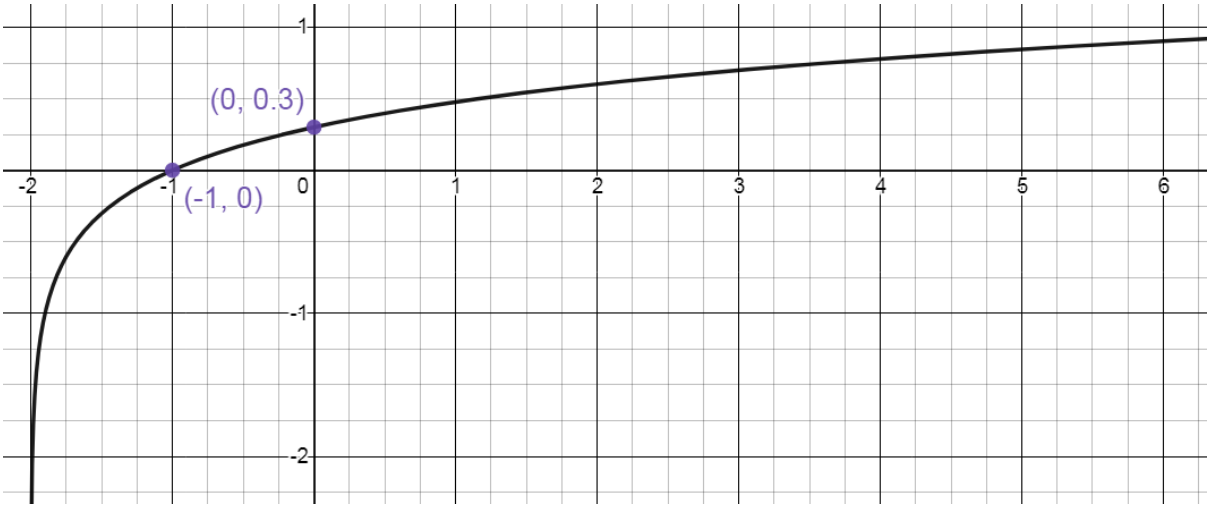
We can find a few ordered pairs for our graph <u>without</u> a calculator by letting $x = -1\frac{9}{10}$, 8, and 98:

$(-1\frac{9}{10}, -1)$ (8, 1) (98, 2)

We can use our calculator to find even more ordered pairs:

(–1.98, –1.7) (3, 0.7) (5, 0.8) (6, 0.9)

[If you're smart, you'll verify each of these ordered pairs.]



We can be pretty sure that the graph goes infinitely down, and we've already determined that log functions go infinitely up (extremely slowly). From these facts, we see that all y-values are obtained by the function. We can summarize these ideas with a pair of limits:

As $x \to -2$ (from the right)**, $y \to -\infty$**

which can be written $\lim_{x \to -2^+} \log_2(x+2) = -\infty$.

As $x \to \infty$, $y \to \infty$

As a final observation, we can see that the line $x = -2$ is a vertical asymptote, but there is NO horizontal asymptote.

Homework

7. Prove that the graph of $y = \log_2 x$ eventually reaches a height of 10. Hint: Find an x that yields a y of 10.

8. Explain why x cannot approach 0 from the left in the function $f(x) = \log x$.

9. Graph $y = \log (x - 1)$ and describe completely.

10. Graph the three functions $\ln x$, $\ln(x-2)$, and $\ln(x+3)$ on the same grid. Describe how the –2 and the 3 affect the graph of $\ln x$.

11. Find the domain and the vertical asymptote of $y = \ln (x - 5)$.

12. Graph the three functions $\ln x$, $\ln x + 2$, and $\ln x - 1$ on the same grid. Describe how the 2 and –1 affect the graph of $\ln x$. Note: the expression $\ln x + 2$ means take the *ln* first and then add 2.

13. Find the domain and the vertical asymptote of $y = \ln x + 9$.

❑ *LOG FUNCTIONS IN A NUTSHELL*

$y = \log_b x$ means $x = b^y$

any real number — **positive, ≠ 1** — **positive**

$b \in (0, \infty) - \{1\}$

Domain = $(0, \infty)$

Homework

14. Let $y = \log_b x$. True or False?

a. x must be a non-negative real number.

b. y can be any real number.

c. b could be 1.

d. The domain of the function is $[0, \infty)$.

e. y can be any real number.

f. The equation is equivalent to $b^x = y$.

g. The graph of the function lies in Quadrants I and II.

h. x could be 1.

i. y could be 1.

j. As $x \to \infty$, $y \to 0$.

k. As $x \to -\infty$, $y \to$ nothing at all.

l. As $x \to 0$ (from the left), $y \to -\infty$.

m. As $x \to 1$ (from the right), $y \to 0$.

n. As $x \to 1$ (from the left), $y \to 0$.

o. x must be a positive number.

p. b could be 0.

q. b could be negative.

❑ THE DECIBEL SCALE

Named after Alexander Graham Bell, the **decibel scale** is a measure of the loudness of sound waves. If I is the intensity of the sound, measured in watts per square meter (W/m^2), then the number of decibels is given by

$$D = 10 \log\left(\frac{I}{10^{-12}}\right).$$

[The denominator 10^{-12}, or 0.000000000001, is the intensity of the softest sound the human ear can detect.]

Calculator Hint: On a TI-30, to enter a number like 7.3×10^{13}, first press 7.3, then press the "EE" button, and then press 13. To enter the number 10^{-12}, press 1, then EE, then 12, and then +/–.

EXAMPLE 5: **The sound produced by a jet engine has an intensity of 8.3×10^2 W/m^2. Find the decibel value of this sound.**

Solution:

$$\begin{aligned} D &= 10\log\left(\frac{I}{10^{-12}}\right) \\ &= 10\log\left(\frac{8.3\times10^{2}}{10^{-12}}\right) \\ &= 10\log(8.3\times10^{14}) \qquad \text{(bring up the negative exponent)} \\ &= 10(14.919) \\ &= 149 \end{aligned}$$

The jet engine sound has a value of

149 decibels

Homework

15. Find the decibel value of a whisper whose intensity is 5.2×10^{-10} W/m^2.

16. Find the decibel value of each sound whose intensity is given:

    a. 1.0×10^{-12} W/m^2
    b. 2.3×10^{-9} W/m^2
    c. 6.3×10^{-7} W/m^2
    d. 8.7×10^{-5} W/m^2

Review Problems

17. Describe the legal values of b in the function $f(x) = \log_b x$.

18. a. If $f(x) = \log_2 x$, calculate $f(1024) + f(1/8)$.
    b. If $g(x) = \log x$, calculate $g(1000) - g(0.01)$.
    c. If $h(x) = \ln x$, calculate $h(e^6) + h(1) - h(\sqrt[3]{e})$.
    d. Calculate $f(512) + g(1 \text{ million}) + h(e)$

19. Find the domain: $f(x) = \log(35 - 7x)$

20. Find the domain of $y = \ln(x^2 - 144)$.

21. Prove that the function $y = \log_2(x - 2)$ has no y-intercept.

22. If $y = \log_5(7x - 21)$, find x if $y = 2$.

23. Graph: $y = \ln(x-2)$ State any intercepts and asymptotes.

24. Let $f(x) = \ln x$ and $g(x) = \ln(x-3)+4$. Describe the graph of g relative to the graph of f.

25. Consider the function $y = \ln x$. Find the following **limits**:

    a. As $x \to \infty$, $y \to$ _____

    b. As $x \to -\infty$, $y \to$ _____

    c. As $x \to 0$ (from the right), $y \to$ _____

    d. As $x \to 0$ (from the left), $y \to$ _____

26. Use the formula $D = 10\log\left(\frac{I}{10^{-12}}\right)$ to find the decibel level of a sound whose intensity is 7.2×10^{-5} W/m^2.

27. True/False:

    a. The base of a log function can be any positive real number.

    b. The base of a log function can be any positive real number except 1.

    c. The domain of the log function $y = \log_b x$ is $x > 0$.

    d. If $g(x) = \log_2(x+10)$, then $g(22) = 5$.

    e. The domain of the function $y = \log_6(x^2-25)$ is $x \le -5$ OR $x \ge 5$.

    f. The x-intercept of the graph of $y = \log(x+2)$ is $(-1, 0)$.

    g. The graph of a log function has a horizontal asymptote.

    h. The graph of a log function has a vertical asymptote.

    i. The graphs of $f(x) = \ln(x+3)$ and $g(x) = \ln x + 3$ are identical.

    j. If the intensity of a sound is 2.765×10^3 W/m^2, the decibel value for that sound is about 154.

Solutions

1. b, the base of the log function, can be any real number greater than 0 but not equal to 1. That is, $b > 0$, $b \neq 1$. [This is exactly the same as what b is allowed to be in the exponential function $y = b^x$.]

2. a. 1 b. 2 c. 4 d. 7 e. 0
f. –1 g. –3 h. Undefined i. Undefined j. Undefined

3. a. 1 b. 0 c. Undefined d. 2 e. Undefined f. –3

4. a. Undefined b. 0 c. 1 d. 2 e. 10
f. x g. –1 h. $\frac{1}{2}$ i. $-\frac{1}{2}$ j. Undefined

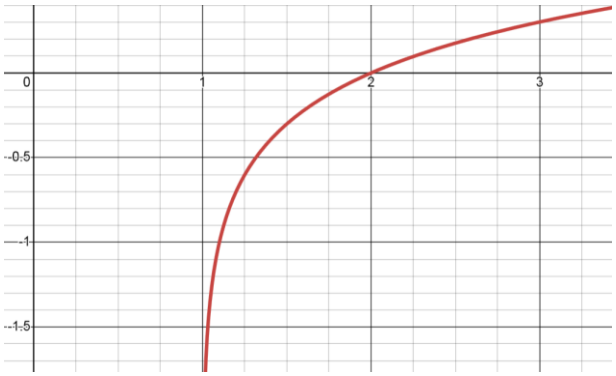
5. c. and g. only, since the quantity $x^2 - 3x - 10$ must be > 0.

6. a. $x > 0$
b. $8 - 2x > 0 \Rightarrow -2x > -8 \Rightarrow x < 4$
c. $x < -3$ OR $x > 1$
d. $x < -7$ OR $x > 7$
e. $\mathbb{R}$
f. $-10 < x < 10$

7. Setting $10 = \log_2 x \Rightarrow x = 2^{10} = 1024$

8. The domain of $\log x$ is $x > 0$. And x can't approach 0 from the left because x can't be negative; that is, there's no graph to the left of the origin.

9.



The domain is all real numbers greater than 1: $x > 1$.

The x-intercept is (2, 0).

There is no y-intercept.

There's a vertical asymptote at $x = 1$.

There is no horizontal asymptote.

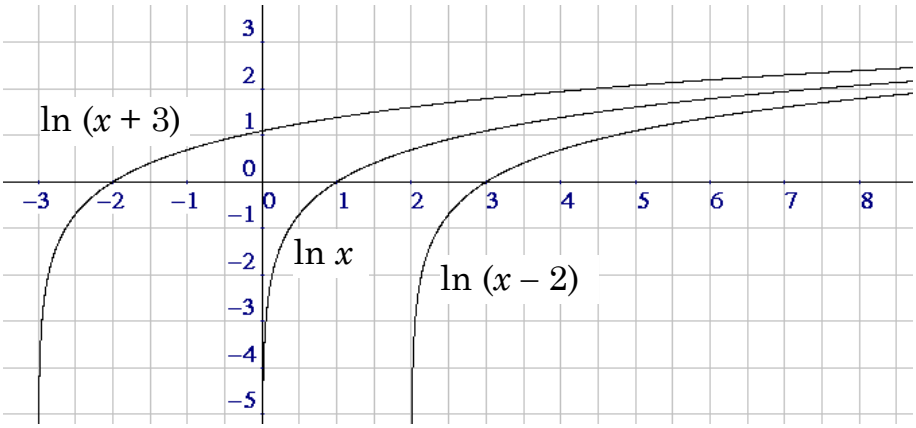
Some important limits:

The limit as $x \to 1$ (from the left) does not exist.

As $x \to 1$ (from the right), $y \to -\infty$.

As $x \to \infty$, $y \to \infty$.

10.

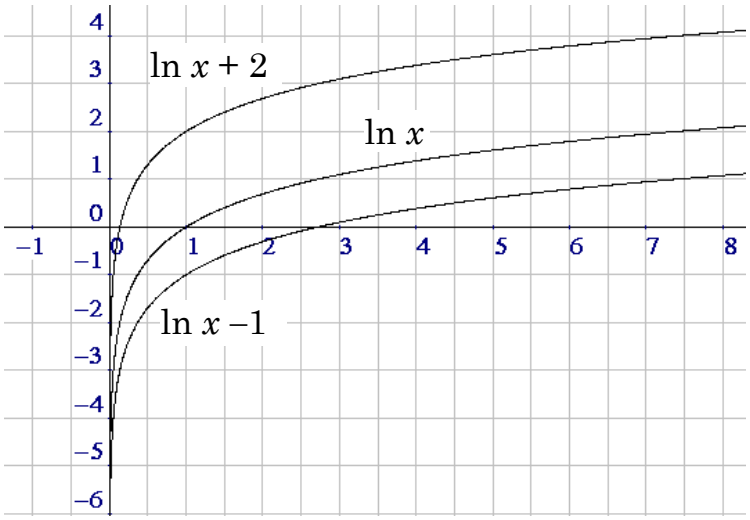


The –2 shifts the graph of $\ln x$ two units to the *right*.

The +3 shifts the graph of $\ln x$ three units to the *left*.

11. Domain: All $x > 5$, or $(5, \infty)$ Vert Asy: $x = 5$

12.



The +2 shifts the graph of ln x two units *up*.
The –1 shifts the graph of ln x one unit *down*.

13. Domain : All $x > 0$, or $(0, \infty)$ Vert Asy: $x = 0$

14. a. F b. T c. F d. F e. T f. F g. F h. T i. T
j. F k. T l. F m. T n. T o. T p. F q. F

15. 27 decibels

16. a. 0 b. 33.6 c. 58 d. 79.4

17. $b > 0$, $b \neq 1$

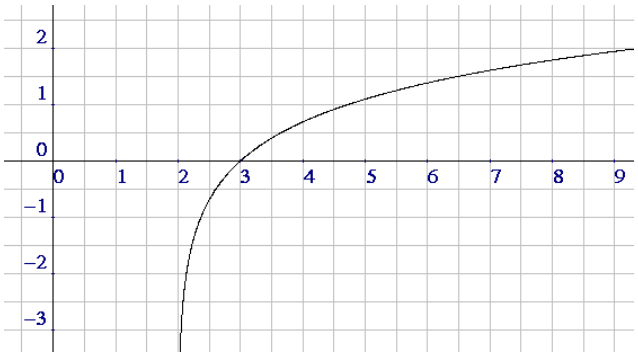
18. a. $10 + (-3) = 7$ b. $3 - (-2) = 5$ c. $6 + 0 - 1/3 = 17/3$
d. $9 + 6 + 1 = 16$

19. $x < 5$ **20**. $x < -12$ OR $x > 12$

21. Setting $x = 0$ yields $\log_2(-2)$, not defined. Therefore, no y-intercept.

22. $2 = \log_5(7x - 21) \Rightarrow 7x - 21 = 5^2 \Rightarrow x = \frac{46}{7}$

23.



x-int: (3, 0)

vert asym: $x = 2$

24. g is obtained by moving f 3 units to the right and 4 units up.

25. a. ∞ b. Does not exist c. $-\infty$ d. Does not exist

26. 78.6 decibels

27. a. F b. T c. T d. T e. F
f. T g. F h. T i. F j. T

Knowledge is the eye of desire and can become the pilot of the soul.

WILL DURANT, American writer, historian, and philosopher

THE LAWS OF LOGS

Since a log is really just an exponent, and since we have laws for exponents, it makes sense that we need to learn some laws for logs, too. These laws of logs will be used to solve new kinds of equations. This chapter also shows us another application of logs, the Richter scale for measuring the magnitude of an earthquake.



Adapazari, Turkey
Aug 17, 1999
Richter 7.8

❑ *THE TWO CANCELATION RULES*

Carefully study the next five examples:

$$\log_9 9^2 = \log_9 81 = 2$$

$$\log_{10} 10^5 = \log_{10} 100{,}000 = 5$$

$$\log_5 5^3 = \log_5 125 = 3$$

$$\log_2 2^1 = \log_2 2 = 1$$

$$\log_e e^0 = \log_e 1 = 0$$

Something's going on here. In each problem, the final answer matches the exponent at the front of the problem. For example,

$$\log_9 9^{\boxed{2}} = \boxed{2}$$

Also notice that the base of the log matches the base of the expression that we're taking the log of. For instance,

$$\log_{\boxed{10}} \boxed{10}^5 = 5$$

Using these two insights, we might now see that $\log_{12} 12^7 = 7$. We can now generalize this whole discussion into the first of two cancelation rules:

$$\log_b b^x = x$$

We call it a canceling rule because the log function (done second) cancels out the exponential function (done first), leaving just the *x*.

To motivate the second cancelation rule, consider the following four calculations:

$$10^{\log_{10} 1000} = 10^3 = 1000$$

$$2^{\log_2 32} = 2^5 = 32$$

$$5^{\log_5 125} = 5^3 = 125$$

$$7^{\log_7 1} = 7^0 = 1$$

For each example, notice that the base of the entire question matches the base of the log — for example, $\boxed{7}^{\log_{\boxed{7}} 1}$. We also see that in every case, the final answer matches the number we're taking the log of — for instance, $2^{\log_2 \boxed{32}} = 2^5 = \boxed{32}$. In a nutshell,

$$b^{\log_b x} = x$$

This is a canceling rule because the exponential function (done second) cancels out the log function (done first), leaving just the *x*.

EXAMPLE 1: **Use the canceling rules to simplify each expression:**

A. $\log_3 3^n = n$

B. $\log 10^{x+y} = x+y$

C. $7^{\log_7(ab)} = ab$

D. $e^{\ln(e+4)} = e+4$

E. $\log_4 5^n$ cannot be simplified by a canceling rule, since the base of the exponential (the 5) does not match the base of the log (the 4).

F. $8^{\log 7}$ also cannot be simplified by a canceling rule, since the base of the exponential (the 8) does not match the base of the log (the 10).

Homework

1. In the second canceling rule, explain why we must restrict x to the positive real numbers.

2. Simplify each expression:

a. $\log 10$ b. $\ln e$ c. $\log 10^{u+v}$ d. $\ln e^{xyz}$

e. $\log_9 9^{500}$ f. $10^{\log(xy)}$ g. $e^{\ln(\log 7)}$ h. $2^{\log_2(\ln e)}$

i. $3^{\log_3(a-b)}$ j. $\log_2 2^R$ k. $\log e^x$ l. $e^{\log Q}$

3. A common student error is to figure that $\log_b(x+y) = \log_b x + \log_b y$. We need to dispel this myth right now. Let $b = 2$, so that we're working with base 2. Then let both x and y equal 8. Show that the left side of the formula results in 4, whereas the right side comes out 6.

4. Prove that the conjecture $\log(xy) = (\log x)(\log y)$ is false. Hint: Let $x = 100$ and $y = 1000$ and work out each side.

❑ THE SUM OF LOGS

We know how to calculate certain logs in our heads; for example, $\log_5 25 = 2$, because $5^2 = 25$. Sometimes in a problem we have the sum or difference of two logs, and our goal is to convert that sum or difference into a single log. For instance, here's how we can

Convert $\log 100 + \log 1000$ *into a single common log.*

Start with the sum of the logs:	$\log 100 + \log 1000$
Calculate each log separately:	$2 + 3$
And add the numbers:	5
Now we have to write 5 as the common log of something.	
Since $10^5 = 100{,}000$, we can finish by writing	$\log 100{,}000$

In short, $\log 100 + \log 1000 = \log 100{,}000$

For a second example, let's

Find the sum of $\ln e^5$ *and* $\ln e^3$.

Start with the sum of the logs:	$\ln e^5 + \ln e^3$
Calculate each log separately:	$5 + 3$
And add the numbers:	8
Now we have to write 8 as the natural log of something.	$\ln e^8$

That is, $\ln e^5 + \ln e^3 = \ln e^8$

Homework

5. Express each sum of logs as a single log:

a. $\log 10 + \log 1000$
b. $\log 10{,}000 + \log 100$
c. $\log 10 + \log 1$
d. $\log 1000 + \log 1000$
e. $\log_2 8 + \log_2 64$
f. $\log_3 9 + \log_3 3$
g. $\log_4 64 + \log_4 4 + \log_4 1$
h. $\log 10 + \log 100 + \log 1000$
i. $\ln e^5 + \ln e^7$
j. $\ln e^{10} + \ln e^{10}$

❑ *THE DIFFERENCE OF LOGS*

To find the difference of logs, we use the same logic as above:

Express $\log_2 32 - \log_2 8$ *as a <u>single</u> log.*

$\log_2 32 - \log_2 8$

= $5 - 3$ (since $2^5 = 32$ and $2^3 = 8$)

= 2 (arithmetic)

= $\log_2 4$ (since $2^2 = 4$)

In short, $\log_2 32 - \log_2 8 = \log_2 4$

For a second difference of logs problem,

Express $\ln e^9 - \ln e^6$ *as a single log.*

$\ln e^9 - \ln e^6$

= $9 - 6$ (take the individual logs)

= 3 (arithmetic)

= $\ln e^3$ (write 3 as an ln)

So, $\ln e^9 - \ln e^6 = \ln e^3$

Homework

6. Express each difference of logs as a single log:

a. $\log 100{,}000 - \log 1000$

b. $\log 1000 - \log 10$

c. $\log_5 125 - \log_5 25$

d. $\ln e^{10} - \ln e^7$

e. $\log_6 36 - \log_6 6$

f. $\ln e^{20} - \ln e^5$

g. $\log 1000 - \log 100$

h. $\log_8 512 - \log_8 64$

i. $\log_2 32 - \log_2 8$

j. $\log_{12} 144 - \log_{12} 1$

❑ *SYNOPSIS OF THE TWO PRECEDING SECTIONS*

Let's look at two of the results from the preceding sections:

1) $\log 100 + \log 1000 = \log 100{,}000$

This shows that the sum of two logs can be combined into a single log — as long as the numbers are multiplied:
$100 \times 1000 = 100{,}000$

2) $\log_2 32 - \log_2 8 = \log_2 4$

This shows that the difference of two logs can be combined into a single log — as long as the numbers are divided:
$\frac{32}{8} = 4$

These two results, that the sum of logs is the log of the product, and that the difference of logs is the log of the quotient, can be written like this:

$$\log_b x + \log_b y = \log_b(xy)$$

$$\log_b x - \log_b y = \log_b \frac{x}{y}$$

Traditionally, these two laws of logs are written the other way around; indeed, that's the way we need one of them in the next section.

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

The log of a product is the sum of the logs.

The log of a quotient is the difference of the logs.

EXAMPLE 2: **Use the two Laws of Logs to expand each expression:**

A. $\log(ABC) = \log A + \log B + \log C$

B. $\ln\left(\frac{xy}{z}\right) = \ln(xy) - \ln z = \ln x + \ln y - \ln z$

C. $\log_2\left(\frac{a}{bc}\right) = \log_2 a - \log_2(bc) = \log_2 a - (\log_2 b + \log_2 c)$
$= \log_2 a - \log_2 b - \log_2 c$

❑ THE LOG OF A POWER

Another type of log expression we'll come across is the log of a power, e.g., $\log(7^3)$. We can break it down and use the notion of exponent and the "log of a product" law above to do the problem. Let's work $\log 7^3$:

$\log 7^3$

$= \log(7 \cdot 7 \cdot 7)$ (definition of exponent)

$= \log 7 + \log 7 + \log 7$ (use the "log of a product" law)

$= \quad \mathbf{3\log 7}$ (combine like terms)

This may not seem very profound, but this maneuver will be necessary to solve exponential equations later in the course. Let's do one more example, and we'll even put a variable into the expression.

$$\ln x^4 = \ln(x \cdot x \cdot x \cdot x) = \ln x + \ln x + \ln x + \ln x = \mathbf{4\ln x}$$

This "log of a power" law can be summarized as

$$\log_b x^n = n \log_b x$$

To calculate the log of a power, bring down the power as a coefficient.

❑ *SUMMARY OF LOGS*

We can summarize our knowledge of **logs** in the following six statements — one definition, two cancelation rules, and three laws:

Statement	Example
$y = \log_b x$ means $b^y = x$	$2 = \log 100$ means $10^2 = 100$
$\log_b b^x = x$	$\ln e^T = T$
$b^{\log_b x} = x$	$2^{\log_2 99} = 99$
$\log_b(xy) = \log_b x + \log_b y$	$\log(100x) = 2 + \log x$
$\log_b \frac{x}{y} = \log_b x - \log_b y$	$\ln \frac{x}{e} = \ln x - 1$
$\log_b x^n = n \log_b x$	$\log_5 \sqrt[7]{Q} = \frac{1}{7} \log_5 Q$

EXAMPLE 3: **Use one of the Laws of Logs to expand each expression:**

A. $\log(7x) = \log 7 + \log x$

B. $\ln(xyz) = \ln x + \ln y + \ln z$

C. $\ln(ex) = \ln e + \ln x = 1 + \ln x$

D. $\log_2 \frac{8}{3} = \log_2 8 - \log_2 3 = 3 - \log_2 3$

E. $\log \frac{A}{B} = \log A - \log B$

F. $\log_{12}[(a+b)(a-b)] = \log_{12}(a+b) + \log_{12}(a-b)$

G. $\log(x^3) = 3\log x$

H. $\log_3 \sqrt{x} = \log_3 x^{1/2} = \frac{1}{2}\log_3 x$

I. $\ln \sqrt[3]{w^2} = \ln w^{2/3} = \frac{2}{3}\ln w$

J. $(\ln x)^2$ cannot be simplified because the power is on the $\ln x$, not the x. Note that $\ln x^2 = 2\ln x$. See the difference?

EXAMPLE 4: **Use the Laws of Logs to expand each expression:**

A. $\ln(ax^3) = \ln a + \ln x^3 = \ln a + 3\ln x$

B. $\log \frac{xy}{z} = \log(xy) - \log z = \log x + \log y - \log z$

C. $\ln \frac{a}{bc} = \ln a - \ln(bc) = \ln a - [\ln b + \ln c] = \ln a - \ln b - \ln c$

D. $\log_2 \dfrac{cd^3}{e^4} = \log_2 cd^3 - \log_2 e^4 = \log_2 c + 3\log_2 d - 4\log_2 e$

E. $\ln\left(\dfrac{\sqrt{x}\ \sqrt[3]{y}}{e^z\ \sqrt[5]{w^3}}\right) = \ln\sqrt{x}\ \sqrt[3]{y} - \ln e^z\ \sqrt[5]{w^3}$

$= \ln\sqrt{x} + \ln\sqrt[3]{y} - \left[\ln e^z + \ln\sqrt[5]{w^3}\right]$

$= \frac{1}{2}\ln x + \frac{1}{3}\ln y - z - \frac{3}{5}\ln w$

EXAMPLE 5: **Use the Laws of Logs to condense each expression into a single log with coefficient 1:**

A. $\log x + \log y = \log(xy)$

B. $\ln a - \ln b = \ln\dfrac{a}{b}$

C. $3\log_2 T = \log_2 T^3$

D. $\frac{3}{7}\log n = \log n^{3/7} = \log\sqrt[7]{n^3}$

E. $\frac{1}{2}\ln x + 5\ln y = \ln\sqrt{x} + \ln y^5 = \ln\left(\sqrt{x}\, y^5\right)$

F. $2\log x + \frac{1}{2}\log y + \frac{2}{5}\log z = \log x^2 + \log\sqrt{y} + \log\sqrt[5]{z^2}$

$= \log\left(x^2\sqrt{y}\ \sqrt[5]{z^2}\right)$

G. $\ln x + \ln y - \ln z = \ln\dfrac{xy}{z}$

H. $2\log x - \frac{1}{3}\log y = \log x^2 - \log y^{1/3} = \log\dfrac{x^2}{\sqrt[3]{y}}$

I. $\ln a - \ln b - \ln c = \ln\dfrac{a}{b} - \ln c = \ln\dfrac{\frac{a}{b}}{c} = \ln\dfrac{a}{bc}$

Alternate approach:

$\ln a - \ln b - \ln c = \ln a - (\ln b + \ln c) = \ln a - \ln(bc) = \ln\dfrac{a}{bc}$

J. $\frac{2}{3}\log(a+b)-\frac{1}{2}\log(a-b)-7\log(ab)$

$= \log\sqrt[3]{(a+b)^2}-\log\sqrt{a-b}-\log(ab)^7$

$= \log\frac{\sqrt[3]{(a+b)^2}}{\sqrt{a-b}}-\log a^7b^7$

$= \log\frac{\frac{\sqrt[3]{(a+b)^2}}{\sqrt{a-b}}}{a^7b^7}$

$= \log\frac{\sqrt[3]{(a+b)^2}}{a^7b^7\sqrt{a-b}}$



Homework

7. Expand each expression:

a. $\ln(ab)$ b. $\log(wyz)$ c. $\log_2(abcd)$

d. $\log\frac{h}{k}$ e. $\log\frac{a+b}{c}$ f. $\log_8\frac{x-y}{u+v}$

g. $\ln x^3$ h. $\log 7^n$ i. $\log_5 5^7$

8. Expand each expression:

a. $\log_7\sqrt{x}$ b. $\log\sqrt[3]{x}$ c. $\ln\sqrt[5]{Q^2}$

d. $\log\frac{1}{x^3}$ e. $\ln\frac{1}{u^{3/2}}$ f. $\log\frac{1}{T^{-4/5}}$

9. Expand each expression:

a. $\ln(a^2b)$ b. $\log(xy^3)$ c. $\ln(a^2b^3c^4)$

d. $\log_6 \frac{x^2}{y^3}$ e. $\ln \frac{ab}{c^9}$ f. $\log \frac{x^2y^5}{z^{10}}$

10. Expand each expression:

a. $\log_3 \frac{a}{bc}$ b. $\ln \frac{x^2y}{wz^3}$ c. $\log \frac{\sqrt{x}}{yz}$

d. $\log(a^2\, b\sqrt[7]{c})$ e. $\ln\left[\frac{wx}{yz}\right]^5$ f. $\log_5 \frac{a^2\sqrt[3]{y}}{wx}$

11. Condense each expression:

a. $\ln x - \ln 7$ b. $\log y + \log 12$

c. $\frac{1}{3}\ln 4 + \ln 2$ d. $\frac{2}{5}\log t - \frac{1}{5}\log t$

e. $\ln(x^2) + \ln x + \ln 7$ f. $\log x - \log y + \log z$

g. $\ln a - \ln b - 3\ln c$ h. $2\log a - \frac{1}{2}\log b - \frac{2}{3}\log c$

12. Evaluate each expression <u>without a calculator</u>:

a. $\log 10^{100}$ b. $e^{\ln 7}$ c. $\log 5 + \log 2$

d. $\ln \frac{e^{10}}{e^2}$ e. $\log_{24} 8 + \log_{24} 3$ f. $\ln x + \ln \frac{1}{x}$

g. $\log 50 + \log 20$ h. $\log_3 \frac{1}{2} + \log_3 162$ i. $\log_5 50 - \log_5 2$

13. True or False:

a. $\log 10^9 = 9$ b. $7^{\log_7 R} = 7^R$

c. $\ln e^{abc} = \ln(abc)$ d. $\ln(xy) = \ln x + \ln y$

e. $\log a^b = (\log a)^b$ f. $\ln \frac{a}{b} = \frac{\ln a}{\ln b}$

g. $\ln(x+y) = \ln x + \ln y$ h. $\log(a-b) = \log a - \log b$

i. $\ln \frac{1}{x} = -\ln x$ j. $\log_{12}(x^a b) = (a\log_{12} x)(\log_{12} b)$

k. $\ln(xy^z) = z\ln(xy)$

l. $\log_3 x - \log_3 y = \log_3 \frac{x}{y}$

m. $\log a + \log b = \log(ab)$

n. $\ln e^t = t$

o. $10^{\log 9} = 9$

p. $\ln \frac{1}{x} = \frac{1}{\ln x}$

q. $e^{\ln(e-3)} = e-3$

r. $\ln \frac{x}{y} = \ln x - \ln y$

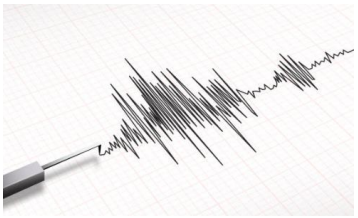
s. $\log(ab^n) = \log a + n\log b$

t. $\ln \frac{a^2}{b^5} = \frac{2\ln a}{5\ln b}$

❑ *THE RICHTER SCALE FOR EARTHQUAKES*

In 1935, Charles Richter devised a scale for earthquakes called the Richter scale (what a coincidence!). If E represents the energy (in joules) of the earthquake, then the Richter magnitude M is given by

$$M = \frac{2}{3}\log\left(\frac{E}{2.5\times10^4}\right)$$



EXAMPLE 6: **The energy release of the 1906 San Francisco earthquake was 5.96×10^{16} joules. Find the Richter magnitude of the quake.**

Solution: According to Richter's formula,

$$\begin{aligned} M &= \frac{2}{3}\log\left(\frac{5.96\times10^{16}}{2.5\times10^4}\right) \\ &= \frac{2}{3}\log\left(2.384\times10^{12}\right) \\ &= \frac{2}{3}(12.3773) \\ &= \boxed{8.25} \end{aligned}$$

Homework

14. An earthquake releases 1.75×10^{11} joules of energy. What is the Richter magnitude of the quake?

15. A more serious quake releases 100 times as much energy as the one in the previous problem. Find the Richter magnitude.

❑ *PROOFS OF THE THREE LAWS OF LOGS*

The laws of logs we've developed and used in this chapter were arrived at using a few examples and noting a consistent pattern. We know that all the examples in the world never constitute a genuine proof, so now it's time to set the record straight and prove the laws the right way. There's no homework for this section, so ask your instructor what you're responsible for on the test.

THEOREM: **The First Law of Logs**

$$\log_b(xy) = \log_b x + \log_b y$$

PROOF: Let $w = \log_b x$ and $z = \log_b y$. Then

$x = b^w$ and $y = b^z$ (from the definition of log).

To prove the First Law of Logs, we'll begin with the left-hand side of the equation, substitute the results just obtained, and work our way to the right-hand side of the equation.

	$\log_b(xy)$	(the left side of the formula)
=	$\log_b(b^w \cdot b^z)$	(substituting from the results above)
=	$\log_b(b^{w+z})$	(the first law of exponents)
=	$w + z$	(the first log cancelation rule)
=	$\log_b x + \log_b y$	(substituting back) **Q.E.D.**

"The log of a product is the sum of the logs."

THEOREM: **The Second Law of Logs**

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

PROOF: Let $w = \log_b x$ and $z = \log_b y$. Then $x = b^w$ and $y = b^z$. We proceed like the previous proof:

	$\log_b\left(\frac{x}{y}\right)$	(the left side of the formula)
=	$\log_b\left(\frac{b^w}{b^z}\right)$	(substituting from the results above)
=	$\log_b(b^{w-z})$	(a law of exponents)
=	$w - z$	(the first log canceling rule)
=	$\log_b x - \log_b y$	(substituting back) **Q.E.D.**

"The log of a quotient is the difference of the logs."

The Third Law of Logs is used whenever there's an exponent on the quantity we're taking the log of.

THEOREM: **The Third Law of Logs**

$$\log_b x^n = n\log_b x$$

PROOF: As in the two previous proofs, let $w = \log_b x$, which implies that $x = b^w$. Thus,

	$\log_b x^n$	(the left side of the formula)

=	$\log_b (b^w)^n$	(substituting the expression for x)	
=	$\log_b (b^{wn})$	(one of the laws of exponents)	
=	wn	(the first log canceling rule)	
=	nw	(commutative property for multiplication)	
=	$n \log_b x$	(substituting back)	**Q.E.D.**

"To calculate the log of a power, bring down the power as a coefficient."

Review Problems

16. $\ln e + \log 1 - \log_3 3 - \log_7 49 =$

17. a. $\log_{12} 12^N =$ b. $9^{\log_9 56} =$
    c. $\ln e^{a-b} =$ d. $\log 10^{25} =$
    e. $\log_3 5^y =$ f. $5^{\log_5 (-5)} =$

18. $\log_b b^x \quad b^{\log_b x} =$

19. $\ln(\log 10) =$

20. Expand: $\log\left(\frac{ab}{10}\right)$

21. Expand: $\ln\left(\frac{x^3}{y\sqrt{z}}\right)$

$y = \log_b x$ means $b^y = x$
$\log_b b^x = x$
$b^{\log_b x} = x$
$\log_b (xy) = \log_b x + \log_b y$
$\log_b \frac{x}{y} = \log_b x - \log_b y$
$\log_b x^n = n \log_b x$

22. Condense: $\ln x + \ln y - \ln z$

23. Condense: $3\log x + \frac{2}{3}\log y - \frac{1}{2}\log z$

24. True/False:

a. $\ln(a-b) = \ln a - \ln b$

b. $\log e^t = t$

c. $\ln\left(\frac{2}{x}\right) = \frac{\ln 2}{\ln x}$

d. $\ln(ab^{10}) = 10\ln(ab)$

25. The energy release of an earthquake is 1.23×10^{17} joules. Use the formula $M = \frac{2}{3}\log\left(\frac{E}{2.5\times10^4}\right)$ to find the Richter magnitude of the quake.

26. True/False:

a. $\log 10^{a+b} = a+b$

b. $\log e^{u-w} = u-w$

c. $6^{\log_6 z} = z$

d. $e^{\log Q} = Q$

e. $\ln(ab) = \ln a + \ln b$

f. $\ln\frac{x}{y} = \frac{\ln x}{\ln y}$

g. $\log ab^y = y\log ab$

h. $\log ab^y = \log a + y\log b$

i. $\ln p - \ln q = \ln\frac{p}{q}$

j. $\log_3 32 - \log_3 8 = \log_3 4$

k. $\ln(x+y+z) = \ln x + \ln y + \ln z$

l. $\log 7^7 = 7\log 7$

m. $(\ln x)^3 = 3\ln x$

n. $\frac{2}{3}\log x + 4\log y = \log\left(\sqrt[3]{x^2}\, y^4\right)$

o. If the energy release of an earthquake is 2.807×10^{15} joules, then the Richter number of the earthquake is about 7.37.

❑ *TO ∞ AND BEYOND*

Prove: $\ln(x^8 + 8x^6 + 24x^4 + 32x^2 + 16) = 4\ln(x^2 + 2)$

Solutions

1. Because the first operation in $b^{\log_b x}$ is $\log_b x$, whose domain is (0, ∞).

2. a. 1 b. 1 c. $u + v$ d. xyz e. 500 f. xy g. $\log 7$ h. 1
i. $a - b$ j. R k. As is l. As is

3. $\log_2(8+8) = \log_2 16 = \mathbf{4}$, whereas
$\log_2 8 + \log_2 8 = 3 + 3 = \mathbf{6}$

4. $\log(xy) = \log(100 \cdot 1{,}000) = \log 100{,}000 = \mathbf{5}$, but
$(\log x)(\log y) = (\log 100)(\log 1{,}000) = 2 \cdot 3 = \mathbf{6}$.

5. a. $\log 10{,}000$ b. $\log 1{,}000{,}000$ c. $\log 10$
d. 1,000,000 e. $\log_2 512$ f. $\log_3 27$
g. $\log_4 256$ h. $\log 1{,}000{,}000$ i. $\ln e^{12}$
j. $\ln e^{20}$

6. a. $\log 100$ b. $\log 100$ c. $\log_5 5$
d. $\ln e^3$ e. $\log_6 6$ f. $\ln e^{15}$
g. $\log 10$ h. $\log_8 8$ i. $\log_2 4$
j. $\log_{12} 144$

7. a. $\ln a + \ln b$ b. $\log w + \log y + \log z$
c. $\log_2 a + \log_2 b + \log_2 c + \log_2 d$ d. $\log h - \log k$
e. $\log(a+b) - \log c$ f. $\log_8(x-y) - \log_8(u+v)$
g. $3\ln x$ h. $n \log 7$ i. 7

8. a. $\frac{1}{2}\log_7 x$ b. $\frac{1}{3}\log x$ c. $\frac{2}{5}\ln Q$
d. $-3\log x$ e. $-\frac{3}{2}\ln u$ f. $\frac{4}{5}\log T$

9. a. $2\ln a + \ln b$ b. $\log x + 3\log y$
c. $2\ln a + 3\ln b + 4\ln c$ d. $2\log_6 x - 3\log_6 y$
e. $\ln a + \ln b - 9\ln c$ f. $2\log x + 5\log y - 10\log z$

10. a. $\log_3 a - \log_3 b - \log_3 c$ b. $2\ln x + \ln y - \ln w - 3\ln z$
c. $\frac{1}{2}\log x - \log y - \log z$ d. $2\log a + \log b + \frac{1}{7}\log c$
e. $5\ln w + 5\ln x - 5\ln y - 5\ln z$ f. $2\log_5 a + \frac{1}{3}\log_5 y - \log_5 w - \log_5 x$

11. a. $\ln \frac{x}{7}$ b. $\log(12y)$ c. $\ln(2\sqrt[3]{4})$ d. $\log \sqrt[5]{t}$
e. $\ln(7x^3)$ f. $\log \frac{xz}{y}$ g. $\ln \frac{a}{bc^3}$ h. $\log \frac{a^2}{\sqrt{b}\sqrt[3]{c^2}}$

12. a. 100 b. 7 c. 1 d. 8 e. 1
f. 0 g. 3 h. 4 i. 2

13. a. T b. F c. F d. T e. F f. F g. F h. F i. T
j. F k. F l. T m. T n. T o. T p. F q. F (tricky!)
r. T s. T t. F

14. $M = 4.6$

15. $M = 5.9$. Notice that this magnitude is only 1.3 Richter points higher than the previous answer, yet the earthquake was 100 times more powerful. Even a small difference in the Richter magnitude represents a huge difference in the actual power of the earthquake.

16. -2

17. a. N b. 56 c. $a - b$ d. 25 e. As is f. Undefined

18. x^2 **19**. 0 **20**. $\log a + \log b - 1$

21. $3\ln x - \ln y - \frac{1}{2}\ln z$ **22**. $\ln\left(\frac{xy}{z}\right)$

23. $\log\left[\frac{x^3\sqrt[3]{y^2}}{\sqrt{z}}\right]$ **24**. They're all false. **25**. 8.46

26. a. T b. F c. T d. F e. T f. F g. F h. T
i. T j. T k. F l. T m. F n. T o. T

"Nothing can stop the man with the right mental attitude from achieving his goal; nothing on earth can help the man with the wrong mental attitude."

Thomas Jefferson



LOGARITHMS

❑ *MELANIE'S ALLOWANCE*



Melanie tells her father that she will pay the full cost (room and board) of her entire college education *if* he agrees to the following plan: He gives her 2¢ on the first day of the month, 4¢ on the second day of the month, 8¢ on the third day of the month, 16¢ on the fourth day of the month, and so on till the 30th of the month. After that month – no more money. Dad (who was a philosophy major) thinks this is a great money-saving idea and accepts Melanie's proposal.

Day	# Pennies
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1,024
⋮	⋮
30	1,073,741,824 (almost $11 million)

In the chart, we have calculated Melanie's earnings for each of the first 10 days of the month. Take your calculator and verify the number of pennies Melanie will receive on the 30^{th} day.

Now let's come up with a direct formula that computes the money earned given the day of the month, <u>without knowing all the previous days' amounts</u>. Notice that each amount of money is simply 2 raised to the power of the day. For example, consider the 9th day. If we raise 2 to the 9th power (use the exponent button on your calculator), we get 512. That is, $2^9 = 512$.

Now, for the 30^{th} day, calculate 2^{30}, and you should get **1,073,741,824** (which, remember, is the number of <u>pennies</u> Melanie gets on day 30);

you can divide this number of pennies by 100, and then round off to see that it's equivalent to about 10.7 million dollars!

Homework

1. a. How many pennies will Melanie earn on the 15th day of the month?

   b. How many pennies will Melanie earn on the 31st of the month if Dad agrees to extend the plan that far?

   c. On which day of the month did Melanie earn <u>half</u> of what she earned on the 30th day?

2. Now for a little practice on the concept to be covered in this chapter. I'll give you a penny amount, and you tell me which day of the month Melanie earned that amount of money.

   a. 512¢ b. 4,096¢ c. 1,048,576¢ d. 33,554,432¢

3. Similar question, but a little more abstract: I'll give you a number, and you tell me the *exponent* that **<u>2</u>** would have to be raised to, in order to get the number I gave you. For example, if I give you the number 2,048, then you say "11" because $2^{\boxed{11}} = 2{,}048$.

   a. 2 b. 256 c. 64 d. 1
   e. 8,192 f. 131,072 g. 524,288 h. 1/2

4. Another question like #3, but now when I give you the number, you tell me the *exponent* that **<u>10</u>** would have to be raised to, in order to get the number I gave you. For example, if I give you the number 1,000, then you say "3" because $10^{\boxed{3}} = 1{,}000$.

   a. 100 b. 10,000 c. 10 d. 1
   e. 100,000 f. 1,000,000 g. 1 billion h. 1 googol

❑ THE MEANING OF A LOG

A ***log*** (short for ***logarithm***) is an exponent. It's the exponent to which one number (called the base) must be raised to get a specified number. This definition is so far off in the clouds that we need an example right now.

> ***logarithm***
> from the Greek:
> *logos* = logic
> *arithmos* = number

For our first example, to calculate

$\log_{10}(1000)$ [read: "log, base 10, of 1000" or "log of 1000, base 10]

we ask ourselves, "10 raised to *what power* equals 1000?" In other words, 10 to the "what" equals 1000? The answer is 3, since $10^3 = 1000$. Therefore,

$\log_{10}(1000) = \mathbf{3}$ [log, base 10, of 1000 is 3.]

For a second example, let's analyze $\log_2 32$, which asks us, "2 raised to the "what" equals 32?" Well, 2 to the 5th power equals 32, and so

$\log_2(32) = \mathbf{5}$ [log, base 2, of 32 is 5.]

Our third example will describe a log a little differently: If you can fill in the box in the equation

$4^{\square} = 16$

then you have found the "log, base 4, of 16," which is 2. That is,

$\log_4(16) = \mathbf{2}$

Notation: A log is a function, so notation like $\log_2(128)$ certainly makes sense, just like when we use parentheses in function notation:

$f(x)$. But if it's clear what we're taking the log of, we don't really need the parentheses; so, for example, $\log_2(128)$ is simply written $\log_2 128$. [Although in computer programming, the parentheses are required.]

Summary: $\log_{10} 1000 = \mathbf{3}$ because $10^{\mathbf{3}} = 1000$

$\log_2 32 = \mathbf{5}$ because $2^{\mathbf{5}} = 32$

$\log_4 16 = \mathbf{2}$ because $4^{\mathbf{2}} = 16$

This is really abstract, isn't it? Let's get right to some homework.

Homework

5. To find $\log_5 25$, which is read "log, base 5, of 25," ask yourself, "5 raised to what power equals 25?" $5^{\square} = 25$

6. To find $\log_2 8$, which is read "log, base 2, of 8," ask yourself, "2 raised to what power equals 8?" $2^{\square} = 8$

7. To find $\log_9 9$, which is read "log, base 9, of 9," ask yourself, "9 raised to what power equals 9?" $9^{\square} = 9$

8. To find $\log_{17} 1$, which is read "log, base 17, of 1," ask yourself, "17 raised to what power equals 1?" $17^{\square} = 1$

9. To find $\log_{100} 10$, which is read "log, base 100, of 10," ask yourself, "100 raised to what power equals 10?" $100^{\square} = 10$

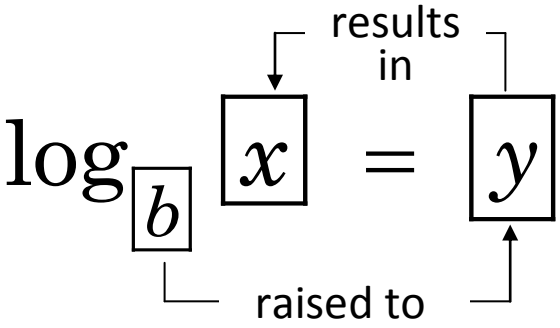
10. To find $\log_6\left(\frac{1}{6}\right)$, which is read "log, base 6, of 1/6," ask yourself, "6 raised to what power equals $\frac{1}{6}$?" $6^{\square} = \frac{1}{6}$

❑ *THE OFFICIAL DEFINITION OF LOGARITHM*

$$\log_b x = y \quad \text{means} \quad b^y = x$$

The notation "$\log_b x$" is read either
"log, base *b*, of *x*" or "log of *x*, base *b*"

Here's another way to visualize the meaning of logarithm:



EXAMPLE 1:

A. $\log_{10} 10{,}000 = 4$ Why? Because $10^4 = 10{,}000$

B. $\log_e e^2 = 2$ because $e^2 = e^2$

C. $\log_{17} 17 = \mathbf{1}$ because $17^1 = 17$

D. $\log_e 1 = \mathbf{0}$ because $e^0 = 1$

E. $\log_{25} 5 = \mathbf{\frac{1}{2}}$ because $25^{1/2} = \sqrt{25} = 5$

F. $\log_{64} 4 = \mathbf{\frac{1}{3}}$ because $64^{1/3} = \sqrt[3]{64} = 4$

G. $\log_8 4 = \mathbf{\frac{2}{3}}$ because $8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

H. $\log_{13}\left(\frac{1}{13}\right) = \mathbf{-1}$ because $13^{-1} = \frac{1}{13}$

I. $\log_6\left(\frac{1}{36}\right) = \mathbf{-2}$ because $6^{-2} = \frac{1}{6^2} = \frac{1}{36}$

J. $\log_{49}\left(\frac{1}{7}\right) = \mathbf{-\frac{1}{2}}$ because $49^{-1/2} = \frac{1}{49^{1/2}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$

K. $\log_{1/2}\left(\frac{1}{8}\right) = \mathbf{3}$ because $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

EXAMPLE 2:

A. $\log_b b = \mathbf{1}$ since $b^1 = b$

B. $\log_b 1 = \mathbf{0}$ since $b^0 = 1$

C. $\log_b \frac{1}{b} = -1$ since $b^{-1} = \frac{1}{b}$

D. $\log_b \sqrt{b} = \frac{1}{2}$ since $b^{1/2} = \sqrt{b}$

E. $\log_b \sqrt[n]{b} = \frac{1}{n}$ since $b^{1/n} = \sqrt[n]{b}$

F. $\log_b b^n = \boldsymbol{n}$ since $b^n = b^n$

Homework

Find the value of each log:

11.	a. $\log_{10} 100$	b. $\log_5 125$	c. $\log_8 64$	d. $\log_2 64$
12.	a. $\log_e e^5$	b. $\log_b b^2$	c. $\log_{\sqrt{2}} \sqrt{2}$	d. $\log_L L$
13.	a. $\log_{10} 1$	b. $\log_e 1$	c. $\log_{\sqrt[5]{99}} 1$	d. $\log_b 1$
14.	a. $\log_{36} 6$	b. $\log_{49} 7$	c. $\log_{144} 12$	d. $\log_b \sqrt{b}$
15.	a. $\log_5 \left(\frac{1}{5}\right)$	b. $\log_e \left(\frac{1}{e}\right)$	c. $\log_{1/e} 1$	d. $\log_n \left(\frac{1}{n}\right)$
16.	a. $\log_Q Q^n$	b. $\log_x 1$	c. $\log_{2.3} 2.3$	d. $\log_9 81$
17.	a. $\log_8 2$	b. $\log_{64} 4$	c. $\log_{125} 5$	d. $\log_a \sqrt[3]{a}$

❑ *CALCULATING LOGS*

The homework problems above were designed to be solved by inspection (with a little experimentation and insight). Some logs aren't easy to do that way. So now we present a longer, but more systematic, method of evaluating a log by solving a certain exponential equation.

EXAMPLE 3: **Calculate: $\log_{27} 9$**

Solution: Let's give our log expression a name — call it y. Now we can write an equation:

$$\log_{27} 9 = y$$

The definition of log shows us how we can convert our log equation into an exponential equation:

$$27^y = 9$$

And now we solve for y. The chapter *Exponential Equations (No Logs)* showed us how:

$$27^y = 9 \Rightarrow (3^3)^y = 3^2 \Rightarrow 3^{3y} = 3^2 \Rightarrow 3y = 2 \Rightarrow y = \frac{2}{3}$$

But y was the name we gave to the original log problem. So we can conclude that

$$\boxed{\log_{27} 9 = \frac{2}{3}}$$

To **check** our result, we can raise 27 to the $\frac{2}{3}$ power and make sure it comes out 9:

$$27^{2/3} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9 \quad \checkmark$$

Homework

Find the value of each log:

18. $\log_{64} 16$
19. $\log_{25}\left(\frac{1}{5}\right)$
20. $\log_{16} 8$
21. $\log_{27}\left(\frac{1}{9}\right)$
22. $\log_{8} 16$
23. $\log_{32}\left(\frac{1}{2}\right)$
24. $\log_{1/3}\left(\frac{1}{9}\right)$
25. $\log_{1/4} 64$
26. $\log_{9} 1$
27. $\log_{\pi} \pi^5$
28. $\log_{\pi} \sqrt{\pi}$
29. $\log_{4}\left(\frac{1}{8}\right)$

30. $\log_5\left(\frac{1}{25}\right)$ 31. $\log_3\left(\frac{1}{\sqrt{3}}\right)$ 32. $\log_4\left(\frac{1}{64}\right)$

33. $\log_2\left(\frac{1}{128}\right)$ 34. $\log_{10}\sqrt{10}$ 35. $\log_{10}\sqrt[3]{100}$

36. $\log_{10}\left(\frac{1}{100}\right)$ 37. $\log_{10}\left(\frac{1}{\sqrt{10}}\right)$ 38. $\log_{10}\left(\frac{1}{\sqrt{1000}}\right)$

❑ *THE pH SCALE FOR ACIDS AND BASES*



One of the uses of logs is defining the **pH scale** for acids and bases. The official definition of the pH of a substance is the negative **logarithm** (base 10) of the hydrogen ion concentration of the substance. Acids (like lemonade) have a pH smaller than 7, while bases (like Drano) have a pH higher than 7. The pH of pure water is a neutral 7. [The word *alkali* is another term for base.]

We'll use the official chemistry notation for the hydrogen ion concentration, $[H^+]$, which has the units of moles/liter. It is not necessary to understand any of the chemistry — indeed, the math takes care of everything. Devised by a biochemist while working on the brewing of beer, the **pH** of a substance is defined to be the negative logarithm (base 10) of the hydrogen ion concentration:

$$\text{pH} = -\log_{10}[H^+]$$

Logs are also used in the definition of the *Richter Scale* for earthquakes, and for the *decibel scale* for measuring the loudness of sound.

New Notation and Calculator Hints:

1. On a TI-30, to enter a number like 1.6×10^{-13}, first press 1.6, then press the "EE" button, and then press 13, and last the +/– key. Your display should then look something like $1.6^{\,-13}$ (the base of 10 is understood).

2. To find $\log_{10} 1000$, enter 1000 into your calculator and then press the ***log*** button. You should, of course, get an answer of 3. On newer calculators, try pressing the *log* button first, followed by 1000. Using 10 as a base for logs is so "common" that it is officially referred to as the **common log**, and we dispense with writing the base, 10 — it's "understood":

$$\log_{10} x \text{ is written } \log x$$

3. [Skip the following if you've never come across the number ***e***.] A base of ***e*** is so important and occurs so "naturally" in the universe that "log, base ***e***" also gets its own name and notation:

$$\log_e x \text{ is written } \ln x$$

You read "ln *x*" either as "el en *x*" or "el en of *x*" or "the **natural log** of *x*." Teachers will many times write it on the whiteboard in cursive:

$\mathcal{ln}\ x$ (*l* for **l**og, *n* for **n**atural)

EXAMPLE 4: **A sample of orange juice has a hydrogen ion concentration of 2.9×10^{-4} moles/liter. Find the pH of the orange juice.**

Solution: According to the definition,

$$\text{pH} = -\log_{10}[\text{H}^+] = -\log(2.9\times10^{-4}) \approx -(-3.54) = 3.54$$

According to the text next to the lemonade stand above, this should mean that orange juice is an acid, as indeed it is (the sour taste is one clue). Thus, the pH of the orange juice is

3.54

Homework

39. The hydrogen ion concentration of household ammonia is 1.26×10^{-12} moles/liter. Find the pH of the ammonia. Is it an acid or a base?

40. Pure water has a hydrogen ion concentration of 1.0×10^{-7} moles/liter. Prove that water has a neutral pH of 7.

41. Find the pH of each substance given its molarity:

    a. 1.3×10^{-2} moles/L
    b. 2.8×10^{-6} moles/L
    c. 0.3×10^{-10} moles/L
    d. 9.2×10^{-12} moles/L
    e. 5.9×10^{-7} moles/L
    f. 8.0×10^{-1} moles/L

Review Problems

42. a. $\log_{10} 1{,}000{,}000 =$
    b. $\log_3 3^7 =$
    c. $\log_e e =$
    d. $\log_7 1 =$
    e. $\log_{16} 4 =$
    f. $\log_9\left(\frac{1}{9}\right) =$
    g. $\log_5\left(\frac{1}{25}\right) =$
    h. $\log_{36}\left(\frac{1}{6}\right) =$
    i. $\log_2 512 =$

43. a. $\log_a a$ = b. $\log_c 1$ = c. $\log_7 7^N$ =

44. The hydrogen ion concentration of an unknown liquid is 3.4×10^{-11} moles/L. Find the pH of the liquid.

45. What kind of music do they play in a lumber camp?

Solutions

1. a. 32,768 b. 2,147,483,648 c. I'm not gonna tell you that.

2. a. day 9 b. day 12 c. day 20 d. day 25

3. a. 1 b. 8 c. 6 d. 0 e. 13 f. 17 g. 19 h. –1

4. a. 2 b. 4 c. 1 d. 0 e. 5 f. 6 g. 9 h. 100

5. $5^{\boxed{?}} = 25$; since $5^2 = 25, \log_5 25 = 2$.

6. $2^{\boxed{?}} = 8$; since $2^3 = 8, \log_2 8 = 3$.

7. 1 **8**. 0

9. $100^{\boxed{?}} = 10$; since $100^{1/2} = \sqrt{100} = 10, \log_{100} 10 = \frac{1}{2}$.

10. –1

11. a. 2 b. 3 c. 2 d. 6

12. a. 5 b. 2 c. 1 d. 1

13. a. 0 b. 0 c. 0 d. 0

14. a. $\frac{1}{2}$ b. $\frac{1}{2}$ c. $\frac{1}{2}$ d. $\frac{1}{2}$

15. a. –1 b. –1 c. 0 d. –1 **16**. a. n b. 0 c. 1 d. 2

17. a. $\frac{1}{3}$ b. $\frac{1}{3}$ c. $\frac{1}{3}$ d. $\frac{1}{3}$

18. Let $y = \log_{64} 16 \Rightarrow 64^y = 16 \Rightarrow (4^3)^y = 4^2$

$\Rightarrow 4^{3y} = 4^2 \Rightarrow 3y = 2 \Rightarrow y = \frac{2}{3}$

19. $-\frac{1}{2}$ **20**. $\frac{3}{4}$ **21**. $-\frac{2}{3}$ **22**. $\frac{4}{3}$ **23**. $-\frac{1}{5}$ **24**. 2

25. –3 **26**. 0 **27**. 5 **28**. $\frac{1}{2}$ **29**. $-\frac{3}{2}$

30. –2 **31**. $-\frac{1}{2}$ **32**. –3 **33**. –7 **34**. $\frac{1}{2}$

35. $\frac{2}{3}$ **36**. –2 **37**. $-\frac{1}{2}$ **38**. $-\frac{3}{2}$

39. pH = 11.9; it's a base (an alkali), since its pH is greater than 7.

40. $\text{pH} = -\log[\text{H}^+] = -\log(1.0\times10^{-7}) = -(-7) = 7$

41. a. 1.89 b. 5.55 c. 10.52 d. 11.04 e. 6.23 f. 0.10

42. a. 6 b. 7 c. 1 d. 0 e. $\frac{1}{2}$ f. –1 g. –2 h. $-\frac{1}{2}$ i. 9

43. a. 1 b. 0 c. N

44. 10.47 (an alkali, or base)

45. I just can't give the answer away.

"Learning is not the product of teaching.

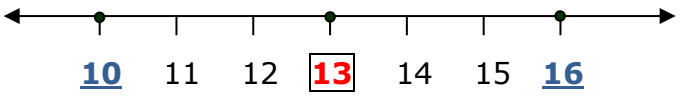
Learning is the product of the activity of learners."

– John Holt, American psychologist and educator

MIDPOINT ON THE LINE AND IN THE PLANE

❑ *MIDPOINT ON THE LINE*

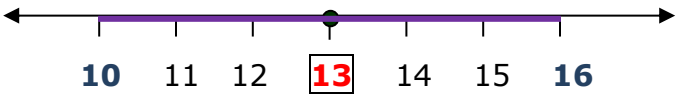
Here's a question for you: What number is *midway* between 10 and 16? You probably know that the number is 13. Why? Because 13 is 3 units away from 10, and 13 is also 3 units away from 16.



Now we need a simple way to find the number that is midway between any two numbers, even when the numbers are not nice, or worse yet, when there are variables involved. Notice: If we take the **average** (officially called the *arithmetic mean*) of 10 and 16 — by adding the two numbers and dividing by 2 — we get

$$\frac{10+16}{2} = \frac{26}{2} = \mathbf{13}, \text{ the midway number}$$

Let's rephrase what we've done with some new terminology. Consider the *line segment* connecting 10 and 16 on the number line:



We can now refer to the 13 as the ***midpoint*** of the line segment connecting 10 and 16.

What is the *midpoint* of the line segment connecting –2.8 and 14.6? Just calculate the average of –2.8 and 14.6:

$$\frac{-2.8+14.6}{2} = \frac{11.8}{2} = \mathbf{5.9}$$

When you see the term ***midpoint***, think *average*!

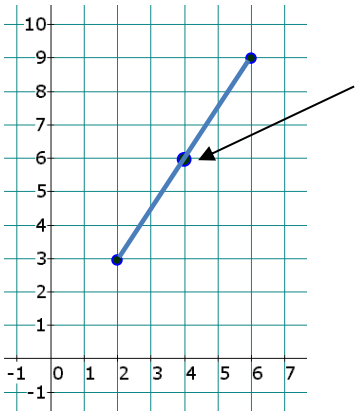
Homework

1. Find the **midpoint** of the line segment connecting the two given numbers (or points) on a number line:

a. 10 and 20	b. 13 and 22	c. –8 and –26
d. –3 and 7	e. –7 and 6	f. π and $-\pi$
g. –21 and –99	h. 0 and 43	i. –50 and 0
j. –44 and 19	k. –41 and 88	l. $3x$ and $-x$

❑ *Midpoint in the Plane*

Now for the more important question: Consider the two points (2, 3) and (6, 9) in the plane and the line segment that connects them. We need to figure out what point is the *midpoint* of the line segment connecting the two points. Recall the advice given above: When you see midpoint, think *average*. So the x-coordinate of the midpoint is the average of the x-coordinates of the two endpoints:



The **midpoint** is found by averaging the x-coordinates and averaging the y-coordinates.

$$x = \frac{2+6}{2} = \frac{8}{2} = \mathbf{4}$$

And the y-coordinate of the midpoint is the average of the y-coordinates of the two endpoints:

$$y = \frac{3+9}{2} = \frac{12}{2} = \mathbf{6}$$

We conclude that the midpoint is **(4, 6)**. That's all there is to it. Now let's do a complete example without plotting any points or drawing any segments.

EXAMPLE 1: **Find the midpoint of the line segment connecting the points (–42, –33) and (90, –10).**

Solution: No graphing needed — we have a formula. The x-coordinate of the midpoint is found by averaging the x-coordinates of the two given points:

$$x = \frac{-42+90}{2} = \frac{48}{2} = 24$$

The y-coordinate of the midpoint is found by averaging the y-coordinates of the two given points:

$$y = \frac{-33+(-10)}{2} = \frac{-43}{2} = -\frac{43}{2}$$

The midpoint is therefore the point

$$\boxed{\left(24, -\frac{43}{2}\right)} \quad \text{or, } (24, -21.5)$$

Homework

2. Find the **midpoint** of the line segment connecting the given pair of points:

a. (–2, 5) and (2, 7)

b. (0, 1) and (0, 6)

c. (–5, 8) and (–5, –8)

d. (–2, 7) and (5, –3)

e. (–9, 2) and (–13, –40)

f. (0, 0) and (–6, –9)

g. (5, 4) and (5, 4)

h. (14, 0) and (0, –9)

i. (8, 8) and (–19, –19)

j. (π, 0) and (–π, 0)

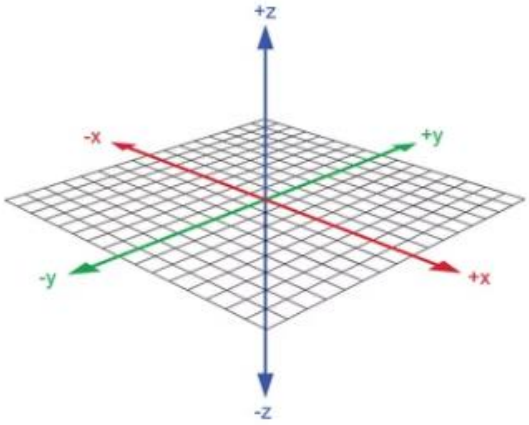
k. $(0, \sqrt{2})$ and $(0, -\sqrt{2})$

l. (a, b) and (c, d)

m. (a, b) and (a, –b)

n. (3a, 3b) and (–3a, b)

❑ *TO ∞ AND BEYOND*



A point on a 1-dimensional line can be described by a single number, for example, 7.

A point in a 2-dimensional plane can be described by an ordered pair, for example, (2, –9).

A point in 3-dimensional space (which has an x-axis, a y-axis, and a ***z*-axis**), can be described by an ordered triple, for example (1, –5, 12).

Find the **midpoint** of the line segment connecting the points (2, –3, 7) and (–5, 17, 20).

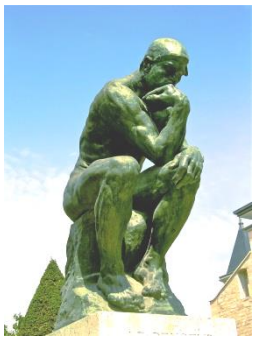
Solutions

1. a. 15 b. 17.5 c. –17 d. 2 e. –0.5 f. 0
g. –60 h. 21.5 i. –25 j. –12.5 k. 23.5 l. x

2. a. (0, 6) b. (0, 7/2) c. (–5, 0)
d. (3/2, 2) e. (–11, –19) f. (–3, –9/2)
g. (5, 4) h. (7, –9/2) i. (–11/2, –11/2)
j. (0, 0) k. (0, 0) l. $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$
m. $(a, 0)$ n. $(0, 2b)$

"What sculpture is to a block of marble, education is to the human soul."

Joseph Addison



MONEY

	Cents	Dollars (Currency Format)	Fraction of a Dollar	Percent of a Dollar
1.	75¢	$0.75	3/4	75%
2.	50¢			
3.			1/4	
4.				100%
5.		$0.10		
6.	20¢			
7.		$0.30		
8.			2/5	
9.				60%
10.	70¢			
11.		$0.80		
12.			9/10	
13.				5%
14.	15¢			
15.		$0.35		
16.			13/20	
17.				95%
18.	22¢			
19.		$0.48		
20.			99/100	
21.			41/50	
22.			1/50	
23.			51/100	
24.			19/50	
25.			13/25	

❑ *SMALLER THAN A PENNY?*

You know what quarters ($0.25), dimes ($0.10), nickels ($0.05), and pennies ($0.01) are. But did you know there's a name for *one-tenth* of a penny? The coin does not physically exist, but one-tenth of a penny is called a ***mil***. Thus, there are 10 mils in a penny, and 1,000 mils in a dollar. One mil can be written in words, in cents notation, and in currency notation,

1 mil = 0.1¢ = $0.001

Notice that the **1** in the currency format (with the dollar sign) is one digit to the right of the pennies place. Here's another example:

7 mils = 0.7¢ = $0.007



1 mil = 1/10 of a penny

You might find a notice on a gas pump that reads, "Price rounded to the nearest **mil**."

Homework

Convert each money amount to *currency format*:

26. $\frac{1}{2}$ dollar
27. $\frac{3}{4}$ dollar
28. $\frac{7}{10}$ dollar
29. $\frac{4}{5}$ dollar
30. $\frac{3}{20}$ dollar
31. $\frac{7}{25}$ dollar
32. $\frac{77}{100}$ dollar
33. $\frac{23}{50}$ dollar
34. $\frac{86}{200}$ dollar
35. 50 cents
36. 25 cents
37. 9 cents
38. 1 cent
39. 37 pennies
40. 16 nickels
41. 7 dimes
42. 3 quarters
43. 145 pennies

44. 44 nickels
45. 137 dimes
46. 37 quarters
47. 0.3 dollar
48. 2.7 dollars
49. 7 dollars
50. $2\frac{3}{4}$ dollars
51. $5\frac{1}{20}$ dollars
52. $7\frac{24}{25}$ dollars
53. $14\frac{3}{5}$ dollars
54. $1\frac{3}{10}$ dollars
55. $7\frac{17}{20}$ dollars
56. 2¢
57. 72¢
58. 7¢
59. 2.3 cents
60. 14.7 cents
61. 33.4¢
62. 3.47 cents
63. 14.09 cents
64. 9.33¢
65. 0.3¢
66. 0.7¢
67. 0.9¢
68. 0.99¢
69. 0.49¢
70. 0.75¢
71. 50¢
72. 0.07¢
73. 0.25¢
74. 5¢
75. 0.5¢
76. 0.05¢
77. 8¢
78. 0.8¢
79. 0.08¢
80. 1¢
81. 0.1¢
82. 0.01¢
83. 3 dollars
84. 0.4 dollar
85. 0.3¢



Solutions

	Cents	Dollars	Fraction of a Dollar	Percent of a Dollar
1.	75¢	$0.75	3/4	75%
2.	50¢	$0.50	1/2	50%
3.	25¢	$0.25	1/4	25%
4.	100¢	$1.00	1	100%
5.	10¢	$0.10	1/10	10%
6.	20¢	$0.20	1/5	20%
7.	30¢	$0.30	3/10	30%
8.	40¢	$0.40	2/5	40%
9.	60¢	$0.60	3/5	60%
10.	70¢	$0.70	7/10	70%
11.	80¢	$0.80	4/5	80%
12.	90¢	$0.90	9/10	90%
13.	5¢	$0.05	1/20	5%
14.	15¢	$0.15	3/20	15%
15.	35¢	$0.35	7/20	35%
16.	65¢	$0.65	13/20	65%
17.	95¢	$0.95	19/20	95%
18.	22¢	$0.22	11/50	22%
19.	48¢	$0.48	12/25	48%
20.	99¢	$0.99	99/100	99%
21.	82¢	$0.82	41/50	82%
22.	2¢	$0.02	1/50	2%
23.	51¢	$0.51	51/100	51%
24.	38¢	$0.38	19/50	38%
25.	52¢	$0.52	13/25	52%

26. $0.50 **27**. $0.75 **28**. $0.70 **29**. $0.80

30. $0.15 **31**. $0.28 **32**. $0.77 **33**. $0.46

34. $0.43 **35**. $0.50 **36**. $0.25 **37**. $0.09

38. $0.01 **39**. $0.37 **40**. $0.80 **41**. $0.70

42. $0.75 **43**. $1.45 **44**. $2.20 **45**. $13.70

46. $9.25 **47**. $0.30 **48**. $2.70 **49**. $7.00

50. $2.75 **51**. $5.05 **52**. $7.96 **53**. $14.60

54. $1.30 **55**. $7.85 **56**. $0.02 **57**. $0.72

58. $0.07 **59**. $0.023 **60**. $0.147 **61**. $0.334

62. $0.0347 **63**. $0.1409 **64**. $0.0933 **65**. $0.003

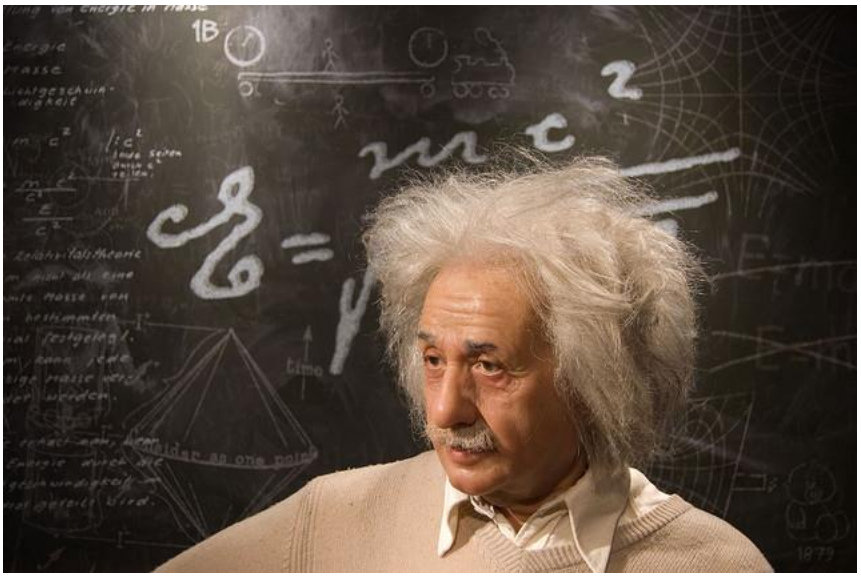
66. $0.007 **67**. $0.009 **68**. $0.0099 **69**. $0.0049

70. $0.0075 **71**. $0.50 **72**. $0.0007 **73**. $0.0025

74. $0.05 **75**. $0.005 **76**. $0.0005 **77**. $0.08

78. $0.008 **79**. $0.0008 **80**. $0.01 **81**. $0.001

82. $0.0001 **83**. $3.00 **84**. $0.40 **85**. $0.003

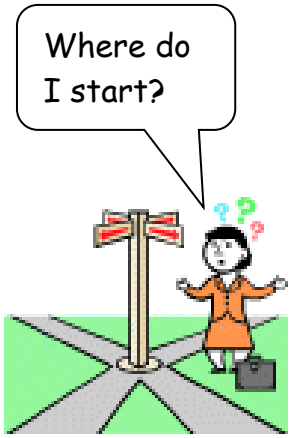


“I think and think for months and years. Ninety-nine times, the conclusion is false. The hundredth time I am right.”

Albert Einstein

THE ORDER OF OPERATIONS

There are a variety of operations we perform in math, including addition, multiplication, and squaring. When we're confronted with an expression that has a variety of these operations, how do we know which operation to perform first? This chapter will show you exactly what steps to take, and the order in which you should take them.



❑ *THE DILEMMA*

Consider the following arithmetic problem:

$$3 + 2 \times 4$$

There are two reasonable methods to work this:

	$\underbrace{3 + 2}_{} \times 4$			$3 + \underbrace{2 \times 4}_{}$	
=	5×4	(add first)	=	$3 + 8$	(multiply first)
=	**20**	(then multiply)	=	**11**	(then add)

The dilemma should now be clear: Which one of these is correct? Neither is innately correct — the answer is determined by the ***Order of Operations***, an agreement that determines the order in which arithmetic operations are to be carried out. This agreement is followed by people in every country of the world, and in almost all computer systems. The above conundrum, $3 + 2 \times 4$, will be resolved in a while.

❑ *ADDITION VS SUBTRACTION*

Adding is the most fundamental operation we have in arithmetic and algebra. Also, since subtracting is merely the reverse of adding, subtracting is considered as fundamental as adding. We say that **adding and subtracting have equal priority**. So, when a problem has only adding and subtracting in it, we do the operations from left to right.

EXAMPLE 1: **Evaluate (or simplify): 20 – 8 + 13 – 22**

Solution: The only operations are adding and subtracting, so we proceed from left to right:

	$20 - 8 + 13 - 22$	(the original expression)
$=$	$12 + 13 - 22$	(1st step: subtract 8 from 20)
$=$	$25 - 22$	(2nd step: add 12 and 13)
$=$	$\boxed{3}$	(3rd step: subtract 22 from 25)

Homework

1. Evaluate (or simplify) each expression:

   a. $45 - 3 - 5$
   b. $50 - 7 - 1 - 2$
   c. $80 + 31 - 10 - 20$
   d. $23 - 23 + 18 - 18$
   e. $10 - 2 - 3 - 5$
   f. $34 + 17 - 1 - 20 + 2 - 10$
   g. $32 + 8 - 4 - 5 + 2 - 1$
   h. $4 + 6 + 10 - 1 - 2 + 1 - 4$
   i. $12 + 34 - 1 - 34 - 11$
   j. $23 + 12 - 1 - 2 - 6 + 7 - 10$

❑ *MULTIPLICATION VS ADDITION*

Since $4 \times 9 = 9 + 9 + 9 + 9$, we see that multiplication is just repeated addition. Since multiplication is based upon addition, we will declare that **multiplication has a higher priority than addition**. Therefore, the agreed-upon way to work the problem discussed at the top of the chapter, $3 + 2 \times 4$, is to perform the multiplication first, giving

$$\begin{aligned} & 3 + 2 \times 4 \\ = \; & 3 + 8 \qquad \text{(multiply BEFORE adding)} \\ = \; & \mathbf{11} \end{aligned}$$

Since we multiply before we add, and since addition and subtraction have equal priority, it follows that **multiplication has priority over subtraction**, also.

EXAMPLE 2: **Evaluate: $5 \times 4 - 2 \cdot 3 + 25 - 4(3)$**

Solution: We do all the multiplications first, since multiplication has priority over addition and subtraction:

$$\begin{aligned} & 5 \times 4 - 2 \cdot 3 + 25 - 4(3) \\ & \quad \downarrow \quad \downarrow \quad \downarrow \\ = \; & 20 - 6 + 25 - 12 && \text{(multiply first)} \\ = \; & 14 + 25 - 12 && \text{(add/subtract left to right)} \\ = \; & 39 - 12 && \text{(add/subtract left to right)} \\ = \; & \boxed{27} \end{aligned}$$

Homework

2. Evaluate (that is, simplify) each expression:

a. $7 + 2 \cdot 9$
b. $12 - 3 \times 4$
c. $10 \times 3 + 20$
d. $2 \cdot 7 + 9 \cdot 1$
e. $5 \times 7 - 8 \cdot 2$
f. $10 + 3 - 4 + 8 \cdot 5$
g. $23 \cdot 2 + 5 \cdot 1 - 4 \times 3$
h. $14 \cdot 3 - 1 - 2 - 10 \cdot 2 + 1$
i. $3 + 6 - 2 \cdot 4 + 1 \times 12$
j. $4(5) - 3(2) + 7(7) - 49$

❑ *MULTIPLICATION VS DIVISION*

Since dividing is the reverse operation of multiplying, we declare that **multiplication and division have equal priority**. This means that when a problem has only multiplication and division in it, we proceed from left to right (unless the fraction bar is used for division).

Two ways of indicating division are the symbols "÷", which is the symbol usually found on a calculator, and "/", used in computer programs such as spreadsheets, word processors, and programming languages. (The fraction bar method will be discussed after the next two examples.)

EXAMPLE 3: **Evaluate: $18 / 2 \times 3 \div 9 \cdot 10$**

Solution: There are two divisions and two multiplications in this expression. Since multiplication and division have equal priority, we work the problem from left to right.

	$18 / 2 \times 3 \div 9 \cdot 10$	(the original problem)
=	$9 \times 3 \div 9 \cdot 10$	(divide 18 by 2)
=	$27 \div 9 \cdot 10$	(multiply 9 by 3)

$= \quad 3 \cdot 10$ (divide 27 by 9)

$= \quad \boxed{30}$ (multiply 3 by 10)

EXAMPLE 4: **Evaluate: $12 - 10 \div 2 + 3 + 3 \times 4 + 20 / 10$**

Solution: Here's the overview: This expression has all four operations in it. Considering the agreement that multiplying and dividing have priority over adding and subtracting, we will first multiply and divide from left to right, and then go back to the front of the problem to add and subtract from left to right. Let's do it:

$12 - 10 \div 2 + 3 + 3 \times 4 + \underbrace{20/10}$ (the original problem)

$= \quad 12 - 5 + 3 + 12 + 2$ (multiply & divide left to right)

$= \quad 7 + 3 + 12 + 2$ (add & subtract left to right)

$= \quad \boxed{24}$

The division symbol usually used in algebra, science, and business is the fraction bar, but the following all have exactly the same meaning:

$$\frac{a}{b} \qquad b\overline{)a} \qquad a \div b \qquad a/b \qquad {}^{a}\!/_{b}$$

For example,

$\frac{8}{2} = 2\overline{)8} = 8 \div 2 = 8/2 = {}^{8}\!/_{2}$, all of which = **4**

When the fraction bar is used, the division occurs after both the top and the bottom have been completely done.

EXAMPLE 5: **Evaluate:** $\mathbf{8 - \frac{6}{2} + \frac{10-3}{6+1} - \frac{2+5(2)}{12-3(3)}}$

Solution: First we simplify all the numerators and denominators. Notice in the last fraction that we will use the priorities we learned earlier — that multiplication has priority over addition and subtraction.

$$8 - \frac{6}{2} + \frac{10-3}{6+1} - \frac{2+5(2)}{12-3(3)}$$ (the original problem)

$$= 8 - \frac{6}{2} + \frac{10-3}{6+1} - \frac{2+10}{12-9}$$ (multiply first)

$$= 8 - \frac{6}{2} + \frac{7}{7} - \frac{12}{3}$$ (simplify tops and bottoms)

$$= 8 - 3 + 1 - 4$$ (perform each division)

$$= 5 + 1 - 4$$ (add/subtract left to right)

$$= 6 - 4$$

$$= \boxed{2}$$

Homework

3. Evaluate each expression:

a. $7 \times 9 \div 3$

b. $100 \div 25 \times 10$

c. $8 \cdot 10 \div 2 \div 2 \div 2$

d. $5 \times 6 \div 2 \times 3 \times 1$

e. $12(10) \div 5 \cdot 2$

f. $100 \div 2 \div 5 \times 3 \div 30$

g. $100 - 20 \div 2 + 8/1$

h. $8 \cdot 2 \cdot 6 / 3 + 2 \cdot 10$

i. $50 + 80 \div 2 \div 4$

j. $\frac{15}{3} + \frac{10}{2} - 2 \cdot 1$

k. $20(10) - 3(14) + 10 \div 2$

l. $200 / 10 + 32 \cdot 2 - \frac{10-2}{2 \cdot 2 \cdot 2}$

m. $\frac{14-4}{12-2} + 8 \times 5 - 2 \times 3$

n. $\frac{32}{8} + \frac{1000}{2} - 8/2 + 22 \div 11 - \frac{5+4}{4-1}$

❑ *EXPONENTS VS MULTIPLICATION*

Recall that in the expression

$$B^E$$

B is called the **base** and E is called the **exponent**. Since an exponent represents repeated multiplication, we have, for example, $9^3 = 9 \cdot 9 \cdot 9$. (Raising the base to an exponent of 3 is called ***cubing***.)

So we declare that **an exponent has a higher priority than multiplication**. It thus follows that exponents have a higher priority than any operation analyzed thus far.

Albert Einstein's famous equation involving the equivalence of mass and energy, $E = mc^2$, where c is the speed of light, was written in this manner because 26-year-old Al knew that anyone reading the formula would understand that squaring is to be done first, followed by the multiplication. That is, only the c is being squared, not the product mc. Do you have an idea what he would have written had he wanted the entire quantity mc to be squared?

A formula you might be familiar with is the area of a circle: $A = \pi r^2$. The answer to the question "What is being squared?" is "just the r." The multiplication by the π is done <u>after</u> the r is squared.

<u>EXAMPLE 6:</u> **Evaluate: 3×5^2**

<u>Solution:</u> This is a small example, but it's an important one! Since exponents have higher priority than multiplication, we perform the exponent operation first:

$$3 \times 5^2 = 3 \times 25 = \boxed{75}$$

Homework

4. Evaluate each expression:

a. $9 \cdot 5^2$

b. 3×10^2

c. 5×4^3

d. $2 + 3 \cdot 4^2$

e. $5^2 - 3^2$

f. $7 \cdot 7 - 7^2$

g. $10 - 3^2$

h. $2 \times 10^2 - 2 \times 10$

i. $3(5)^2$

j. $1000 \div 5^2 - 18$

k. $100 / 2^2 + 8 \cdot 3$

l. $10^2 + 3 \cdot 5^3$

m. $3^2 + 4^2 - 2^3$

n. $5^2 - 4(2)(3)$

o. $10^3 + 3(4) + 1$

p. $4^2 - 4(1)(1)$

q. $8 + 5^2 - 5 \times 4$

r. $1^2 + 2^2 + 3^2 + 4^2$

s. $8^2 \cdot 10 - 3 \times 10^2$

t. $4^3 - 8^2 + 0 \cdot 99$

u. $10^2 \div 5^2 + 3 \times 4$

v. $18 + 2 \cdot 9^2 - 12 \div 2 + 50/2$

w. $32 \div 2^2 + 50/2 \div 5 \cdot 1^3$

x. $10^2 - (4)(2)(10)$

❑ *PARENTHESES HAVE VETO POWER*

Recall the problem which started this whole discussion:

$$3 + 2 \times 4$$

First we showed that if we do the addition first, the answer is 20; but when the multiplication is done first, the result is 11. Then we decided that multiplication has priority over addition — that multiplication is to be done first — and therefore, the proper answer is 11. That is,

$3 + 2 \times 4 = 3 + 8 = \mathbf{11}$ This is the proper way to do it.

But what if we really wanted to add the 3 and the 2 first in the expression $3 + 2 \times 4$? Is there a way to override all the rules we've developed so far? Yes — we just add a new rule: Use parentheses ( ) or brackets [ ] or braces { } to mean "Do me first!" Now the addition will take place first, because it's in the parentheses:

$$(3 + 2) \times 4 \;=\; 5 \times 4 \;=\; \mathbf{20}$$

For another example of the use of parentheses, recall Example 6, where we decided that $3 \times 5^2 \;=\; 3 \times 25 \;=\; 75$, because exponents have priority over multiplication. But if we really intended to do the multiplication first, we could use parentheses and get a totally different answer:

$$(3 \times 5)^2 \;=\; 15^2 \;=\; \mathbf{225}$$

❑ *Parentheses Are Used to Change How Things Are Associated*

Since parentheses are always done first, you should be confident in the following two calculations:

$(3 + 4) + 5 \;=\; 7 + 5 \;=\; 12$ Here we "associate" the 3 and 4.

$3 + (4 + 5) \;=\; 3 + 9 \;=\; 12$ Here we "associate" the 4 and 5.

We got the same answer to the problem $3 + 4 + 5$ whether we added the 3 and the 4 first, or added the 4 and the 5 first. Thus, when three quantities are to be added, we can "associate" the first two or the last two by putting parentheses around them. In short, if a, b, and c represent any numbers at all,

$$\boldsymbol{(a + b) + c \;=\; a + (b + c)}$$

and we say that **addition is an *associative* operation**.

Similarly, since $(4 \cdot 5) \cdot 3 \;=\; 20 \cdot 3 \;=\; 60$ and $4 \cdot (5 \cdot 3) \;=\; 4 \cdot 15 \;=\; 60$, for instance, we conclude that multiplication is also an associative operation:

$\boldsymbol{(ab)c \;=\; a(bc)}$ for any numbers a, b, and c.

❑ *PARENTHESES AND BRACKETS*

When expressions are "nested" (inside one another), we can use parentheses for both levels, or we can use parentheses inside brackets. For example, the following two expressions are completely equivalent:

$$(5 + 3(9 - 2))^2 \qquad\qquad [5 + 3(9 - 2)]^2$$

The first version is appropriate for a formula for a spreadsheet or computer language; the second is easier to read, and is common in math.

❑ *THE ORDER OF OPERATIONS*

The following chart summarizes this chapter, and shows the order in which to carry out math operations — the higher something is in the chart, the higher its priority. In other words, start at the top of the chart and move down as you do the problem.

Order of Operations

(Parentheses) and [Brackets]

Exponents

Multiply & Divide (left to right)

Add & Subtract (left to right)

EXAMPLE 7: $\mathbf{2 \cdot 10^2 - (7 - 4)^3 - 3 \times 5}$

$= 2 \cdot 10^2 - 3^3 - 3 \times 5$ (parentheses first)

$= 2 \cdot 100 - 27 - 3 \times 5$ (exponents next)

$= 200 - 27 - 15$ (then multiply)

$= 173 - 15$ (subtract from left to right)

$= \boxed{158}$

EXAMPLE 8: $2[12 - 3(5 - 3)]$

$= 2[12 - 3(2)]$ (innermost grouping first (parens))

$= 2[12 - 6]$ (multiply before subtract)

$= 2[6]$ (now the outer grouping [brackets])

$= \boxed{12}$

A Note Regarding Fractions

Even though a fraction indicates division, the order of operations does not include an explicit reference to fractions. This is because it's reasonably clear how to handle them: First use the order of operations to simplify the top, then use the order of operations to simplify the bottom, and last do the division.

Homework

5. Evaluate (simplify) each expression:

a. $3 \cdot 10^2 - (8 - 4)^3 - 3 \times 2$

b. $(5 - 3)^2 + (10 - 7)^3$

c. $[3 + 2(5)] - 1 + 3 \cdot 10$

d. $2(10 - 5)^2 - 12 \div 3$

e. $[2(10 - 5)]^3 \div (10 \cdot 10^2)$

f. $(1 + 4)^2 - (4 + 1)^2$

g. $[(3^2 - 2^2)^3 - 80] \div (36 / 4)$

h. $3 \cdot 4^2 - (13 - 12)^3$

i. $10 + 8(8 - 1)^2 - 3 - 2 - 1$

j. $[8^2 - 2^3 + 3 \cdot 4 - 2(7)]^2$

k. $[20 - (5 - 2)^2]^2 - 2 \cdot 3 \cdot 4$

l. $[13 - (8 - 3) + (10 - 2)]^3$

6. Evaluate each expression for the given values:

a. $(x + y)^2$ for $x = 2$ and $y = 1$

b. $x^2 + y^2$ for $x = 10$ and $y = 5$

c. $x^2 + xy + y^2$ for $x = 3$ and $y = 6$

d. $(x + y)(x - y)$ for $x = 10$ and $y = 2$

e. $x^2 - y^2$ for $x = 12$ and $y = 10$

❑ *First and Final Operations*

Homework

7. In each formula, indicate what operation would be performed <u>first</u>:

a. $P = 2l + 2w$

b. $P = 2(l + w)$

c. $A = \pi r^2$

d. $E = mc^2$

e. $k = a^2 + b^2$

f. $D = b^2 - 4ac$

g. $B = (a + b)(c + d)$ h. $V = (x - h)^2$

i. $r = \frac{d + a}{b}$ j. $N = a + \frac{b}{c}$

k. $A = \frac{c}{x - y}$ l. $a = b + cd$

8. Now, using the same 12 formulas from the previous problem, indicate what operation is the final one (that is, the operation which is performed last).

❑ *NUMBER OF TERMS*

Determining the *number of terms* in an expression is one of the most important skills you can acquire in algebra, allowing you to better understand a myriad of concepts coming up in this and future courses. The idea is somewhat subtle, however, so don't get discouraged if it takes a while for it to sink in.

If the final operation in an expression is multiplication, then the expression consists of one ***term***. Each of the following consists of one term:

ab $x(y + z)$ $(a - 2)(a - 3)^2$ $(z - w + u)(A + B)$

Each expression consists of one term
because the final operation is multiplication.

If the final operation in an expression consists of additions and/or subtractions, then the expression consists of at least two ***terms***. The number of terms is found by counting the number of things being added or subtracted in the final operations. The following examples are comprised of two or more terms:

$a + b$ 2 terms

$x - y + z$ 3 terms

$2 + mnp$ 2 terms

$a + b - c + def$	4 terms
$(y - x - m)^2 A + B$	2 terms
$(a - b)^2 + (c + d)^3 + e$	3 terms
$ab - cd + x - y + wxyz$	5 terms

Homework

9. Determine the number of ***terms*** in each expression:

a. abc
b. $a + b - c$
c. $xyz - w$
d. $(x - 3)^2$
e. $x^2 + 25$
f. $ab + ac - xy$
g. $(a + b)(x - y)$
h. $rst - qrw$
i. $(rst)(qrw)$
j. $(rst)(qrw) - 1$
k. $[(rst)(qrw) - 1]^5$
l. $25 - (x + y - z)^2$
m. $a - b$
n. $x + y^2 + z$
o. $20 - mnpq$
p. $a + x - c + QT$
q. $(y + xm)^2 A + B$
r. $(a - b)^3 - (c - d)^3 - w$
s. $abcd + x - y + wxyz$
t. $[a(b - c)ed - 7]^4$
u. $w(u - x)^3 - ab(de - mq)$
v. $a + b - c[wx + y - z]$

Review Problems

10. Evaluate each expression:

a. $3^2 - 2^2 + (5 - 1)^3$
b. $(7 - 2 + 1)^2 - \frac{8 + 2}{12 - 2}$

c. 3×4^2 d. $2 \cdot 5^3$

e. $2 \cdot 3 + 9 \div 3 - 8 / 8 + 3^2$ f. $\dfrac{3^3 - 3^2 - 3}{4^3 - 9 \cdot 7}$

g. $(2 \cdot 8)^2 - 2 \cdot 8^2$ h. $20 + 5 \times 3 - 2 \times 1 + 0(99)$

11. How many terms are there in each expression?

a. $xyz + w$ b. $x(yz + w)$ c. $abc - def + 7$

d. $(ab - h)^2$ e. $(x - y)^2 + abc^2 - ab - c$

Solutions

1. a. 37 b. 40 c. 81 d. 0 e. 0 f. 22 g. 32
h. 14 i. 0 j. 23

2. a. 25 b. 0 c. 50 d. 23 e. 19 f. 49 g. 39
h. 20 i. 13 j. 14

3. a. 21 b. 40 c. 10 d. 45 e. 48 f. 1 g. 98
h. 52 i. 60 j. 8 k. 163 l. 83 m. 35 n. 499

4. a. 225 b. 300 c. 320 d. 50 e. 16 f. 0 g. 1
h. 180 i. 75 j. 22 k. 49 l. 475 m. 17 n. 1
o. 1013 p. 12 q. 13 r. 30 s. 340 t. 0 u. 16
v. 199 w. 13 x. 20

5. a. 230 b. 31 c. 42 d. 46 e. 1 f. 0
g. 5 h. 47 i. 396 j. 2916 k. 97 l. 4096

6. a. $(x + y)^2 = (2 + 1)^2 = 3^2 = 9$
b. $x^2 + y^2 = 10^2 + 5^2 = 100 + 25 = 125$
c. 63 d. 96 e. 44

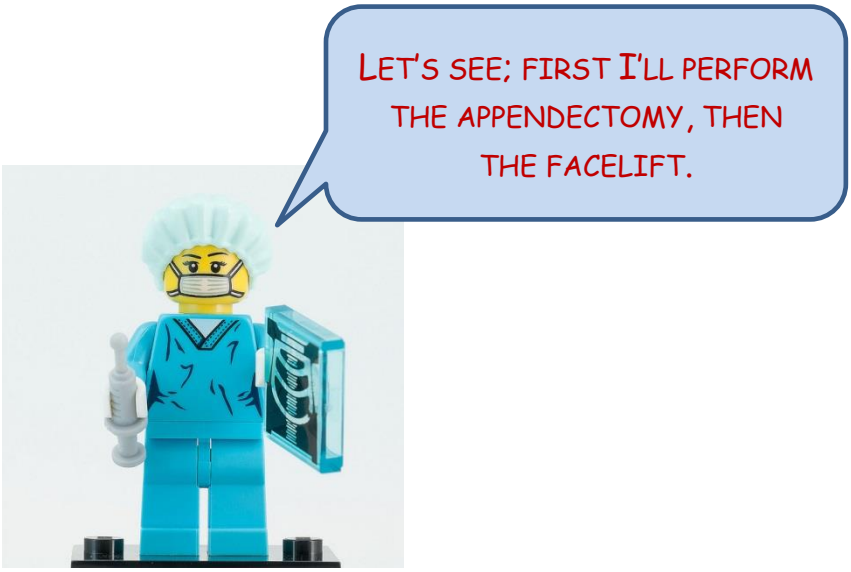
7. a. Multiply b. Add c. Square d. Square
e. Square f. Square g. Add h. Subtract
i. Add j. Divide k. Subtract l. Multiply

8. a. Add b. Multiply c. Multiply d. Multiply
e. Add f. Subtract g. Multiply h. Square
i. Divide j. Add k. Divide l. Add

9. a. 1 b. 3 c. 2 d. 1 e. 2 f. 3 g. 1 h. 2
i. 1 j. 2 k. 1 l. 2 m. 2 n. 3 o. 2 p. 4
q. 2 r. 3 s. 4 t. 1 u. 2 v. 3

10. a. 69 b. 35 c. 48 d. 250 e. 17 f. 15 g. 128 h. 33

11. a. 2 b. 1 c. 3 d. 1 e. 4



What the Order of Operations ISN'T!

"Not with what we have –
but with what we enjoy –
do we find our
true abundance."

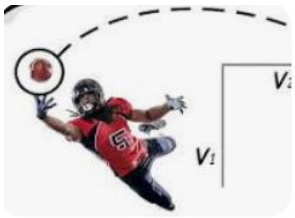
Tibetan quote

GRAPHING PARABOLAS



The **parabola** (accent on the *rab*) is a very special shape used in searchlights and satellite dishes. Even football sports reporters use parabolic reflectors to listen in on comments made by coaches on the sidelines and players in the huddle. In fact, when a football is thrown or punted, its path is that of a parabola.





❑ *GRAPHING A PARABOLA*

EXAMPLE 1: **Graph: $y = x^2 - 4x + 3$**

Solution: Who's to say that the graph of this formula isn't a straight line? Let's work it out. We'll find some points on our graph by choosing some values of x, and then calculating the corresponding y-values — and we'll see what points we get.

If $x = -1$, then $y = (\mathbf{-1})^2 - 4(\mathbf{-1}) + 3 = 1 + 4 + 3 = 8 \Rightarrow$ **(–1, 8)**

If $x = 0$, then $y = (\mathbf{0})^2 - 4(\mathbf{0}) + 3 = 0 - 0 + 3 = 3 \Rightarrow$ **(0, 3)**

If $x = 1$, then $y = (\mathbf{1})^2 - 4(\mathbf{1}) + 3 = 1 - 4 + 3 = 0 \Rightarrow$ **(1, 0)**

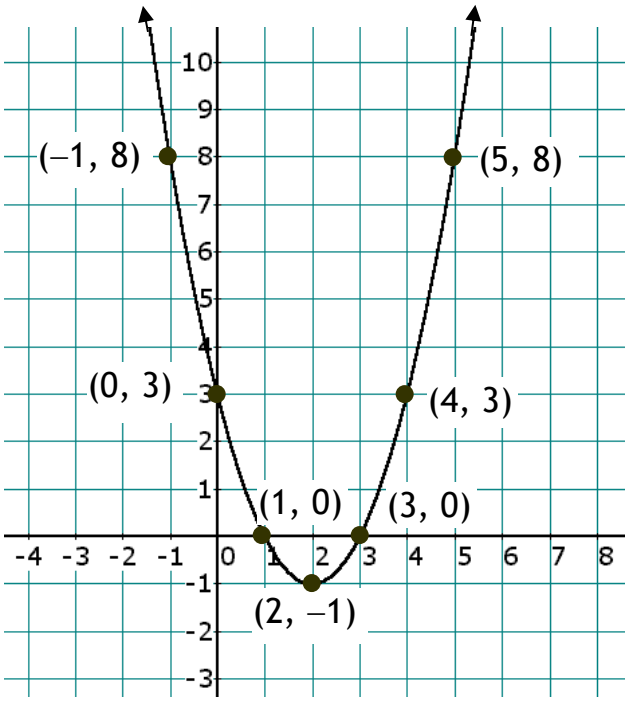
If $x = 2$, then $y = (\mathbf{2})^2 - 4(\mathbf{2}) + 3 = 4 - 8 + 3 = -1 \Rightarrow$ **(2, –1)**

If $x = 3$, then $y = (\mathbf{3})^2 - 4(\mathbf{3}) + 3 = 9 - 12 + 3 = 0 \Rightarrow$ **(3, 0)**

If $x = 4$, then $y = (\mathbf{4})^2 - 4(\mathbf{4}) + 3 = 16 - 16 + 3 = 3 \Rightarrow$ **(4, 3)**

If $x = 5$, then $y = (\mathbf{5})^2 - 4(\mathbf{5}) + 3 = 25 - 20 + 3 = 8 \Rightarrow$ **(5, 8)**

Plotting these seven points leads us to the following graph:



What do we notice about the graph? It's a curve, not a straight line. We notice that x can be any real number (but notice that y never goes below –1). Also note that the graph has one y-intercept but two x-intercepts. In addition, there is no highest point on the parabola, but the lowest point on the parabola is (2, –1), and we call this point the ***vertex*** of the parabola.

This is the shape called the ***parabola***. We say that the parabola just graphed "**opens *up***." The equation of a parabola is characterized by the fact that one variable (the x) is *squared* while the other variable (the y) is raised to the first power.

EXAMPLE 2: **Graph: $y = -x^2 - 2x - 1$**

Solution: First we notice that the quadratic term, the $-x^2$, has a leading negative sign. And we remember that, due to the Order of Operations, exponents have a higher priority than minus signs, so we know that to evaluate $-x^2$, we square the x first, and apply

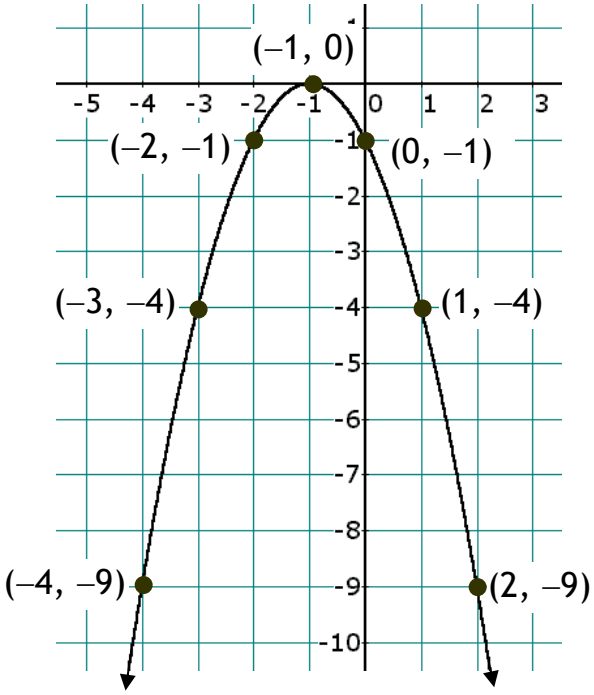
the minus sign second. [For example, $-7^2 = -49$.] We'll calculate together two points on the parabola and leave the rest of the points for you to do.

If $x = 2$, then $y = -(\mathbf{2})^2 - 2(\mathbf{2}) - 1 = -4 - 4 - 1 = -9$. Thus, the point **(2, –9)** is on the graph of our parabola.

If $x = -3$, then $y = -(\mathbf{-3})^2 - 2(\mathbf{-3}) - 1 = -9 + 6 - 1 = -4$. Therefore, the point **(–3, –4)** is on the graph.

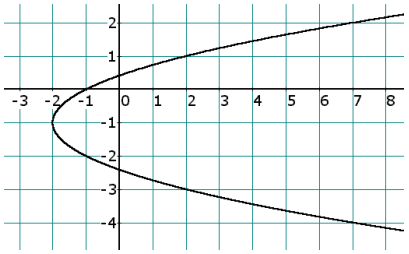
You should now do the calculations to show that each of the following points is also on the graph:

(–4, –9) **(–2, –1)** **(–1, 0)** **(0, –1)** **(1, –4)**



From the formula (and kind of from the graph) we note that x can be any real number, but that y never gets above 0. As for intercepts, there's one x-intercept and one y-intercept. There is no lowest point on the graph, but the highest point, the **vertex**, is the point (–1, 0). We say that this parabola **opens down**.

There are also "sideways" parabolas, but in this chapter only parabolas which open up or down will be discussed.



Homework

1. In Example 1 we saw that the graph of $y = x^2 - 4x + 3$ is a parabola opening up. Example 2 showed us that the graph of $y = -x^2 - 2x - 1$ is a parabola opening down. Take a guess what property of these equations determines whether the parabola opens up or down.

2. True/False: (Recall that all parabolas in this chapter open up or down.)



   a. Every parabola has a y-intercept.

   b. Every parabola has an x-intercept.

   c. Every parabola has a vertex.

   d. The vertex of a parabola is always the highest point of the parabola.

   e. The vertex of a parabola is always the lowest point of the parabola.

3. Graph each parabola by plotting points. Then use your graph to determine the **intercepts** and the **vertex** of your parabola:

   a. $y = 9 - x^2$

   b. $y = x^2 + 6x + 5$

   c. $y = x^2 + 2x - 8$

   d. $y = -x^2 + 4x - 4$

   e. $y = 0.5x^2$

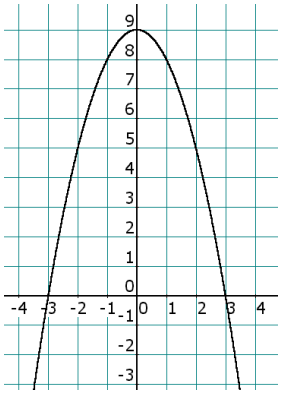
   f. $y = -0.2x^2 + 5$

Solutions

1. The coefficient of the quadratic term in Example 1 is positive, while that of the quadratic term in Example 2 is negative. That's the clue which determines whether a parabola opens up or down. Therefore, the parabola $y = \pi x^2 - 13x + 2$ opens up, whereas the parabola $y = -0.7x^2 + 99x + 14$ opens down.

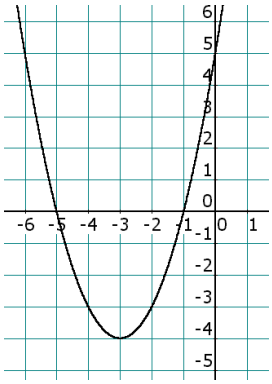
2. a. True b. False c. True d. False e. False

3. a.



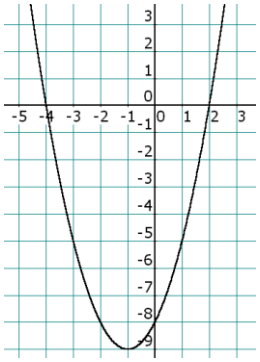
Intercepts:
(–3, 0), (3, 0), (0, 9)
Vertex: (0, 9)

b.



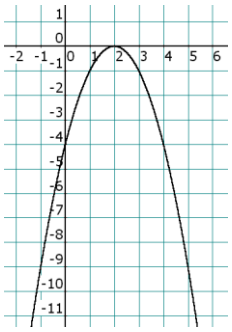
Intercepts:
(–5, 0), (–1, 0), (0, 5)
Vertex: (–3, –4)

c.



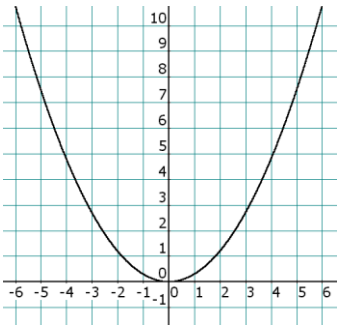
Intercepts:
(–4, 0), (2, 0), (0, –8)
Vertex: (–1, –9)

d.



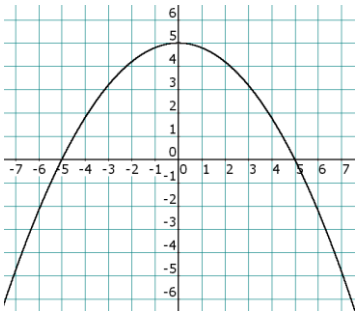
Intercepts:
(2, 0), (0, –4)
Vertex: (2, 0)

e.



Intercepts: (0, 0)
Vertex: (0, 0)

f.



Intercepts:
(–5, 0), (5, 0), (0, 5)
Vertex: (0, 5)

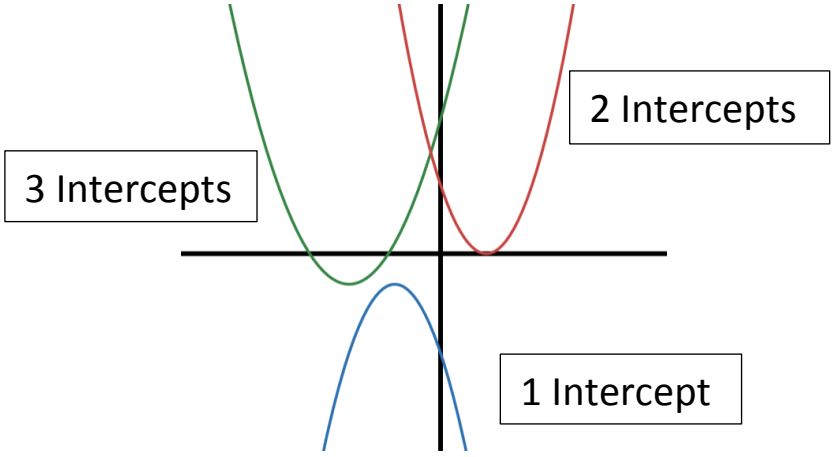
"It is our choices that show what we truly are, far more than our abilities."

– *J.K. Rowling*



INTERCEPTS OF A PARABOLA – FACTORABLE

Most LINES have just two intercepts (or just one intercept if it's a horizontal or vertical line that's not one of the axes, or even infinitely many intercepts if the line is one of the axes.) But a PARABOLA may have one intercept, perhaps two intercepts, or possibly three intercepts:



It is impossible for a parabola to have NO intercepts, mainly because in a parabola opening up or down, the *x*-value can be any real number, meaning that the graph goes infinitely to the right and infinitely to the left. If follows that the parabola MUST pass through the *y*-axis somewhere, thus providing at least one intercept.

As in the chapter Graphing Parabolas, we're assuming parabolas that open up or down.

❑ How Do We Find the Intercepts of a Parabola?

Remember the method for finding the intercepts of a line? Well, an intercept is an intercept, so the rules for finding the intercepts of a parabola (or any graph at all!) are identical to the rules we learned before:

> ***x*-intercepts are found by setting *y* to 0.**
>
> ***y*-intercepts are found by setting *x* to 0.**

Also recall that an intercept is a point in the plane, and should be written as an ordered pair like (2, 0) or (0, –3). Ask your instructor if you need to write your intercepts this way.

EXAMPLE 1: **Find the intercepts of $y = x^2 - x - 6$.**

Solution:

x-intercepts: Setting $y = 0$

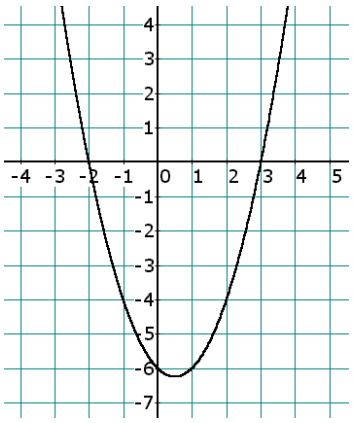
$\Rightarrow \quad \mathbf{0} = x^2 - x - 6$

$\Rightarrow \quad 0 = (x + 2)(x - 3)$

$\Rightarrow \quad x + 2 = 0 \text{ or } x - 3 = 0$

$\Rightarrow \quad x = -2 \text{ or } x = 3$ We conclude that

the *x*-intercepts are (–2, 0) and (3, 0)



y-intercepts: Setting $x = 0$

$\Rightarrow \quad y = \mathbf{0}^2 - 6(\mathbf{0}) - 6 = -6$

Therefore,

the *y*-intercept is (0, –6)

EXAMPLE 2: **Find the intercepts of $y = x^2 - 6x + 9$.**

Solution:

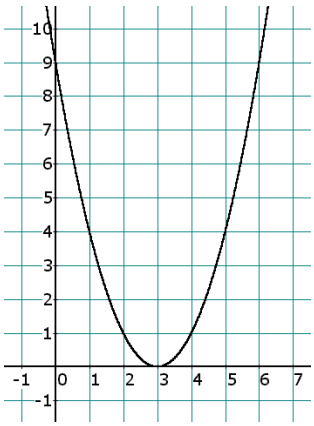
x-intercepts: Setting $y = 0$

$$\Rightarrow \quad \mathbf{0} = x^2 - 6x + 9$$
$$\Rightarrow \quad 0 = (x - 3)(x - 3)$$
$$\Rightarrow \quad x - 3 = 0$$
$$\Rightarrow \quad x = 3,$$

only one solution for x.

We conclude that

the x-intercept is (3, 0)



y-intercepts: Setting $x = 0$

$$\Rightarrow \quad y = (\mathbf{0})^2 - 6(\mathbf{0}) + 9 = 9$$

Therefore,

the y-intercept is (0, 9)

Homework

Find all the **intercepts** of each parabola:

1. $y = x^2 - 25$
2. $y = 2x^2 - 3x - 35$
3. $y = 9x^2 + 6x + 1$
4. $y = -49x^2 + 28x - 4$
5. $y = 9x^2 - 1$
6. $y = -14x^2 + x + 3$
7. $y = 10x^2 + 27x + 14$
8. $y = 15x^2 - 34x + 15$

Solutions

1. $(5, 0)$ $(-5, 0)$ $(0, -25)$

2. $(5, 0)$ $\left(-\frac{7}{2}, 0\right)$ $(0, -35)$

3. $\left(-\frac{1}{3}, 0\right)$ $(0, 1)$

4. $\left(\frac{2}{7}, 0\right)$ $(0, -4)$

5. $\left(\frac{1}{3}, 0\right)$ $\left(-\frac{1}{3}, 0\right)$ $(0, -1)$

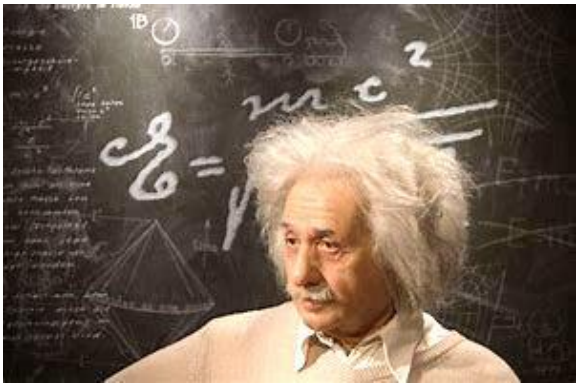
6. $\left(-\frac{3}{7}, 0\right)$ $\left(\frac{1}{2}, 0\right)$ $(0, 3)$

7. $\left(-\frac{7}{10}, 0\right)$ $(-2, 0)$ $(0, 14)$

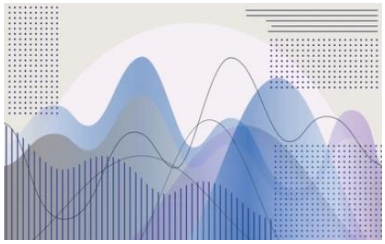
8. $\left(\frac{5}{3}, 0\right)$ $\left(\frac{3}{5}, 0\right)$ $(0, 15)$

"Any fool can know. The point is to understand."

– Albert Einstein



INTERCEPTS OF A PARABOLA – NON-FACTORABLE



This chapter, like an earlier one, helps you find the **intercepts** of a parabola, but the quadratic equations you will obtain will not be factorable, so another method must be utilized. You could use Completing the Square, but we'll use the Quadratic Formula.

❑ *EXAMPLES*

EXAMPLE 1: **Find the intercepts of $y = -3x^2 + 5x - 1$.**

Solution:

x-intercepts: Setting $y = 0$ turns the parabola equation into

$$\mathbf{0} = -3x^2 + 5x - 1$$

Bringing all the terms to the left side gives us the following quadratic equation in standard form:

$$3x^2 - 5x + 1 = 0$$

The left side of the equation is not factorable (give it a try!), so let's apply the Quadratic Formula, where in this case $a = 3$, $b = -5$, and $c = 1$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(1)}}{2(3)}$$

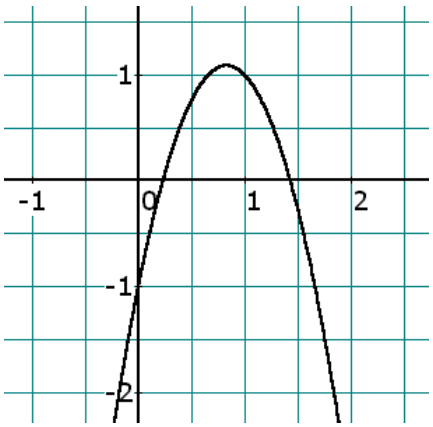
$$= \frac{5 \pm \sqrt{25 - 12}}{6} = \frac{5 \pm \sqrt{13}}{6}$$

These two solutions are certainly correct, and we could even write our two x-intercepts like this:

$$\left(\frac{5+\sqrt{13}}{6}, 0\right) \text{ and } \left(\frac{5-\sqrt{13}}{6}, 0\right)$$

However, this form of the x-intercepts is not very useful for plotting them on a grid. It's better to use your calculator to convert the two exact radical answers into approximate decimal answers; our x-intercepts are roughly

(1.434, 0) and (0.232, 0)



y-intercepts: Setting $x = 0 \Rightarrow$

$$y = -3(\mathbf{0})^2 + 5(\mathbf{0}) - 1 = -1$$

The y-intercept is therefore

(0, –1)

EXAMPLE 2: **Find the intercepts of $y = x^2 + x + 2$.**

Solution: Seems easy enough, but this is a strange one.

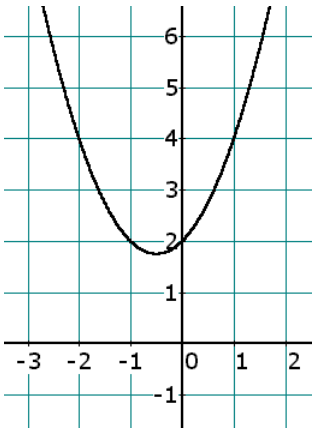
x-intercepts: Setting $y = 0$ yields the quadratic equation $x^2 + x + 2 = 0$. First, this quadratic won't factor, but that's okay; we have the Quadratic Formula to rescue us:

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(2)}}{2(1)} = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm \sqrt{-7}}{2}$$

We're in trouble; the –7 in the radical sign indicates that these two solutions for x are <u>not</u> real numbers. This means that wherever these non-real (imaginary) numbers are, they are absolutely <u>not</u> on the x-axis (which is just the set of real numbers).

The conclusion of all this? This parabola has

no x-intercept



<u>y-intercepts:</u> Setting $x = 0$ gives $y = 2$, and so the y-intercept is

(0, 2)

Homework

1. Find all the **intercepts** of each parabola in exact form — no calculator, no approximations:

   a. $y = 2x^2 + 8x + 5$ b. $y = 3x^2 - 6x + 4$

   c. $y = x^2 - 13$ d. $y = 2x^2 + 9$

2. Find all the **intercepts** of each parabola, rounded to 3 digits past the point:

   a. $y = x^2 + 7x + 1$ b. $y = -2x^2 + 5x + 4$

   c. $y = 3x^2 - 6x - 2$ d. $y = 5x^2 + 3x + 1$

Solutions

1. a. $\left(\frac{-4+\sqrt{6}}{2}, 0\right)$ $\left(\frac{-4-\sqrt{6}}{2}, 0\right)$ (0, 5)

b. No x-intercepts (0, 4)

c. $\left(\pm\sqrt{13}, 0\right)$ (0, –13)

d. No x-intercepts (0, 9)

2. a. (–0.146, 0) (–6.854, 0) (0, 1)

b. (3.137, 0) (–0.637, 0) (0, 4)

c. (2.291, 0) (–0.291, 0) (0, –2)

d. No x-intercepts (0, 1)

"Your attitude,

not your aptitude,

will determine your altitude."

–Zig Ziglar

THE VERTEX OF A PARABOLA

If a parabola opens up, then the ***vertex*** of the parabola is the point at the bottom, the minimum point of the parabola. If the parabola opens down, then the vertex is the top, the maximum point of the parabola.



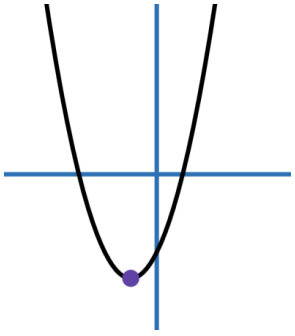
To find the vertex of a parabola in this chapter, we'll need the Midpoint Formula:

*If (a, b) and (c, d) are two points in the plane, then the **midpoint** of the line segment connecting those points is*

$$\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$$

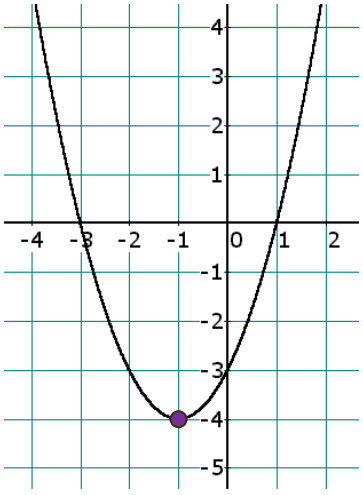
❑ *FINDING THE VERTEX*

The goal is to devise a way to determine the x-coordinate of the **vertex**, the point at the bottom of the graph of the parabola $y = x^2 + 2x - 3$. Once we find the x-coordinate, we can find the y-coordinate by plugging the x-coordinate into the parabola equation.



Here's the secret to finding the x-coordinate of the vertex: Go straight up from the vertex to the x-axis. That x-value is midway between the x-values of the two x-intercepts; in this case it's midway between $x = -3$ and $x = 1$. Recalling the Midpoint Formula reviewed in the Introduction, the x-coordinate of the vertex is

$$x = \frac{-3+1}{2} = \frac{-2}{2} = -\mathbf{1},$$



a fact which should be clear by looking at the graph. Since the parabola's equation is $y = x^2 + 2x - 3$, we find the y-coordinate of the vertex by simply placing $x = -1$ into the equation and finding y:

$$y = x^2 + 2x - 3 = (-\mathbf{1})^2 + 2(-\mathbf{1}) - 3$$
$$= 1 - 2 - 3 = -4,$$

which is consistent with the picture. We conclude that the vertex of the parabola is the point **(–1, –4)**.

To find the *x*-coordinate of the vertex of a parabola, find the <u>average</u> of the *x*-coordinates of the parabola's *x*-intercepts.

❑ *EXAMPLES AND FURTHER ANALYSIS*

<u>EXAMPLE 1:</u> **Find the vertex of $y = -2x^2 - 4x + 30$.**

<u>Solution:</u> According to the rule in the box above, we first calculate the x-coordinates of the x-intercepts of the parabola. This, of course, is accomplished by setting y to 0:

$$\mathbf{0} = -2x^2 - 4x + 30$$
$$\Rightarrow \quad 2x^2 + 4x - 30 = 0 \qquad \text{(move everything over)}$$
$$\Rightarrow \quad 2(x^2 + 2x - 15) = 0 \qquad \text{(factor out the GCF)}$$
$$\Rightarrow \quad 2(x + 5)(x - 3) = 0 \qquad \text{(factor the trinomial)}$$
$$\Rightarrow \quad (x + 5)(x - 3) = 0 \qquad \text{(divide both sides by 2)}$$

$\Rightarrow \quad x + 5 = 0 \text{ or } x - 3 = 0$ (set each factor to 0)

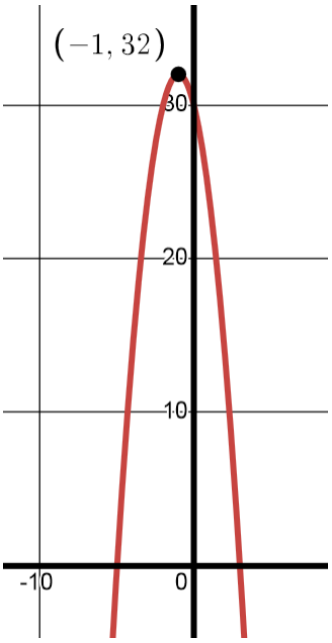
$\Rightarrow \quad x = -5 \text{ or } x = 3$ (solve each linear equation)

The average of these x-values is $\frac{-5+3}{2} = \frac{-2}{2} = -1$, and so we have $x = -1$ as the x-coordinate of the vertex. The y-value of the vertex is

$$y = -2x^2 - 4x + 30 = -2(\mathbf{-1})^2 - 4(\mathbf{-1}) + 30$$
$$= -2 + 4 + 30 = 32$$

and we conclude that the vertex of the parabola is the point

$$\boxed{(-1, 32)}$$



EXAMPLE 2: **Find the vertex of the parabola $y = x^2 - 6x + 9$.**

Solution: Find the x-intercepts by setting y to 0:

$\mathbf{0} = x^2 - 6x + 9$

$\Rightarrow \quad 0 = (x - 3)(x - 3)$

$\Rightarrow \quad x - 3 = 0 \text{ or } x - 3 = 0$

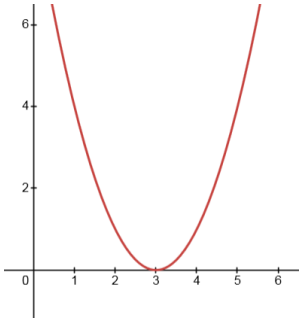
$\Rightarrow \quad x = 3 \text{ or } x = 3$

There's only one x-intercept, (3, 0), but it did occur twice! So we'll go through the motions and calculate the average of 3 and 3, which is **3**, the x-coordinate of the vertex of the parabola. And the y-coordinate of the vertex is

$$y = \mathbf{3}^2 - 6(\mathbf{3}) + 9 = 9 - 18 + 9 = \mathbf{0}$$

We conclude that the vertex of the parabola is

$$\boxed{(3, 0)}$$



EXAMPLE 3: **Find the vertex of $y = 3x^2 - 7x + 3$.**

Solution: Set y to 0 to calculate the x-intercepts. This will be followed by averaging those x-intercepts to find the x-coordinate of the vertex of the parabola.

$$\mathbf{0} = 3x^2 - 7x + 3$$

Factoring will prove fruitless, so it's time for the Quadratic Formula (we could also Complete the Square) — the values of a, b, and c are 3, –7, and 3:

$$x = \frac{-(-\mathbf{7}) \pm \sqrt{(-\mathbf{7})^2 - 4(\mathbf{3})(\mathbf{3})}}{2(\mathbf{3})} = \frac{7 \pm \sqrt{49 - 36}}{6} = \frac{7 \pm \sqrt{13}}{6}$$

Thus, the x-coordinates of the x-intercepts are

$$\frac{7 + \sqrt{13}}{6} \quad \text{and} \quad \frac{7 - \sqrt{13}}{6}$$

It is these two numbers that we average to obtain the x-coordinate of the vertex:

$$x = \frac{\dfrac{7 + \sqrt{13}}{6} + \dfrac{7 - \sqrt{13}}{6}}{2} \qquad \text{(average = sum divided by 2)}$$

$$= \frac{\dfrac{7 + \sqrt{13} + 7 - \sqrt{13}}{6}}{2} \qquad \text{(adding the two fractions)}$$

$$= \frac{\dfrac{14}{6}}{2} = \frac{\dfrac{7}{3}}{2} = \frac{7}{3} \div 2 = \frac{7}{3} \times \frac{1}{2} = \frac{7}{6}$$

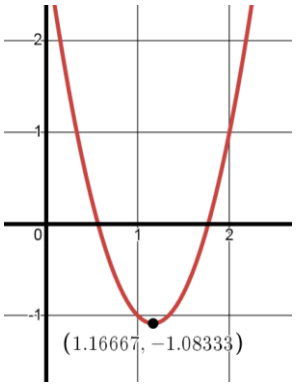
Now we know that the x-coordinate of the vertex is $\frac{7}{6}$, and so the y-coordinate of the vertex is

$$y = 3\left(\frac{\mathbf{7}}{\mathbf{6}}\right)^2 - 7\left(\frac{\mathbf{7}}{\mathbf{6}}\right) + 3 = 3\left(\frac{49}{36}\right) - 7\left(\frac{7}{6}\right) + 3 = \frac{49}{12} - \frac{49}{6} + 3$$

$$= \frac{49}{12} - \frac{98}{12} + \frac{36}{12} = -\frac{13}{12}$$

We're finally able to state our conclusion: The vertex of the parabola is

$$\boxed{\left(\frac{7}{6}, -\frac{13}{12}\right)}$$



EXAMPLE 4: **Find the vertex of $y = x^2 + 3x + 4$.**

Solution: Seems innocent enough — what could possibly go wrong? (Ever heard of Murphy's Law?) To find x-intercepts, we set y to 0 and obtain the quadratic equation

$$\mathbf{0} = x^2 + 3x + 4$$

Since this quadratic is not factorable, we employ the Quadratic Formula:

$$x^2 + 3x + 4 = 0 \qquad a = 1;\ b = 3;\ c = 4$$

$$\Rightarrow \quad x = \frac{-\mathbf{3} \pm \sqrt{(\mathbf{3})^2 - 4(\mathbf{1})(\mathbf{4})}}{2(\mathbf{1})} = \frac{-3 \pm \sqrt{9-16}}{2} = \frac{-3 \pm \sqrt{-7}}{2}$$

Gee, these two numbers are not real numbers; you might know that we call them "imaginary" numbers, and that they don't exist anywhere on the x-axis (which is an axis of real numbers). We deduce that the parabola has no x-intercepts. So how in the heck do we average the x-intercepts to find the vertex when they don't even exist? The answer is, trust me! That is, let's calculate the average of these two "phantom" intercepts and see what happens.

There are two issues to consider if we want to calculate the average of two imaginary numbers. First, the two solutions of the quadratic equation above can be written separately as

$$\frac{-3+\sqrt{-7}}{2} \quad \text{and} \quad \frac{-3-\sqrt{-7}}{2}$$

Second, the average of two numbers (even these imaginary numbers!) is found by dividing their sum by 2 — here goes:

$$x = \frac{\frac{-3+\sqrt{-7}}{2} + \frac{-3-\sqrt{-7}}{2}}{2}$$ (average = sum divided by 2)

$$= \frac{\frac{-3+\sqrt{-7}-3-\sqrt{-7}}{2}}{2}$$ (adding the two fractions)

$$= \frac{\frac{-6}{2}}{2}$$ (combine like terms)

$$= \frac{-3}{2} = -\mathbf{\frac{3}{2}}$$

Thus, assuming you still trust me, we will accept this number to be the x-coordinate of the vertex. The y-coordinate would be

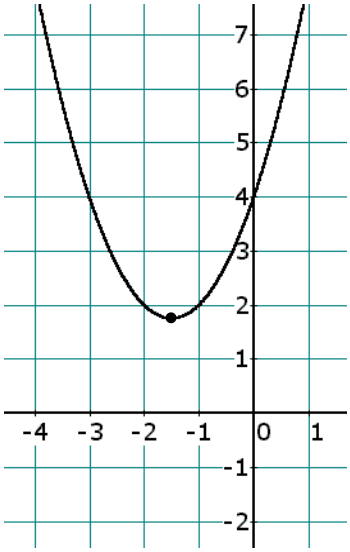
$$y = x^2+3x+4 = \left(-\mathbf{\frac{3}{2}}\right)^2+3\left(-\mathbf{\frac{3}{2}}\right)+4 = \frac{9}{4}-\frac{9}{2}+4$$

$$= \frac{9}{4}-\frac{18}{4}+\frac{16}{4} = \mathbf{\frac{7}{4}}$$

Our best theory, therefore, is that the vertex of the parabola is

$$\left(-\frac{3}{2}, \frac{7}{4}\right)$$

Let's confirm that this vertex we just calculated of the parabola with no x-intercepts makes some sense by sketching the parabola. It looks pretty reasonable. Notice that the parabola truly has no x-intercepts, just as predicted by the algebra. Moreover, if you look carefully at the vertex of the parabola, you can estimate that the x-coordinate is between –2 and –1, which the number $-\frac{3}{2}$ is. Also, the



y-coordinate of the vertex appears to be a little shy of 2, which the number $\frac{7}{4}$ is. I'm convinced — are you?

Homework

1. By averaging the x-coordinates of the x-intercepts, find the **vertex** of each parabola:

   a. $y = x^2 + 2x - 48$ b. $y = x^2 + 10x + 25$

   c. $y = 2x^2 + 8x + 5$ d. $y = 3x^2 - 6x + 4$

❑ *PUTTING IT ALL TOGETHER*

EXAMPLE 5: **Graph $y = x^2 - 6x + 5$ using all the concepts of we've learned about parabolas.**

Solution: The leading coefficient, 1, is positive; thus, the parabola **opens up**.

To find x-intercepts, set y to 0:

$$x^2 - 6x + 5 = \mathbf{0}$$
$$\Rightarrow \quad (x - 5)(x - 1) = 0$$
$$\Rightarrow \quad x - 5 = 0 \text{ or } x - 1 = 0$$
$$\Rightarrow \quad x = 5 \text{ or } x = 1$$

Therefore, **the x-intercepts are (1, 0) and (5, 0)**.

As for y-intercepts, we set x to 0:

$$y = \mathbf{0}^2 - 6(\mathbf{0}) + 5 = 5$$

Thus, **the y-intercept is (0, 5)**.

To find the x-coordinate of the vertex we calculate the average of the two x-coordinates of the two x-intercepts:

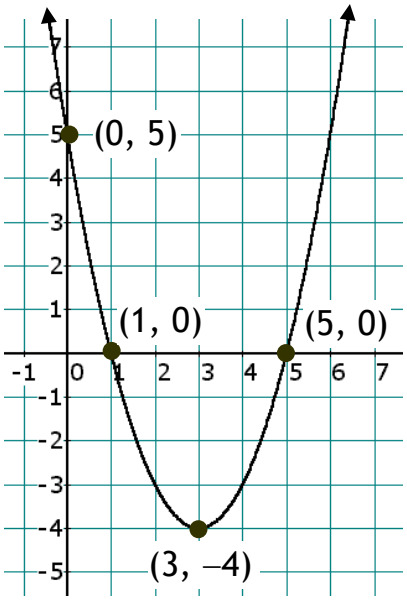
$$x = \frac{1+5}{2} = \frac{6}{2} = 3,$$

which implies that

$$y = \mathbf{3}^2 - 6(\mathbf{3}) + 5 = 9 - 18 + 5 = -4$$

So **the vertex of the parabola is (3, –4)**.

We now have our parabola:



Homework

2. Find the y-intercept, the x-intercept(s), and the vertex of $y = 2x^2 + 7x - 5$. Sketch the parabola.

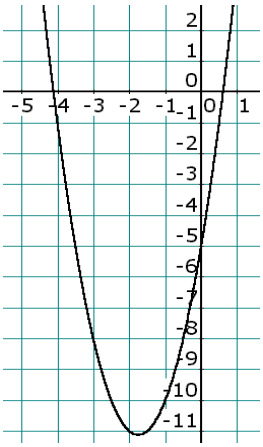
3. Find the y-intercept, the x-intercept(s), and the vertex of $y = -16x^2 + 24x - 9$. Sketch the parabola.

4. Find the y-intercept, the x-intercept(s), and the vertex of $y = x^2 - 3x + 5$. Sketch the parabola.

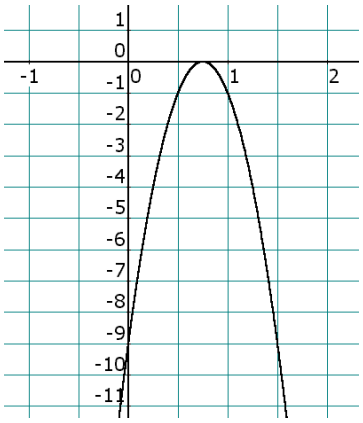
Solutions

1. a. (–1, –49) b. (–5, 0) c. (–2, –3) d. (1, 1)

2. *y*-int: (0, –5) *x*-int: $\left(\frac{-7+\sqrt{89}}{4}, 0\right), \left(\frac{-7-\sqrt{89}}{4}, 0\right)$ $V\left(-\frac{7}{4}, -\frac{89}{8}\right)$

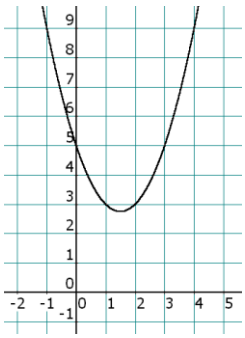


3. y-int: (0, –9) x-int: $\left(\frac{3}{4}, 0\right)$ $V\left(\frac{3}{4}, 0\right)$



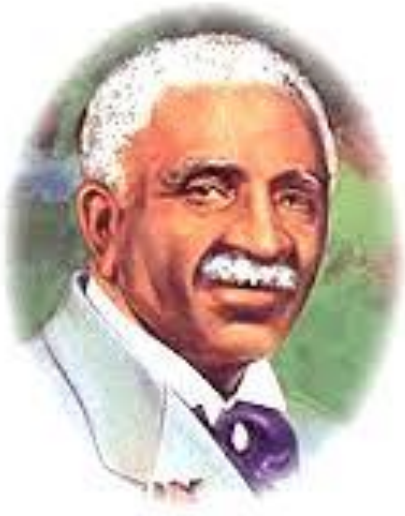
Note that the axes are scaled differently. In actuality, the parabola is much skinnier than the one shown.

4. y-int: (0, 5) x-int: None $V\left(\frac{3}{2}, \frac{11}{4}\right)$



"EDUCATION IS THE KEY TO UNLOCK THE GOLDEN DOOR OF FREEDOM."

– George Washington Carver



DIVIDING POLYNOMIALS

❑ *INTRODUCTION*

First we need the proper terminology. When written as a fraction, a division problem has two parts:

$$\frac{\textbf{dividend}}{\text{divisor}}$$

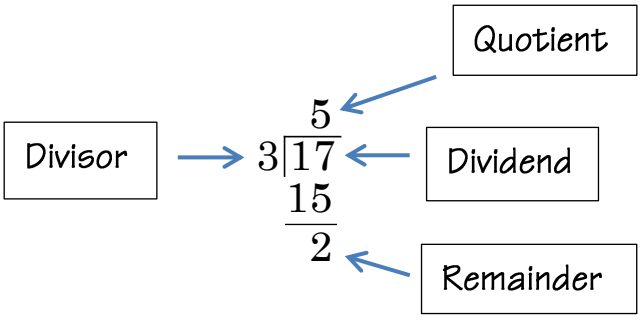
When written in the standard "long division" format, we write

$$\text{divisor}\,\overline{)\,\textbf{dividend}}$$

And we can use the " ÷ " symbol for division:

$$\textbf{dividend} \div \text{divisor}$$

The result of dividing is called the ***quotient***, and the leftover is called the ***remainder***. For example,



We can then write the answer as $5+\frac{2}{3}$ $\left(\text{quotient} + \frac{\text{remainder}}{\text{divisor}}\right)$, or as the mixed number $5\frac{2}{3}$ when we're dealing with numbers.

❑ *DIVIDING A POLYNOMIAL BY A MONOMIAL*

Just as $\frac{1}{7}+\frac{3}{7} = \frac{4}{7}$, we can work the problem $\frac{a}{b}+\frac{c}{b}$ by adding the numerators, and placing that sum over the common denominator b:

$$\frac{a}{b}+\frac{c}{b} = \frac{a+c}{b}$$

By reversing this reasoning, we can take the fraction $\frac{a+c}{b}$ and, if we like, *split* it into the sum of two fractions:

$$\frac{a+c}{b} = \frac{a}{b}+\frac{c}{b}$$

This is the trick we need to divide a polynomial by a monomial.

EXAMPLE 1: **Divide:** $\boldsymbol{\frac{ab^2+a^2b+ab}{ab}}$

Solution: Separate the fraction into three separate fractions, keeping the denominator for all three fractions:

$$\frac{ab^2}{ab} + \frac{a^2b}{ab} + \frac{ab}{ab}$$

and then simplify (reduce) each fraction:

$$\boxed{b+a+1}$$

EXAMPLE 2: **Divide:** $\boldsymbol{\frac{12x^2y^3-8x^3y^2+7xy^3}{2x^2y}}$

Solution: Split the fraction into three separate fractions:

$$\frac{12x^2y^3}{2x^2y} - \frac{8x^3y^2}{2x^2y} + \frac{7xy^3}{2x^2y}$$

and then simplify (reduce) each fraction:

$$6y^2 - 4xy + \frac{7y^2}{2x}$$

Homework

1. Perform each division problem, where the divisor (the bottom) is a monomial:

a. $\frac{x^3 - x^2 + x}{x}$ b. $\frac{14xy + 21x^2y - 28xy^2}{7xy}$ c. $\frac{x^2 + 3x + 1}{x}$

d. $\frac{a+b}{b}$ e. $\frac{x-y}{y}$ f. $\frac{ax + bx}{x}$

❑ *Dividing a Polynomial by a Polynomial*

Think back when you were a kid and learned long division of numbers. Though I've seen different ways of doing this, the following method works well in algebraic long division. It boils down to a 4-step process, a process that is repeated until the problem is finished:

1. Divide the divisor into the first part of the dividend.
2. Multiply the part of the quotient calculated in step 1 by the divisor.
3. Subtract.
4. Bring down the next digit.

And then repeat steps 1–4 as many times as necessary.

We use the same process for polynomial long division in algebra.

EXAMPLE 3: **Perform the long division:** $\dfrac{3x^3 - 5x - 2}{x - 1}$

Solution: The first step is to fill in the missing term in the dividend. Since there is no x^2 term, we insert $\mathbf{0x^2}$ as a "placeholder" between the cubic term and the linear term, giving us a dividend of $3x^3 + \mathbf{0x^2} - 5x - 2$. So our long division problem is

$$x - 1 \overline{)\, 3x^3 + 0x^2 - 5x - 2}$$

1. Divide x into $3x^3$; it goes in $3x^2$ times (since $3x^2 \cdot x = 3x^3$):

$$\begin{array}{r} \mathbf{3x^2} \\ \mathbf{x} - 1 \overline{)\, \mathbf{3x^3} + 0x^2 - 5x - 2} \end{array}$$

2. Multiply $3x^2$ by the divisor, $x - 1$:

$$\begin{array}{r} \mathbf{3x^2} \\ \mathbf{x} - \mathbf{1} \overline{)\, 3x^3 + 0x^2 - 5x - 2} \\ \underline{\mathbf{3x^3 - 3x^2}} \end{array}$$

3. Subtract; $3x^3 - 3x^3 = 0$; $0x^2 - (-3x^2) = 3x^2$:

$$\begin{array}{r} 3x^2 \\ x - 1 \overline{)\, \mathbf{3x^3} + \mathbf{0x^2} - 5x - 2} \\ \underline{-\,(\mathbf{3x^3 - 3x^2})} \\ \mathbf{0} \quad + \mathbf{3x^2} \end{array}$$

4. Bring down the next term, $-5x$:

$$\begin{array}{r} 3x^2 \\ x - 1 \overline{)\, 3x^3 + 0x^2 - \mathbf{5x} - 2} \\ \underline{-\,(3x^3 - 3x^2)} \\ 0 \quad + 3x^2 - \mathbf{5x} \end{array}$$

1. And repeat: Divide x into $3x^2$:

$$\begin{array}{r} 3x^2 \ \mathbf{+3x} \\ \mathbf{x-1}\overline{\left| 3x^3+0x^2-5x-2 \right.} \\ \underline{-(3x^3-3x^2)} \\ 0 \ \mathbf{+3x^2}-5x \end{array}$$

2. Multiply $3x$ by $x - 1$, the divisor:

$$\begin{array}{r} 3x^2 \ \mathbf{+3x} \\ \mathbf{x-1}\overline{\left| 3x^3+0x^2-5x-2 \right.} \\ \underline{-(3x^3-3x^2)} \\ 0 \ +3x^2-5x \\ \mathbf{3x^2-3x} \end{array}$$

3. Subtract; $3x^2 - 3x^2 = 0$; $-5x - (-3x) = -2x$:

$$\begin{array}{r} 3x^2 \ +3x \\ x-1\overline{\left| 3x^3+0x^2-5x-2 \right.} \\ \underline{-(3x^3-3x^2)} \\ 0 \ \mathbf{+3x^2-5x} \\ \underline{\mathbf{-(3x^2-3x)}} \\ \mathbf{0 \ -2x} \end{array}$$

4. Bring down the next (and last) term, –2:

$$\begin{array}{r} 3x^2 \ +3x \\ x-1\overline{\left| 3x^3+0x^2-5x-\mathbf{2} \right.} \\ \underline{-(3x^3-3x^2)} \\ 0 \ +3x^2-5x \\ \underline{-(3x^2-3x)} \\ 0 \ -2x-\mathbf{2} \end{array}$$

1. Divide x into $-2x$:

$$\begin{array}{r} 3x^2+3x-\mathbf{2} \\ \boldsymbol{x}-1\overline{\left|\,3x^3+0x^2-5x-2\right.} \\ \underline{-(3x^3-3x^2)} \\ 0\ +3x^2-5x \\ \underline{-(3x^2-3x)} \\ 0\ \ -\mathbf{2x}-2 \end{array}$$

2. Multiply -2 by $x-1$:

$$\begin{array}{r} 3x^2+3x-\mathbf{2} \\ \boldsymbol{x}-\mathbf{1}\overline{\left|\,3x^3+0x^2-5x-2\right.} \\ \underline{-(3x^3-3x^2)} \\ 0\ +3x^2-5x \\ \underline{-(3x^2-3x)} \\ 0\ \ -2x-2 \\ \underline{\quad -\mathbf{2x}+\mathbf{2}} \end{array}$$

3. Subtract; $-2x-(-2x)\ =\ 0$; $-2-(+2)\ =\ -4$

$$\begin{array}{r} 3x^2+3x-2 \\ x-1\overline{\left|\,3x^3+0x^2-5x-2\right.} \\ \underline{-(3x^3-3x^2)} \\ 0\ \ +3x^2-5x \\ \underline{-(3x^2-3x)} \\ 0\ \ -\mathbf{2x}-\mathbf{2} \\ \underline{\quad -(-\mathbf{2x}+\mathbf{2})} \\ \mathbf{0}\ \ -\mathbf{4} \end{array}$$

There are no terms left to bring down in the dividend, so we write the remainder (the -4) over the divisor and add it to the quotient. The final answer to the long division problem is

$$\boxed{3x^2+3x-2+\frac{-4}{x-1}}$$

Homework

2. Perform each polynomial long division problem, expressing any remainder as a fraction added to the quotient:

a. $\frac{x^2+5x+6}{x+3}$ b. $\frac{x^2-9}{x-3}$ c. $\frac{x^2+2x+1}{x+1}$

d. $\frac{n^2+n-4}{n+5}$ e. $\frac{2a^2-5a+2}{a+3}$ f. $\frac{3w^2+10}{w+5}$

g. $\frac{6b^2+b-15}{2b+3}$ h. $\frac{3y^2-9}{y+5}$ i. $\frac{10x^2+3x-7}{2x-1}$

j. $\frac{x^3+1}{x+1}$ Hint: $x^3 + 1 = x^3 + 0x^2 + 0x + 1$

k. $\frac{n^3-8}{n-2}$ l. $\frac{a^3+27}{a^2-3a+9}$

3. Perform each polynomial long division problem (Hint: there is no remainder):

a. $\frac{40x^3+97x^2+60x+27}{5x+9}$ b. $\frac{8w^3+22w^2+13w+2}{2w^2+5w+2}$

c. $\frac{40r^3-4r^2-7r-3}{8r^2+4r+1}$ d. $\frac{63m^3+43m^2+13m+1}{7m^2+4m+1}$

Review Problems

4. Divide: $\frac{4x^3 - 8x^2 + 6x - 10}{4x^2}$

5. Divide: $\frac{x^2 + 9}{x - 5}$

6. Divide: $\frac{x^3 - 3x + 8}{x + 3}$

7. Divide: $\frac{x^4 - 1}{x + 1}$

8. Divide: $\frac{n^3 + 8}{n + 2}$

Solutions

1. a. $x^2 - x + 1$ b. $2 + 3x - 4y$ c. $x + 3 + \frac{1}{x}$

d. $\frac{a}{b} + 1$ e. $\frac{x}{y} - 1$ f. $a + b$

2. a. $x + 2$ b. $x + 3$ c. $x + 1$

d. $n - 4 + \frac{16}{n + 5}$ e. $2a - 11 + \frac{35}{a + 3}$ f. $3w - 15 + \frac{85}{w + 5}$

g. $3b - 4 + \frac{-3}{2b + 3}$ h. $3y - 15 + \frac{66}{y + 5}$ i. $5x + 4 + \frac{-3}{2x - 1}$

j. $x^2 - x + 1$ k. $n^2 + 2n + 4$ l. $a + 3$

3. a. $8x^2 + 5x + 3$ b. $4w + 1$ c. $5r - 3$

d. $9m + 1$

4. $x-2+\frac{3}{2x}-\frac{5}{2x^2}$

5. $x+5+\frac{34}{x-5}$

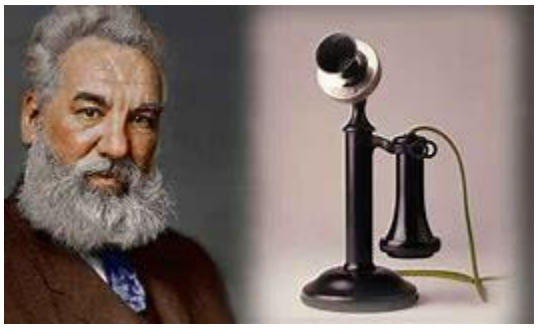
6. $x^2-3x+6+\frac{-10}{x+3}$

7. x^3-x^2+x-1

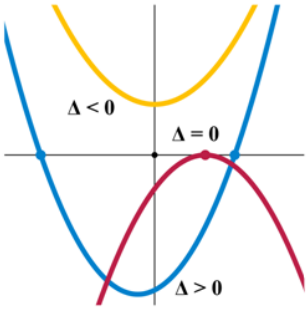
8. n^2-2n+4

“When one door closes, another opens; but we often look so long and so regretfully upon the closed door that we do not see the one which has opened for us.”

- Alexander Graham Bell



POLYNOMIALS



It might be difficult to define a polynomial at this point in your algebra studies, depending on what you've learned about exponents. Suffice it to say that a typical polynomial expression looks like

$$3x^5 - \pi x^3 + x^2 - 9x + 0.8$$ [Note: 5 terms]

The central theme of a ***polynomial*** is that all of the exponents on the *x* (or whatever variable) must be one of the whole numbers 0, 1, 2, 3, …. The following are <u>not</u> polynomials: $8x^{-2}$ and $3x^{1/4}$, because the exponents –2 and 1/4 are not whole numbers.

❑ *WORKING WITH MONOMIALS*

A polynomial with one term, such as $7x^3$, is called a ***monomial***. Let's look at a couple of examples of multiplying monomials together, a skill we'll need when we multiply binomials together (Can you guess what a <u>bi</u>nomial is?). The key to multiplying monomials is that each monomial is a single term whose final operation is multiplication.

For example, to find the product of $7x$ and $9x$, we proceed the long way — you don't ever actually do it this way, but it's important to see:

	$(7x)(9x)$	(the original expression)
=	$(7 \cdot x) \cdot (9 \cdot x)$	(it's all multiplication)
=	$(7 \cdot 9) \cdot (x \cdot x)$	(rearrange the factors)
=	$63 \cdot x^2$	(something times itself is squaring)
=	$\mathbf{63x^2}$	(remove the dot)

Another example is $3(-10n) = (3)(-10)n = \mathbf{-30n}$.

And don't forget that adding and subtracting don't follow the same rules as multiplication. Two monomials can be added or subtracted only if they are *like terms*. See if the homework sorts all of this out.

Homework

1. Simplify each expression:

a. $3(7L)$	b. $-5(2x)$	c. $-6(-2T)$
d. $20(-3w)$	e. $3 + 7L$	f. $-5 + 2x$
g. $-6 - 2T$	h. $20 - 3w$	i. $(7y)5$
j. $(-2p)(-5)$	k. $(-3a)(10)$	l. $(5n)(-2)$
m. $7y + 5$	n. $(4x)(3x)$	o. $4x + 3x$
p. $(2n)(-3n)$	q. $2n - 3n$	r. $(-8x)(-7x)$
s. $(7u)(-u)$	t. $(-4c)(4c)$	u. $-4c + 4c$
v. $(7m)(6n)$	w. $7m - 6n$	x. $(13k)(-13k)$
y. $13k - 13k$	z. $-14x + 20x$	

2. Find the **product** of

a. $3x$ and 4	b. $7y$ and 5	c. $-3n$ and 2
d. $2n$ and -10	e. 6 and 7	f. -8 and -8
g. z and z	h. $2n$ and $3n$	i. $7m$ and $-2m$
j. $2p$ and $-2p$	k. $3a$ and $-5a$	l. $-3a$ and -7
m. -10 and 10	n. $-8g$ and -10	o. 4 and $-12x$

❑ *More Like Terms*

To simplify the 6-term polynomial

$$5x^2 + 7x - 8 - 2x^2 - 9x - 20$$

we can rearrange the terms like this (since addition is *commutative*):

$$5x^2 - 2x^2 + 7x - 9x - 8 - 20$$ (This step is optional)

and then combine like terms to reach a result of 3 terms:

$$\mathbf{3x^2 - 2x - 28}$$ ⟵ This is called a **trinomial**.

For a second example, let's simplify an expression containing parentheses:

$$3(y^2 - 8y + 9) - (3y^2 - 7y + 1)$$ (This polynomial has 2 terms)

This next step is optional, but use it if you need to:

$$3(y^2 - 8y + 9) - \mathbf{1}(3y^2 - 7y + 1)$$

Now distribute:

$$3y^2 - 24y + 27 - 3y^2 + 7y - 1$$

Rearrange the terms into pairs of like terms:

$$3y^2 - 3y^2 - 24y + 7y + 27 - 1$$ (This step is optional)

And then combine them:

$$\mathbf{-17y + 26}$$ ⟵ This is called a **binomial**.

Homework

3. Suppose a friend believes that $4n^2$ and $7n$ are like terms, and that their sum should be $11n^3$. Prove your friend wrong by letting $n = 2$.

4. Simplify each expression by combining like terms:

a. $3x^2 - 7x + 5x^2 + 9$

b. $n^2 - 9 + 9 - n^2$

c. $1 - 3u - u^2 - 3u^2 + 7u - 1$

d. $7a^2 - 8a + 7 - 9a^2 + 7a - 7$

e. $x^2 - 3x - 1 + 7x^2 - 3x + 1$

f. $3y^2 - 2 + 3y^2 - 2$

g. $1 - 3x - x^2 + 5 - 7x + x^2$

h. $-5w^2 + 2 - 3w + 8w - 2 - w^2$

5. Simplify each expression by distributing and then combining like terms:

a. $(3c^2 - 2c - 1) + 2(c^2 + 5c - 7)$

b. $3(x^2 - 8x + 1) - 5(2x^2 + 7x - 1)$

c. $-(a^2 - a - 1) + 3(-a^2 + a)$

d. $7w^2 - 13w + 8 - (5w^2 - 3w - 2)$

e. $-(7u^2 - 7u - 6) - (-6u^2 + 3u + 5)$

f. $(3x^2 - x - 1) - (3x^2 - x - 1)$

g. $-2(x^2 - 3x + 7) - (3x^2 + 10x - 1)$

h. $-(3n^2 + 8n - 1) - 3(n^2 + 2n - 1)$

❑ *MORE DISTRIBUTING*

A polynomial with one term is called a ***monomial***, and a polynomial with two terms is called a ***binomial***. A typical problem where we must multiply a monomial by a binomial is the following:

$3x(2x + 10)$ ($3x$ is the monomial; $2x + 10$ is the binomial)

Finding the product of these two polynomials is pretty easy — just distribute:

$3x(2x + 10)$

$= (3x)(2x) + (3x)(10)$

$= \mathbf{6x^2 + 30x}$, and it's done

Homework

6. Simplify each expression by distributing:

a. $9a(a + 9)$ b. $10x(3x - 1)$ c. $-5y(1 - y)$

d. $n(5n + 1)$ e. $5a(5a + 5)$ f. $3u(7u - 8)$

g. $-5m(m + 1)$ h. $-3t(10t + 12)$ i. $-(6x^2 - 9)$

Review Problems

7. Simplify each expression:

a. $4(3n)$ b. $-8(3x)$ c. $2b + 3b$

d. $(8y)(-2y)$ e. $8y - 2y$ f. $(2x)(-3z)$

8. Find the product of

a. $7x$ and 4 b. $3a$ and $5a$ c. $2n$ and $3m$

d. 7 and -7 e. 3 and $-13t$ f. $7x$ and $-7x$

9. Simplify each expression:

a. $7x^2 - 3x + 7 - 7x^2 - 3x - 7$ b. $-8(3y^2 - 4y - 1)$

c. $2(a^2 - 8) - (a^2 - 2a - 1)$ d. $-(4n^2 - 4n) - (4n - 4n^2)$

e. $3(4g^2 - g + 3) - 2(6g^2 + g - 1)$ f. $-8x(-7x - 5)$

Solutions

1. a. $21L$ b. $-10x$ c. $12T$ d. $-60w$ e. As is f. As is
g. As is h. As is i. $35y$ j. $10p$ k. $-30a$ l. $-10n$
m. As is n. $12x^2$ o. $7x$ p. $-6n^2$ q. $-n$ r. $56x^2$
s. $-7u^2$ t. $-16c^2$ u. 0 v. $42mn$ w. As is x. $-169k^2$
y. 0 z. $6x$

2. a. $12x$ b. $35y$ c. $-6n$ d. $-20n$ e. 42 f. 64
g. z^2 h. $6n^2$ i. $-14m^2$ j. $-4p^2$ k. $-15a^2$ l. $21a$
m. -100 n. $80g$ o. $-48x$

3. $4n^2 + 7n = 4(\mathbf{2})^2 + 7(\mathbf{2}) = 4(4) + 7(2) = 16 + 14 = 30$,
whereas $11n^3 = 11(\mathbf{2})^3 = 11(8) = 88$
Therefore, $4n^2 + 7n \neq 11n^3$

4. a. $8x^2 - 7x + 9$ b. 0 c. $-4u^2 + 4u$ d. $-2a^2 - a$
e. $8x^2 - 6x$ f. $6y^2 - 4$ g. $-10x + 6$ h. $-6w^2 + 5w$

5. a. $5c^2 + 8c - 15$ b. $-7x^2 - 59x + 8$ c. $-4a^2 + 4a + 1$
d. $2w^2 - 10w + 10$ e. $-u^2 + 4u + 1$ f. 0
g. $-5x^2 - 4x - 13$ h. $-6n^2 - 14n + 4$

6. a. $9a^2 + 81a$ b. $30x^2 - 10x$ c. $-5y + 5y^2$
d. $5n^2 + n$ e. $25a^2 + 25a$ f. $21u^2 - 24u$
g. $-5m^2 - 5m$ h. $-30t^2 - 36t$ i. $-6x^2 + 9$

7. a. $12n$ b. $-24x$ c. $5b$ d. $-16y^2$ e. $6y$ f. $-6xz$

8. a. $28x$ b. $15a^2$ c. $6mn$ d. -49 e. $-39t$ f. $-49x^2$

9. a. $-6x$ b. $-24y^2 + 32y + 8$ c. $a^2 + 2a - 15$
d. 0 e. $-5g + 11$ f. $56x^2 + 40x$

❑ *TO ∞ AND BEYOND*

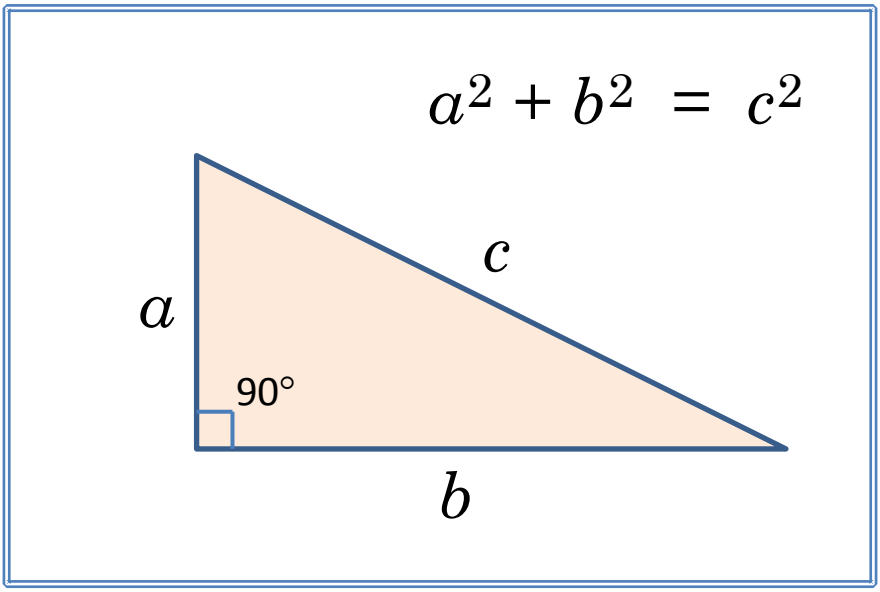
If n is any whole number ($n = 0, 1, 2, 3, \ldots$), prove that the polynomial $n^2 + 5n + 6$ is <u>always</u> an <u>even</u> number.

"Formal education is but an incident in the lifetime of an individual. Most of us who have given the subject any study have come to realize that education is a continuous process ending only when ambition comes to a halt."

– R.I. Rees

THE PYTHAGOREAN THEOREM – FINDING A LEG

❑ *RECALLING THE PYTHAGOREAN THEOREM*



The 90° angle is called the **right** angle of the right triangle. The other two angles of the right triangle are called the **acute** angles, because each of them must be less than 90°. The **legs** are *a* and *b*, and the **hypotenuse** is *c*.

EXAMPLE 1: **One leg of a right triangle is 7, while the hypotenuse is 25. Find the length of the other leg.**

Solution: Prior to this chapter we were always given the two legs and asked to find the hypotenuse. This problem throws us a curve, but we now have the skills needed to solve it. Since the

sum of the squares of the legs must equal the square of the hypotenuse, we start with the Pythagorean Theorem,

$a^2 + b^2 = c^2$ *a* and *b* are legs; *c* is the hypotenuse

Putting in a leg of 7 and a hypotenuse of 25, the formula becomes

	$a^2 + \mathbf{7}^2 = \mathbf{25}^2$	(the leg can be either *a* or *b*)
$\Rightarrow$	$a^2 + 49 = 625$	(square the numbers)
$\Rightarrow$	$a^2 + 49 - \mathbf{49} = 625 - \mathbf{49}$	(subtract 49 from each side)
$\Rightarrow$	$a^2 = 576$	(simplify each side)
$\Rightarrow$	$a = 24$	(calculator or guessing)

We conclude that the other leg has a length of $\boxed{24}$

Check: If we're claiming that a right triangle can have legs of 24 and 7 and a hypotenuse of 25, then these three values had better satisfy the Pythagorean Theorem. Let's see if they do:

$$(\text{leg})^2 + (\text{leg})^2 \stackrel{?}{=} (\text{hypotenuse})^2$$
$$24^2 + 7^2 \stackrel{?}{=} 25^2$$
$$576 + 49 \stackrel{?}{=} 625$$
$$625 = 625 \checkmark$$



EXAMPLE 2: **Find the hypotenuse of a right triangle whose legs are 5 and 7.**

Solution:

$a^2 + b^2 = c^2$	(the Pythagorean Theorem)
$\mathbf{5}^2 + \mathbf{7}^2 = c^2$	(substitute the known values)
$25 + 49 = c^2$	(square each leg)
$74 = c^2$	(simplify)

Is there a whole number whose square is 74? No, because $8^2 = 64$, which is smaller than 74, while $9^2 = 81$, which is bigger than 74. We see, therefore, that the solution is somewhere between the consecutive numbers 8 and 9; but where between 8 and 9?

$8^2 = 64$ ✓
$what^2 = 74$??
$9^2 = 81$ ✓

Finding a number whose square is 74 is the kind of problem that has plagued and enticed mathematicians, scientists, and philosophers for literally thousands of years. They'd really be mad if they knew that we can find an excellent approximation of this number using a cheap calculator. Enter the number 74 followed by the *square root* key (or the other way around, depending on the calculator), which is labeled something like $\sqrt{x}$. Thus, the hypotenuse is $\sqrt{74}$ (read: *the positive square root of 74*), and your calculator should have the result 8.602325267, or something close to that.

But your calculator doesn't tell the whole story. The fact is, the square root of 74 has an <u>infinite</u> number of digits following the decimal point, and they <u>never</u> have a repeating pattern (very similar to the number π). Thus, we'll have to round off the answer to whatever's appropriate for the problem. For this problem, we'll round to the third digit past the point.

The number $\sqrt{74}$ is called **irrational**, due to its infinite, non-repeating decimal expansion.

The hypotenuse is 8.602

Notice that $8.602^2 = 73.994404$, which is quite close to 74.

EXAMPLE 3: **The hypotenuse of a right triangle is 15 and one of its legs is 4. Find the length of the other leg (rounded to 3 digits).**

Solution: Let's get right to it:

$$a^2 + b^2 = c^2$$

$$\Rightarrow \quad \mathbf{4}^2 + b^2 = \mathbf{15}^2$$

$$\Rightarrow \quad 16 + b^2 = 225$$

$$\Rightarrow \quad b^2 = 209$$

$$\Rightarrow \quad b = \boxed{14.457} \text{ (approximately)}$$

This result was found using a calculator; if no calculator is allowed, then you'd write the answer like this:

The leg is between 14 and 15.

Homework

1. In each problem, one leg and the hypotenuse of a right triangle are given (l = leg and h = hypotenuse). Find the other leg — NO calculator.

a. $l = 3$; $h = 5$ b. $l = 5$; $h = 13$ c. $l = 6$; $h = 10$

d. $l = 9$; $h = 41$ e. $l = 12$; $h = 13$ f. $l = 24$; $h = 25$

2. In each problem, one leg and the hypotenuse of a right triangle are given (l = leg and h = hypotenuse). Use your calculator to find the other leg, rounded to 3 digits. If no calculator is allowed, tell what two consecutive whole numbers the answer is between.

   a. $l = 3$; $h = 10$ b. $l = 2$; $h = 9$ c. $l = 5$; $h = 6$

   d. $l = 4$; $h = 20$ e. $l = 7$; $h = 19$ f. $l = 13$; $h = 55$

3. A triangle has sides of 6, 10, and 8. Prove that it's a *right triangle*. Hint: See the Check in Example 1.

4. A triangle has sides of 5, 9, and 7. Prove that it's not a *right triangle*.

   [Note: Whether you're trying to prove that a triangle is – or is not – a right triangle, logic dictates that you must assume that the longest of the three given sides is the hypotenuse.]

Review Problems

5. The hypotenuse of a right triangle is 65 and one of its legs is 60. Find the other leg.
6. Use a calculator (if allowed) to find the leg of a right triangle if its hypotenuse is 89 and one of its legs is 1. Round to the 3rd digit.
7. Verify that a triangle with sides 24, 10, and 26 is a right triangle.
8. Is a triangle with sides 15, 12, and 8 a right triangle?

Solutions

1. a. 4 b. 12 c. 8 d. 40
e. 5 f. 7

2. a. 9.539 b. 8.775 c. 3.317 d. 19.596
e. 17.664 f. 53.442

3. Verify that $6^2 + 8^2 = 10^2$.

4. Show that $5^2 + 7^2 \neq 9^2$. Notice that even though the 9 was listed second in the list of the three sides of the triangle, we used the 9 as the *potential* hypotenuse.

5. 25 **6**. 88.99

7. Verify that $10^2 + 24^2 = 26^2$.

8. No; be sure you assume that the hypotenuse would be the 15.

"It is possible to store the mind with a million facts and still be entirely uneducated."

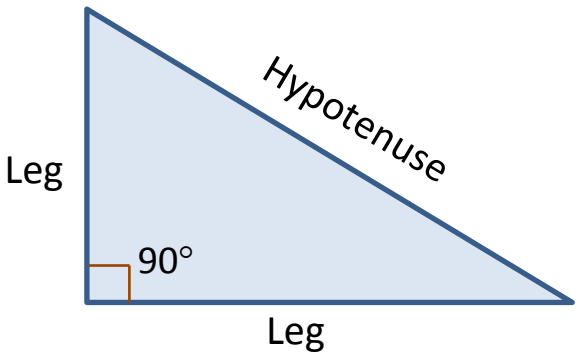
– Alec Bourne, M.D.

THE PYTHAGOREAN THEOREM – FINDING THE HYPOTENUSE

❑ THE RIGHT TRIANGLE

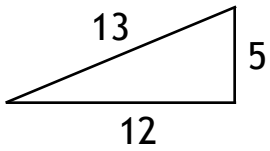
An angle of 90° is called a ***right angle***, and when two things meet at a right angle, we say they are ***perpendicular***. For example, the angle between a flagpole and the ground is 90°, and so the flagpole is perpendicular to the ground.

If we have a *triangle* with a 90° angle in it, we call the triangle a ***right triangle***. The two sides which form the right angle (90°) are called the ***legs*** of the right triangle, and the side opposite the right angle is called the ***hypotenuse***. Is it clear that the hypotenuse is always the longest side of the right triangle?



❑ THE PYTHAGOREAN THEOREM

Ancient civilizations discovered that a triangle with sides 5, 12, and 13 (whether the unit of measurement was inches, cubits, or anything) would always be a right triangle — that is, a triangle with a 90° angle in it. [Note that the hypotenuse must be the side of length 13.] But what if just the two legs are known? Is there a way to calculate the length of the hypotenuse? The answer is yes, and the formula dates back to 500 BC, the time of Pythagoras.



A Classic Right Triangle



To discover this formula, let's rewrite the three sides of the above right triangle:

leg = 5, leg = 12, hypotenuse = 13

Since $5 + 12 \neq 13$, it's clear that the hypotenuse is not simply the sum of the two legs. [See the sidebar.]

Here's the secret: Use the idea of squaring. If we square the 5, the 12, and the 13, we get

$5^2 = 25$ $12^2 = 144$ $13^2 = 169$

and we notice (if we're as clever as Pythagoras was!) that the sum of 25 and 144 is 169:

$$25 + 144 = 169$$

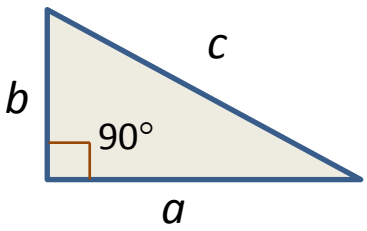
In other words, a triangle with sides 5, 12, and 13 forms a right triangle precisely because

$$5^2 + 12^2 = 13^2$$

Now let's try to express this relationship in words — it appears that

When you square the legs of a right triangle and add them together, you get the square of the hypotenuse.

As a formula, we can state it this way:



If a and b are the legs of a right triangle and c is the hypotenuse, then

$$a^2 + b^2 = c^2$$

It should be clear that it can't possibly be the case that adding the two legs gives the hypotenuse. After all, the hypotenuse is the shortest distance between the two corners (vertices) of the triangle, so it must be less than "going the long way" along the legs.

❑ *SOLVING RIGHT TRIANGLES*

EXAMPLE 1: **The legs of a right triangle are 6 m and 8 m. Find the hypotenuse.**

Solution: We begin by writing the Pythagorean Theorem. Then we plug in the known values, and finally determine the hypotenuse of the triangle.

$a^2 + b^2 = c^2$	(the Pythagorean Theorem)
$\mathbf{6}^2 + \mathbf{8}^2 = c^2$	(substitute the known values)
$36 + 64 = c^2$	(square each leg)
$100 = c^2$	(simplify)

This last statement is called an **equation**, and states that the square of some number is 100. Well, what positive number, when squared, results in 100? A little experimentation yields the solution 10 (since $10^2 = 100$). Our conclusion is that $c = 10$, and thus,

The hypotenuse is 10 m

Note: The equation $100 = c^2$ has another solution: $c = -10$ [since $(-10)^2 = 100$]. But a length cannot be negative, so we discard it and keep just the positive solution.

Homework

1. In each problem, the two legs of a right triangle are given. Find the hypotenuse. Ask your teacher whether you may use a calculator.

a. 3, 4	b. 5, 12	c. 10, 24	d. 30, 16
e. 7, 24	f. 12, 16	g. 30, 40	h. 9, 40
i. 12, 35	j. 20, 21	k. 48, 55	l. 13, 84

2. In each problem, the length and width in inches of an LCD TV screen are given. Find the diagonal measure of the screen.

   a. $l = 8$, $w = 6$

   b. $l = 48$, $w = 14$

   c. $l = 48$, $w = 20$

3. In each problem, two traveling distances in miles are given. How many miles is the person from his starting point?

   a. first 9 miles North; then 40 miles West

   b. first 7 miles South; then 24 miles East

   c. first 10 miles East; then 24 miles South

   d. first 40 miles West; then 42 miles North

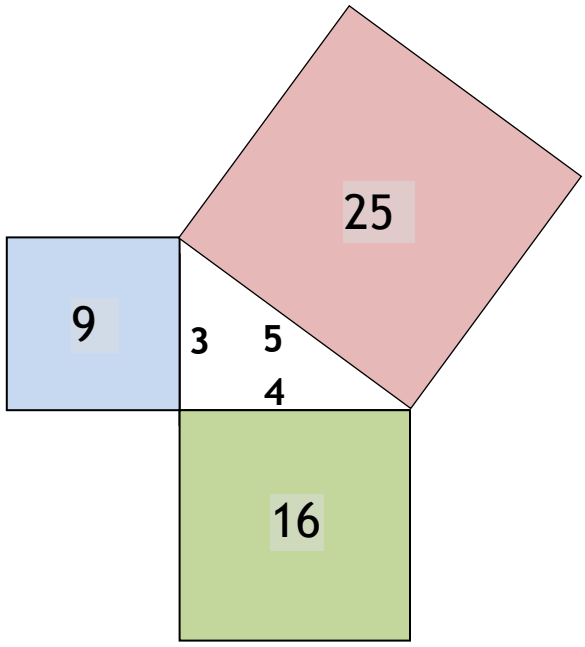
❑ *FINAL NOTES*

A Greek Perspective on the Right Triangle

As stated in Book I, Proposition 47, of Euclid's *Elements*:

> *"In right-angled triangles, the square on the side subtending the right angle is equal to the squares on the sides containing the right angle."*

Notice that the Pythagorean Theorem is stated geometrically — it's not the algebraic formula we wrote. For example, consider the following right triangle with squares attached to each of the three sides:



The Greeks would <u>not</u> have claimed the numerical statement $3^2 + 4^2 = 5^2$, but instead that the **<u>areas</u>** <u>of the squares</u> attached to the legs (the 9 and the 16) add up to the **<u>area</u>** <u>of the square</u> attached to the hypotenuse (the 25).

Using a Right Triangle in Construction

To see if a wall is truly vertical (straight up) relative to the floor, a contractor will do something like the following: Take a board (like a two-by-four) and lean it (diagonally) against the wall from some spot on the floor. Assume that the length of the board is 9 ft, that the distance from the bottom of the board to the wall (along the floor) is 6 ft., and that the board touches the wall 7 ft. above the floor. If the floor and the wall are truly perpendicular, then the numbers 6, 7, and 9 had better satisfy the Pythagorean Theorem (thus guaranteeing a 90° angle, indicating a vertical wall):

$6^2 + 7^2 = 36 + 49 =$ **85**, whereas $9^2 =$ **81**.

We do NOT have a right triangle, and so the angle between the floor and the wall is NOT 90° — time for some reconstruction.

Review Problems

4. The legs of a right triangle are 15 and 36. Find the hypotenuse.

5. The legs of a right triangle are 44 and 483. Find the hypotenuse. [You may use a calculator on this one.]

6. The length and width of an LCD TV set are 42 in and 40 in. Find the diagonal measure of the screen.

7. Maria walks 7 km north and then 24 km west. How far is Maria from her starting point?

8. Why should you never argue with a 90° angle?

Solutions

1. a. 5 b. 13 c. 26 d. 34 e. 25 f. 20
g. 50 h. 41 i. 37 j. 29 k. 73 l. 85

2. a. 10 in b. 50 in c. 52 in

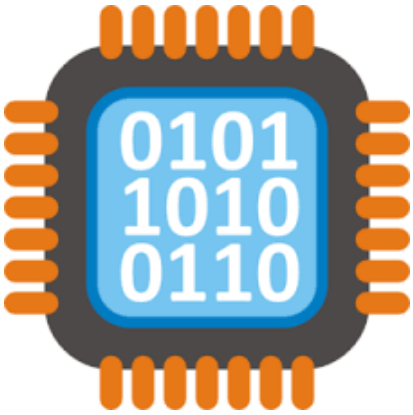
3. a. 41 mi b. 25 mi c. 26 mi d. 58 mi

4. 39 **5**. 485 **6**. 58 in **7**. 25 km

8. Because they're always *right*!

The Pythagorean Theorem With Radicals

The last time we studied the Pythagorean Theorem we may have used our calculator to round off square roots that didn't come out whole numbers. But rounding off square roots is not always a good idea in the real world.



After we've done some applied mathematics, we may give the results to an engineer. Suppose we've rounded to the hundredths place (good enough for business), but the engineer is designing the Core 9 Quad microprocessor, and she needs all the digits of the calculations out to the billionths place. We haven't done her any good at all by rounding off for her. The safest thing to do is to give her the exact results and let her round off according to the requirements of her project.

We're now going to work with the Pythagorean Theorem again, but this time we're going to leave our answers exact. To this end, we need to review the key concept regarding square roots.

❑ *AN IMPORTANT FACT ABOUT SQUARE ROOTS*

If we square the square root of "something" (that's not negative), we get the "something":

$$(\sqrt{25})^2 = 5^2 = 25$$

$$(\sqrt{34})^2 \approx (5.830951895)^2 \approx 34$$

And we can write the general formula:

$$\left(\sqrt{x}\right)^2 = x$$

Squaring Undoes Square-Rooting

We assume $x \geq 0$, lest we take the square root of a negative number, resulting in a non-real number.

Another skill we'll need for this chapter is the ability to simplify an expression like $(5\sqrt{3})^2$. Recall one of our five laws of exponents: $(ab)^2 = a^2b^2$.

This rule means that we can simplify $(5\sqrt{3})^2$ as follows:

$$(5\sqrt{3})^2 = 5^2(\sqrt{3})^2 = 25 \cdot 3 = \mathbf{75}$$

Homework

1. While it's the case that $(ab)^2 = a^2b^2$, is this rule true for addition? That is, does $(a + b)^2$ always equals $a^2 + b^2$?

2. Simplify each expression:

a. $(\sqrt{7})^2$ b. $(\sqrt{29})^2$ c. $(\sqrt{233})^2$ d. $(\sqrt{101})^2$

e. $(2\sqrt{3})^2$ f. $(5\sqrt{5})^2$ g. $(9\sqrt{2})^2$ h. $(12\sqrt{7})^2$

3. Example: $\sqrt{200} = \sqrt{100 \cdot 2} = \sqrt{100} \cdot \sqrt{2} = \mathbf{10\sqrt{2}}$

Simplify each radical:

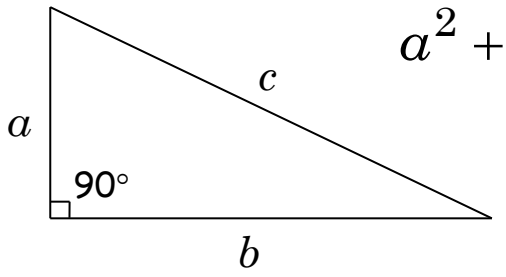
a. $\sqrt{144}$ b. $\sqrt{225}$ c. $\sqrt{1}$ d. $\sqrt{0}$

e. $\sqrt{-49}$ f. $\sqrt{14}$ g. $\sqrt{200}$ h. $\sqrt{45}$

i. $\sqrt{98}$ j. $\sqrt{841}$ k. $\sqrt{48}$ l. $\sqrt{112}$

❑ *SOLVING RIGHT TRIANGLES*

Recall the Pythagorean Theorem:



$$a^2 + b^2 = c^2$$

EXAMPLE 1: **The legs are 6 and 10. Find the hypotenuse.**

$$a^2 + b^2 = c^2 \Rightarrow \mathbf{6}^2 + \mathbf{10}^2 = c^2 \Rightarrow 36 + 100 = c^2$$
$$\Rightarrow c^2 = 136 \Rightarrow c = \sqrt{136} = \sqrt{4 \cdot 34} = \mathbf{2\sqrt{34}}$$

Notice that $c^2 = 136$ is a *quadratic equation*, and therefore probably has two solutions; in fact, the solutions are $\pm\sqrt{136}$, or $\pm 2\sqrt{34}$. But in this problem, we're talking about the hypotenuse of a triangle, whose length must be positive. So we immediately discard the negative solution and retain the positive one.

EXAMPLE 2: **One leg is 20 and the hypotenuse is 30. Find the other leg.**

$$a^2 + b^2 = c^2 \Rightarrow \mathbf{20}^2 + b^2 = \mathbf{30}^2$$

$$\Rightarrow 400 + b^2 = 900 \Rightarrow b^2 = 500$$

$$\Rightarrow b = \sqrt{500} = \sqrt{100 \cdot 5} = \mathbf{10\sqrt{5}}$$

EXAMPLE 3: **The legs are $2\sqrt{7}$ and $3\sqrt{13}$. Find the hypotenuse.**

$$a^2 + b^2 = c^2 \Rightarrow (\mathbf{2\sqrt{7}})^2 + (\mathbf{3\sqrt{13}})^2 = c^2$$

$$\Rightarrow 28 + 117 = c^2 \Rightarrow c^2 = 145 \Rightarrow c = \mathbf{\sqrt{145}}$$

EXAMPLE 4: **The hypotenuse is $\sqrt{72}$ and one of the legs is 8. Find the other leg.**

$$a^2 + b^2 = c^2$$

$$\Rightarrow a^2 + \mathbf{8}^2 = (\mathbf{\sqrt{72}})^2$$

$$\Rightarrow a^2 + 64 = 72$$

$$\Rightarrow a^2 = 8$$

$$\Rightarrow a = \sqrt{8} = \sqrt{4 \cdot 2} = \mathbf{2\sqrt{2}}$$

EXAMPLE 5: **One of the legs is $3\sqrt{2}$ and the hypotenuse is $\sqrt{42}$. Find the other leg.**

$$a^2 + b^2 = c^2 \Rightarrow a^2 + (\mathbf{3\sqrt{2}})^2 = (\sqrt{\mathbf{42}})^2$$
$$\Rightarrow a^2 + 18 = 42$$
$$\Rightarrow a^2 = 24$$
$$\Rightarrow a = \sqrt{24} = \mathbf{2\sqrt{6}}$$

Homework

Calculate the EXACT answer to each Pythagorean problem:

4. The legs of a right triangle are 8 and 12. Find the hypotenuse.
5. The legs of a right triangle are 6 and 9. Find the hypotenuse.
6. Find the hypotenuse of a right triangle given that its legs have lengths of 5 and 7.
7. One leg of a right triangle is 15 and the hypotenuse is 20. Find the other leg.
8. The hypotenuse of a right triangle is 18 and one of its legs is 10. Find the other leg.
9. One leg of a right triangle is 5 and the hypotenuse is $\sqrt{43}$. Find the other leg.
10. Find the missing leg of a right triangle if the hypotenuse is $\sqrt{18}$ and one of its legs is 3.
11. The legs of a right triangle are $2\sqrt{3}$ and $3\sqrt{2}$. Find the hypotenuse.
12. The hypotenuse of a right triangle is $5\sqrt{7}$ and one of its legs is $2\sqrt{3}$. Find the length of the other leg.

13. A leg of a right triangle is $2\sqrt{8}$ and its hypotenuse is 10. Find the length of the other leg.

14. The leg of a right triangle is 5 and its hypotenuse is 4. Find the other leg.

Review Problems

15. In each problem, two of the sides of a right triangle are given (l = leg and h = hypotenuse). Find the third side.

a. $l = 7;\ l = 10$

b. $l = 1;\ l = 3$

c. $l = 4;\ l = 6$

d. $l = \sqrt{8};\ l = 2\sqrt{2}$

e. $l = 4;\ l = \sqrt{10}$

f. $l = \sqrt{2};\ l = \sqrt{14}$

g. $l = 5;\ h = 12$

h. $l = \sqrt{5};\ h = 2\sqrt{3}$

i. $l = 6;\ h = 5\sqrt{2}$

j. $l = \sqrt{7};\ h = \sqrt{31}$

k. $l = \sqrt{2};\ h = 2\sqrt{5}$

l. $l = \sqrt{5};\ h = 4\sqrt{5}$

m. $l = \sqrt{2};\ l = \sqrt{7}$

n. $l = 7;\ h = 8$

o. $l = \sqrt{5};\ h = \sqrt{13}$

p. $l = 5;\ h = \sqrt{26}$

q. $l = \sqrt{10};\ h = \sqrt{41}$

r. $l = \sqrt{11};\ h = 8$

s. $l = \sqrt{13};\ l = \sqrt{13}$

t. $l = 5;\ l = \sqrt{6}$

u. $l = 4;\ l = 5$

v. $l = 1;\ h = 2$

w. $l = 1;\ l = 1$

x. $l = 2\sqrt{2};\ h = 2\sqrt{5}$

y. $l = \sqrt{5};\ h = \sqrt{21}$

z. $l = 3\sqrt{2};\ h = 20$

Solutions

1. NO! You should know quite well that

$$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ba + b^2 = \underline{a^2 + 2ab + b^2}$$

2. a. 7 b. 29 c. 233 d. 101
e. 12 f. 125 g. 162 h. 1008

3. a. 12 b. 15 c. 1 d. 0
e. Not a Real Number f. $\sqrt{14}$ g. $10\sqrt{2}$ h. $3\sqrt{5}$
i. $7\sqrt{2}$ j. 29 k. $4\sqrt{3}$ l. $4\sqrt{7}$

4. $4\sqrt{13}$ **5**. $3\sqrt{13}$ **6**. $\sqrt{74}$ **7**. $5\sqrt{7}$ **8**. $4\sqrt{14}$ **9**. $3\sqrt{2}$

10. 3 **11**. $\sqrt{30}$ **12**. $\sqrt{163}$ **13**. $2\sqrt{17}$

14. This scenario is impossible, since the hypotenuse of a right triangle must be longer than either of its legs. Here's a mathematical proof:

$a^2 + b^2 = c^2$

$\Rightarrow$ $a^2 + 5^2 = 4^2$ (presuming that the leg is 5 and the hypotenuse is 4)

$\Rightarrow$ $a^2 + 25 = 16$

$\Rightarrow$ $a^2 = -9$ (subtract 25 from each side of the equation)

$\Rightarrow$ $a = \sqrt{-9}$, which is not a number in this class. And even if it becomes a number in a later class, it still couldn't be considered the length of a side of a triangle.

15. a. $\sqrt{149}$ b. $\sqrt{10}$ c. $2\sqrt{13}$ d. 4 e. $\sqrt{26}$

f. 4 g. $\sqrt{119}$ h. $\sqrt{7}$ i. $\sqrt{14}$ j. $2\sqrt{6}$

k. $3\sqrt{2}$ l. $5\sqrt{3}$ m. 3 n. $\sqrt{15}$ o. $2\sqrt{2}$

p. 1 q. $\sqrt{31}$ r. $\sqrt{53}$ s. $\sqrt{26}$ t. $\sqrt{31}$

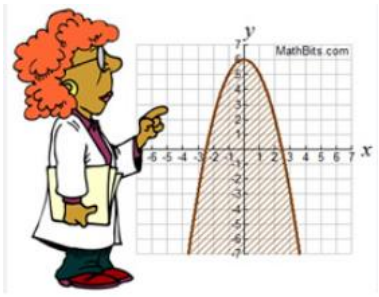
u. 3 v. $\sqrt{3}$ w. $\sqrt{2}$ x. $2\sqrt{3}$ y. 4

z. $\sqrt{382}$

"He who opens a school door, closes a prison."

– Victor Hugo

QUADRATIC INEQUALITIES



For linear inequalities such as

$$7n + 4 \geq 7,$$

the Basic Principle of Inequalities (If you multiply or divide by a negative number, *switch* the inequality sign) is sufficient. But other types of inequalities are best solved by a technique called the Boundary Point Method. This will help us solve inequalities such as

$$x^2 - 3x - 10 \geq 0.$$

❑ *THE BOUNDARY POINT METHOD*

1) Change the *inequality* to an *equation* and solve the equation.

2) The solutions of that equation are called the *boundary points*.

3) Plot the boundary points on the real line. This will create one or more *intervals* on the line.

4) Now choose a *test point* from each interval (any convenient number in the interval) and plug it into the <u>original inequality</u>. If the test point satisfies the inequality, then the entire interval in which the test point lies <u>is</u> a part of the solution of the inequality. If the test point does not satisfy the inequality, then the associated interval is <u>not</u> part of the solution.

5) Lastly, check the boundary points themselves in the inequality.

❑ *SOLVING QUADRATIC INEQUALITIES*

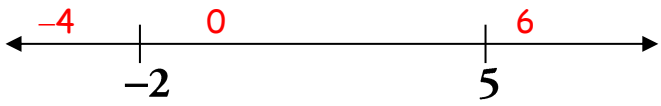
EXAMPLE 1: **Solve for x: $x^2 - 3x - 10 \geq 0$**

Solution: We will solve this kind of problem using the Boundary Point Method (there are other ways). First we find the boundary points:

$$x^2 - 3x - 10 = 0 \Rightarrow (x + 2)(x - 5) = 0 \Rightarrow x = \mathbf{-2} \text{ or } x = \mathbf{5}.$$

Now we mark these boundary points on a number line, and then choose a test point in each interval to test in the original inequality $x^2 - 3x - 10 \geq 0$:

The **boundary** points are **–2** and **5**.



The **test** points are –4, 0, and 6.

–4 $\in (-\infty, -2)$ **0** $\in (-2, 5)$ **6** $\in (5, \infty)$

$(-4)^2 - 3(-4) - 10 = 16 + 12 - 10 = 18$, which is ≥ 0. Thus, $(-\infty, -2)$ is part of the solution.

$0^2 - 3(0) - 10 = 0 - 0 - 10 = -10$, which is not ≥ 0. So, $(-2, 5)$ is not part of the solution.

$6^2 - 3(6) - 10 = 36 - 18 - 10 = 8$, which is ≥ 0. Therefore, $(5, \infty)$ is part of the solution.

Last, you can check the boundary points yourself — they work in the original inequality. Our final answer is the combination of the left interval and the right interval:

$$\boxed{(-\infty, -2] \cup [5, \infty)}$$

The answer can also be written

$$\{x \in \mathbb{R} \mid x \leq -2 \text{ OR } x \geq 5\}$$

EXAMPLE 2: **Solve for w: $9w - w^2 \geq 14$**

Solution: Solve the associated equation: $9w - w^2 = 14$. Rearranging the terms and multiplying through by -1 gives the equation $w^2 - 9w + 14 = 0$, whose solutions are 2 and 7.

These two boundary points yield the three intervals to check: $(-\infty, 2)$, $(2, 7)$ and $(7, \infty)$. Choose a test point (your choice) in each interval and substitute that test point into the original inequality. You'll find that the only interval that works is the center one: $(2, 7)$.

Finally, check the boundary points themselves. They both work in the original inequality. The final answer, then, is $\{ w \in \mathbb{R} \mid 2 \leq w \leq 7 \}$, which we write as

$$\boxed{[2, 7]}$$

EXAMPLE 3: **Solve for y: $y^2 - 12 > -4y$**

Solution: To solve this inequality, convert it to an equation, write it in standard quadratic form, factor, and you'll find that the boundary points are -6 and 2. Pick some test points and see that the intervals $(-\infty, -6)$ and $(2, \infty)$ work, but the interval $(-6, 2)$ doesn't. Also, the boundary points themselves don't work. The final answer is therefore

$$\boxed{(-\infty, -6) \cup (2, \infty)}$$

EXAMPLE 4: **Solve for x: $x^2 + x + 4 > 0$**

Solution: The associated equation is $x^2 + x + 4 = 0$, which won't factor, so let's use the quadratic formula:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(4)}}{2(1)} = \frac{-1 \pm \sqrt{-15}}{2}$$

There are no real solutions to this equation; therefore, there are no boundary points, and so there is only one interval to check, namely the entire real line, $(-\infty, \infty)$.

Choose any real number [that is, any number in the interval $(-\infty, \infty)$], for instance, **2**, as a test point, and plug it into the original inequality: $2^2 + 2 + 4 = 10$, which is clearly > 0. Since the test point worked, the interval in which the test point lies must work; however, this interval was the entire real line. Therefore, the solution to the inequality is the set of real numbers

$\mathbb{R}$

EXAMPLE 5: **Solve for x: $x^2 + 6x + 9 < 0$**

Solution: Solving the associated equation, $x^2 + 6x + 9 = 0$, we obtain one solution: $x = -3$. That produces two intervals: $(-\infty, -3)$ and $(-3, \infty)$. Pick a test point in each interval and you'll discover that both test points fail; even the boundary point $x = -3$ fails. In other words, everything fails! The inequality has **NO solution** at all. We can write this using the null set (empty set):

$\varnothing$

EXAMPLE 6: **Solve for x: $x^2 + 6x + 9 \leq 0$**

Solution: Solving the associated equation gives us the same boundary point as in the previous example. The two intervals also fail, as in the previous example. BUT this time, if we check the boundary point, $x = -3$, it works. So our final solution of the inequality is

$\boxed{x = -3}$ [which can also be written $\{-3\}$]

Homework

Solve each quadratic inequality:

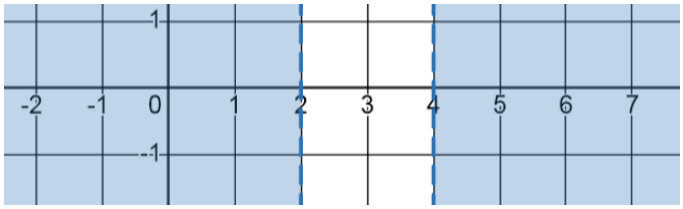
1. $a^2 - 9a - 10 > 0$
2. $2x^2 \leq 5 - 9x$
3. $x^2 + 9 < 6x$
4. $y^2 + 10y + 25 \leq 0$
5. $z^2 > 12z - 36$
6. $w^2 + 14w + 49 \geq 0$
7. $x^2 \geq 9$
8. $2x^2 - 15x + 7 < 0$
9. $5a - a^2 \geq 6$
10. $-3(3n^2 + 8n) \leq 16$

❑ *USING DESMOS TO SOLVE QUADRATIC INEQUALITIES*

You can use Desmos to solve these problems. For example, type

$$x^2 - 6x + 8 > 0$$

and you'll get the following graph:



Look at the two shaded regions. We see shading to the left of 2 and to the right of 4. This means that the inequality is satisfied whenever $x < 2$ or $x > 4$, giving us a final answer of

$$(-\infty, -2) \cup (4, \infty)$$

Review Problems

Solve each inequality:

11. $2(x - 5) - 9(2x + 1) > 0$

12. a. $x^2 + 7x + 12 < 0$
    b. $x^2 - 100 \geq 0$
    c. $x^2 + 10x + 25 > 0$
    d. $x^2 - 6x + 9 \leq 0$
    e. $4x^2 - 12x + 9 < 0$
    f. $x^2 \geq 2x - 1$

13. $x^2 - 5x \geq -4$

14. $x^2 < 20x - 100$

Solutions

1. $(-\infty, -1) \cup (10, \infty)$

2. $\left[-5, \frac{1}{2}\right]$

3. $\varnothing$

4. $\{ -5 \}$

5. $(-\infty, 6) \cup (6, \infty)$; or $\mathbb{R} - \{6\}$

6. $\mathbb{R}$

7. $(-\infty, -3] \cup [3, \infty)$

8. $\left(\frac{1}{2}, 7\right)$

9. $[2, 3]$

10. $\mathbb{R}$

11. $\left(-\infty, -\frac{19}{16}\right)$

12. a. $(-4, -3)$ b. $(-\infty, -10] \cup [10, \infty)$ c. $\mathbb{R} - \{-5\}$
d. $\{3\}$ e. $\varnothing$ f. $\mathbb{R}$

13. $(-\infty, 1] \cup [4, \infty)$

14. $\varnothing$

“A child without education is like a bird without wings.”

—Tibetan Proverb



PREPARING FOR COMPLETING THE SQUARE

❑ *INTRODUCTION*

I assume that the only methods you have learned for solving a *quadratic equation* are the **factoring method (**and perhaps the **Quadratic Formula)**. For example, to solve the quadratic equation

$$2x^2 + 3x - 20 = 0$$

by factoring, you would proceed as follows:

$2x^2 + 3x - 20 = 0$ (the given equation)

$\Rightarrow \quad (2x - 5)(x + 4) = 0$ (factor the quadratic)

$\Rightarrow \quad 2x - 5 = 0 \quad \text{OR} \quad x + 4 = 0$ (set each factor to 0)

$\Rightarrow \quad x = \frac{5}{2} \quad \text{OR} \quad x = -4$, and we're done — two solutions.

So factoring is a good enough method, you say? I cannot agree with you; I submit to you the following quadratic equation:

$$x^2 + 7x + 5 = 0$$

It's not a complicated equation — the numbers are small, and there aren't even any negative numbers to mess with. So go ahead, I dare you to factor that thing.

See my point? Trust me, that equation does have two solutions, but factoring is not the method that will yield those two solutions. We need a new method for solving quadratic equations, and this chapter will give you some of the prerequisite skills required for a future chapter that covers the technique called *Completing the Square*.

❑ *FACTORING PERFECT SQUARE TRINOMIALS*

Let's review the term ***perfect square***. The number 100 is a perfect square because 100 can be written as the square of 10: $100 = 10^{\mathbf{2}}$. Also, the expression $(x + 5)^{\mathbf{2}}$ is a perfect square, since it is the square of $x + 5$. Even n^6 is a perfect square because it is the square of n^3: $n^6 = (n^3)^{\mathbf{2}}$.

One of the steps in solving a quadratic equation by *completing the square* is factoring a ***perfect square trinomial***. Let's look at four examples.

Example 1:

$x^2 + 14x + 49$ is a perfect square trinomial because it's a trinomial that factors into the square of a binomial:

$$x^2 + 14x + 49 = (x + 7)(x + 7) = (x + 7)^2$$

[perfect square trinomial] [square of a binomial]

Example 2:

$n^2 - 20n + 100$ is a perfect square trinomial:

$$n^2 - 20n + 100 = (n - 10)(n - 10) = (n - 10)^2$$

Here are a couple of examples with fractions:

Example 3:

$$a^2 + 3a + \frac{9}{4} = \left(a + \frac{3}{2}\right)\left(a + \frac{3}{2}\right) = \left(a + \frac{3}{2}\right)^2$$

Check:

$$\left(a + \frac{3}{2}\right)^2 = \left(a + \frac{3}{2}\right)\left(a + \frac{3}{2}\right) = a^2 + \underbrace{\frac{3}{2}a + \frac{3}{2}a}_{\frac{3}{2}+\frac{3}{2}=\frac{6}{2}=3} + \frac{9}{4} = a^2 + 3a + \frac{9}{4}$$ ✓

Example 4:

$$y^2 - \frac{2}{5}y + \frac{1}{25} = \left(y - \frac{1}{5}\right)\left(y - \frac{1}{5}\right) = \left(y - \frac{1}{5}\right)^2$$

Check:

$$\left(y - \frac{1}{5}\right)\left(y - \frac{1}{5}\right) = y^2 - \frac{1}{5}y - \frac{1}{5}y + \frac{1}{25} = y^2 - \frac{2}{5}y + \frac{1}{25}$$ ✓

Homework

1. Factor each perfect square trinomial:

a. $x^2 + 10x + 25$

b. $y^2 - 18y + 81$

c. $a^2 + a + \frac{1}{4}$

d. $m^2 - \frac{4}{3}m + \frac{4}{9}$

e. $z^2 + \frac{2}{5}z + \frac{1}{25}$

f. $w^2 - \frac{5}{3}w + \frac{25}{36}$

g. $b^2 + \frac{9}{5}b + \frac{81}{100}$

h. $u^2 + \frac{3}{2}u + \frac{9}{16}$

i. $n^2 - \frac{4}{7}n + \frac{4}{49}$

j. $x^2 + \frac{10}{11}x + \frac{25}{121}$

❑ *Review of Solving Quadratics by Taking Square Roots*

Another skill required for completing the square is taking square roots to solve quadratic equations. You might want to review *Quadratic Equations by Taking Square Roots*. Here's an example from that chapter.

EXAMPLE 5: **Solve the quadratic equation: $(x + 7)^2 = 81$**

Solution: According to the Square Root Theorem, we can remove the squaring by taking the square root of both sides of the equation, remembering that the number 81 has two square roots:

	$(x+7)^2 = 81$	(the original equation)
$\Rightarrow$	$x+7 = \pm\sqrt{81}$	(the Square Root Theorem)
$\Rightarrow$	$x+7 = \pm 9$	($\sqrt{81} = 9$)
$\Rightarrow$	$x = -7 \pm 9$	(subtract 7 from each side)

Remember!
81 has two square roots.

Using the plus sign yields $x = -7 + 9 = 2$.

Using the minus sign yields $x = -7 - 9 = -16$.

$$x = 2 \text{ or } -16$$

❑ THE "MAGIC NUMBER"



Consider the trinomial $x^2 + 10x + 25$. We've learned that its factorization is

$$x^2 + 10x + 25 = (x+5)^2, \text{ which is the square of a binomial.}$$

Let's look carefully at the numbers in this equality. Notice that the 5 is half of the 10, and that the 25 is the square of the 5.

Let's do one more example. Consider the factorization

$$n^2 - 14n + 49 = (n-7)^2, \text{ which is the square of a binomial.}$$

We note that the –7 is half of the –14, and that the 49 is the square of the –7.

So now imagine that I give you

$$x^2 + 6x$$

and I ask you to add a third term to this binomial so that the resulting trinomial will factor into the square of a binomial:

$$x^2 + 6x + \underline{???} \;=\; (x + ?)(x + ?) \;=\; (x + ?)^2$$

Each single "?" must be 3, since 3 is half of 6. Also, the "???" must be 9, since 9 is the square of the 3. In other words,

$$x^2 + 6x + \boxed{9} \;=\; (x + \underline{\mathbf{3}})(x + \underline{\mathbf{3}}) \;=\; (x + \underline{\mathbf{3}})^2$$

We shall call 9 the "magic number." It can be calculated for this problem using the following two-step rule:

1) Calculate <u>half</u> of 6, which is 3.
2) <u>Square</u> the 3, which is **9**, the "magic number."

Note: When we convert $x^2 + 6x$ to $x^2 + 6x + 9$ by adding the "magic number" **9**, we are <u>not</u> saying that they're equal, but there will always be a way of adding the magic number without violating any of the laws of algebra.

To become proficient in completing the square, we must be really good at finding the *magic number*. The following chart gives more examples of how this is done. We'll let b represent the number in front of the variable (the coefficient of the linear term).

Original Quadratic	Value of b	<u>Half</u> of b	Half of b <u>Squared</u> = **The Magic Number**	New Quadratic	New Quadratic in Factored Form
$x^2 + 22x$	22	11	**121**	$x^2 + 22x + \mathbf{121}$	$(x + 11)^2$
$A^2 - 14A$	–14	–7	**49**	$A^2 - 14A + \mathbf{49}$	$(A - 7)^2$
$n^2 + 9n$	9	$\frac{9}{2}$	$\frac{\mathbf{81}}{\mathbf{4}}$	$x^2 + 9x + \frac{\mathbf{81}}{\mathbf{4}}$	$\left(x + \frac{9}{2}\right)^2$
$y^2 - \frac{2}{5}y$	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{\mathbf{1}}{\mathbf{25}}$	$y^2 - \frac{2}{5}y + \frac{\mathbf{1}}{\mathbf{25}}$	$\left(y - \frac{1}{5}\right)^2$
$3u^2 - 7u$	This problem can't be done yet; the leading coefficient is <u>not</u> 1.				

Homework

2. Find the **magic number** for each quadratic binomial:

a. $a^2 + 8a$ b. $x^2 - 12x$ c. $y^2 + 28y$

d. $b^2 + 3b$ e. $c^2 - 7c$ f. $t^2 + 11t$

g. $x^2 + x$ h. $y^2 - y$ i. $z^2 + 2z$

j. $g^2 + \frac{2}{3}g$ k. $a^2 - \frac{5}{7}a$ l. $x^2 - \frac{7}{13}x$

Review Problems

3. How many solutions does each quadratic equation have?

a. $(x + 3)(x + 4) = 0$ ___ b. $(x + 9)^2 = 0$ ___

c. $(x - 1)^2 = 10$ ___ d. $(x + 6)^2 = -9$ ___

4. Factor each trinomial:

a. $n^2 + 10n + 25$ b. $x^2 - 18x + 81$

c. $a^2 + 5a + \frac{25}{4}$ d. $x^2 - \frac{4}{3}x + \frac{4}{9}$

e. $y^2 + \frac{6}{7}y + \frac{9}{49}$ f. $t^2 - \frac{1}{5}t + \frac{1}{100}$

5. Find the magic number for $w^2 - \frac{7}{9}w$.

Solutions

1. a. $(x+5)^2$ b. $(y-9)^2$ c. $\left(a+\frac{1}{2}\right)^2$ d. $\left(m-\frac{2}{3}\right)^2$

e. $\left(z+\frac{1}{5}\right)^2$ f. $\left(w-\frac{5}{6}\right)^2$ g. $\left(b+\frac{9}{10}\right)^2$ h. $\left(u+\frac{3}{4}\right)^2$

i. $\left(n-\frac{2}{7}\right)^2$ j. $\left(x+\frac{5}{11}\right)^2$

2. a. $\frac{1}{2}\cdot 8 = 4$, and $4^2 = \mathbf{16}$

b. $\frac{1}{2}\cdot -12 = -6$, and $(-6)^2 = \mathbf{36}$

c. $\frac{1}{2}\cdot 28 = 14$, and $14^2 = \mathbf{196}$

d. $\frac{1}{2}\cdot 3 = \frac{3}{2}$, and $\left(\frac{3}{2}\right)^2 = \mathbf{\frac{9}{4}}$

e. $\frac{1}{2}\cdot -7 = -\frac{7}{2}$, and $\left(-\frac{7}{2}\right)^2 = \mathbf{\frac{49}{4}}$

f. $\frac{1}{2}\cdot 11 = \frac{11}{2}$, and $\left(\frac{11}{2}\right)^2 = \mathbf{\frac{121}{4}}$

g. $\frac{1}{2}\cdot 1 = \frac{1}{2}$, and $\left(\frac{1}{2}\right)^2 = \mathbf{\frac{1}{4}}$

h. $\frac{1}{2}\cdot -1 = -\frac{1}{2}$, and $\left(-\frac{1}{2}\right)^2 = \mathbf{\frac{1}{4}}$

i. $\frac{1}{2}\cdot 2 = 1$, and $1^2 = \mathbf{1}$

j. $\frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$, and $\left(\frac{1}{3}\right)^2 = \mathbf{\frac{1}{9}}$

k. $\frac{1}{2}\cdot -\frac{5}{7} = -\frac{5}{14}$, and $\left(-\frac{5}{14}\right)^2 = \mathbf{\frac{25}{196}}$

l. $\frac{1}{2}\cdot -\frac{7}{13} = -\frac{7}{26}$, and $\left(-\frac{7}{26}\right)^2 = \mathbf{\frac{49}{676}}$

3. a. 2 b. 1 c. 2 d. 0

4. a. $(n + 5)^2$ b. $(x - 9)^2$ c. $\left(a + \frac{5}{2}\right)^2$

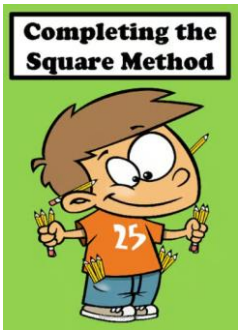
d. $\left(x - \frac{2}{3}\right)^2$ e. $\left(y + \frac{3}{7}\right)^2$ f. $\left(t - \frac{1}{10}\right)^2$

5. $\frac{49}{324}$

Education is not a preparation for life; education is life itself.

John Dewey

COMPLETING THE SQUARE



If you understand all the concepts in the chapter *Preparing for Completing the Square*, you're ready to tackle this chapter.

❑ THE FIVE STEPS IN COMPLETING THE SQUARE

We're now ready to combine many of our algebra skills into solving non-factorable quadratic equations, like $x^2 + 3x + 1 = 0$.

We start by assuming that the quadratic equation is in standard form. If it's not, we know we can always move things around to convert it to standard form:

$$ax^2 + bx + c = 0 \qquad \textbf{(where } a \neq 0\textbf{)}$$

1. Make sure that the leading coefficient (the *a*) is 1. If it's not, divide each side of the equation (all terms) by *a*.
2. Move the constant to the other side of the equation.
3. Compute the "magic number" and add it to both sides of the equation. This step "*Completes the Square*."
4. Factor the left side, and then simplify the right side.
5. Solve the resulting equation by taking square roots, remembering that every positive number has two square roots (the Square Root Theorem).

Note: Steps 1 and 2 can be done in either order.

❑ *EXAMPLES OF COMPLETING THE SQUARE*

EXAMPLE 1: **Solve by Completing the Square:**

$$x^2 + 8x - 20 = 0$$

Solution: Even though this quadratic equation is factorable, we'll solve it by completing the square — a technique that will also work on problems that aren't factorable, as well as allow us to solve some problems with circles.

Step 1:

We must ensure that the leading coefficient (the a) is 1. It already is 1 ($x^2 = 1x^2$), so step 1 is done, and the equation remains the same:

$$x^2 + 8x - 20 = 0$$

Step 2:

Move the constant (the –20) to the right side of the equation by adding 20 to each side of the equation:

$$x^2 + 8x = 20$$

Step 3:

Now we calculate the *magic number*:

a) Calculate half of 8: $\frac{1}{2}(8) = 4$

b) Square that 4; the magic number is **16**.

Add the magic number to each side of the equation:

$$x^2 + 8x + \boxed{16} = 20 + \boxed{16}$$

Step 4:

Factor the left side and simplify the right side:

$$(x + 4)^2 = 36$$

Step 5:

Solve by taking square roots:

$x + 4 = \pm\sqrt{36}$ (36 has **two** square roots)

$x + 4 = \pm 6$ (simplify the radical)

$x = -4 \pm 6$ (subtract 4 from both sides)

Using the plus sign $\Rightarrow$ $x = -4 + 6 = 2$

Using the minus sign $\Rightarrow$ $x = -4 - 6 = -10$

We have now solved our first equation by completing the square, and its solutions are

$$\boxed{x = 2 \text{ or } -10}$$

Note that we could have solved the quadratic equation by factoring. No matter the method, we would get the same solutions.

EXAMPLE 2: **Solve by Completing the Square:**

$$\mathbf{3x^2 - 5x + 1 = 0}$$

Solution: We proceed as we did above with the 5-step plan.

Step 1:

The first requirement for completing the square is a leading coefficient of 1. Since the leading coefficient in this

problem is 3, we will have to divide both sides of the equation by 3:

$$\frac{3x^2 - 5x + 1}{3} = \frac{0}{3}$$

or, $x^2 - \frac{5}{3}x + \frac{1}{3} = 0$ (divide <u>all</u> terms by 3)

Step 2:

Move the constant to the right side, resulting in

$$x^2 - \frac{5}{3}x = -\frac{1}{3}$$ (subtract $\frac{1}{3}$ from both sides)

Step 3:

It's now time for the magic number, the number that will complete the square. We calculate half of $-\frac{5}{3}$, square that result, and we'll have the "magic number" that will be added to each side of the equation.

Magic Number Calculation:

$$\frac{1}{2}\left(-\frac{5}{3}\right) = -\frac{5}{6}, \text{ and then } \left(-\frac{5}{6}\right)^2 = \mathbf{\frac{25}{36}}$$

Add this number to each side of the equation:

$$x^2 - \frac{5}{3}x + \boxed{\frac{25}{36}} = -\frac{1}{3} + \boxed{\frac{25}{36}}$$

Step 4:

Factoring the left side and adding the fractions on the right side gives us:

$$\left(x - \frac{5}{6}\right)^2 = \frac{13}{36} \qquad \left[-\frac{1}{3} + \frac{25}{36} = -\frac{12}{36} + \frac{25}{36} = \frac{13}{36}\right]$$

Step 5:

Now we take the square root of each side of the equation, remembering that the right side has **<u>two</u>** square roots:

$$x - \frac{5}{6} = \pm\sqrt{\frac{13}{36}}$$

Next, we isolate the x by adding $\frac{5}{6}$ to each side of the equation:

$$x = \frac{5}{6} \pm \sqrt{\frac{13}{36}}$$

Split the radical:

$$x = \frac{5}{6} \pm \frac{\sqrt{13}}{\sqrt{36}} \qquad \left[\text{from the rule: } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}\right]$$

And simplify the bottom radical:

$$x = \frac{5}{6} \pm \frac{\sqrt{13}}{6}$$

Combining the fractions into a single fraction (the LCD is 6) produces the final answer:

$$\boxed{x = \frac{5 \pm \sqrt{13}}{6}}$$

Our quadratic has two solutions.

Homework

1. At what point in Completing the Square could you determine that the quadratic equation you're trying to solve has NO solution?

2. Solve each quadratic equation by Completing the Square:

a. $y^2 - 6y + 5 = 0$

b. $x^2 + 13x + 30 = 0$

c. $z^2 + 5z - 14 = 0$

d. $2n^2 - n - 1 = 0$

e. $12t^2 - 5t - 3 = 0$

f. $10a^2 + 7a + 1 = 0$

g. $x^2 + 25 = 10x$

h. $4u^2 + 20u + 25 = 0$

i. $w^2 = -w - 5$

j. $2h^2 + 1 = h$

3. Solve each quadratic equation by Completing the Square:

a. $x^2 + 3x + 1 = 0$

b. $y^2 - 4y + 2 = 0$

c. $2a^2 + 6a + 3 = 0$

d. $n^2 + 8n - 2 = 0$

e. $3u^2 - 4u - 2 = 0$

f. $t^2 + 10t + 3 = 0$

g. $5w^2 + w + 1 = 0$

h. $2x^2 = -5x - 1$

i. $g^2 = 3g - 5$

j. $3m^2 = 1 - 4m$

4. Solve each quadratic equation by Completing the Square:

a. $q^2 + 8q + 15 = 0$

b. $2x^2 - 7x + 1 = 0$

c. $3n^2 + n - 5 = 0$

d. $4a^2 - 5a = -3$

e. $5w^2 = -2 - 7w$

f. $z^2 + 1 = 2z$

Solutions

1. As soon as you notice that you've reached the square root of a negative number, you can stop and conclude that the quadratic equation has no solution (in the real numbers).

2. a. $y = 1, 5$

b. $x = -3, -10$

c. $z = 2, -7$

d. $n = 1, -\frac{1}{2}$

e. $t = \frac{3}{4}, -\frac{1}{3}$

f. $a = -\frac{1}{2}, -\frac{1}{5}$

g. $x = 5$

h. $u = -\frac{5}{2}$

i. No solution

j. No solution

3. a. $x = \frac{-3 \pm \sqrt{5}}{2}$ b. $y = 2 \pm \sqrt{2}$ c. $a = \frac{-3 \pm \sqrt{3}}{2}$

d. $n = -4 \pm 3\sqrt{2}$ e. $u = \frac{2 \pm \sqrt{10}}{3}$ f. $t = -5 \pm \sqrt{22}$

g. No solution h. $x = \frac{-5 \pm \sqrt{17}}{4}$ i. No solution

j. $m = \frac{-2 \pm \sqrt{7}}{3}$

4. a. $q = -5, -3$ b. $x = \frac{7 \pm \sqrt{41}}{4}$ c. $n = \frac{-1 \pm \sqrt{61}}{6}$

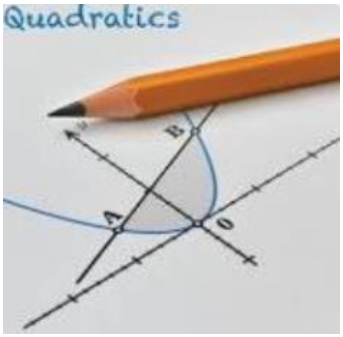
d. No solution e. $w = -\frac{2}{5}, -1$ f. $z = 1$

"The true sign of intelligence is not knowledge, but *imagination*."

Albert Einstein



SOLVING QUADRATICS BY FACTORING



We're pretty good by now at solving equations like $2(3x - 4) + 8 = -10(x + 1)$, and you've probably had your fill of word problems which can be solved by such equations. But some of the most important applications in algebra, science, and business involve equations where the variable is *squared*. For instance, the following is called a ***quadratic equation***, due simply to the fact that the variable is being squared:

$$x^2 - 10x + 16 = 0$$

Now that's an equation!

❑ *CHECKING THE SOLUTIONS OF A QUADRATIC EQUATION*

First, let's verify that $\underline{x = 2}$ is a solution of the above equation (don't worry about where the 2 came from):

$x^2 - 10x + 16 = 0$	(the given equation)
$\mathbf{2}^2 - 10(\mathbf{2}) + 16 \stackrel{?}{=} 0$	(change each x into a 2)
$4 - 20 + 16 \stackrel{?}{=} 0$	(exponents and multiplication first)
$-16 + 16 \stackrel{?}{=} 0$	(add and subtract – left to right)
$0 = 0$ ✓	($x = 2$ works!)

Fine — we have a solution: $x = 2$. Here comes the (possibly) surprising fact: This equation has another solution, namely $x = 8$. Watch this:

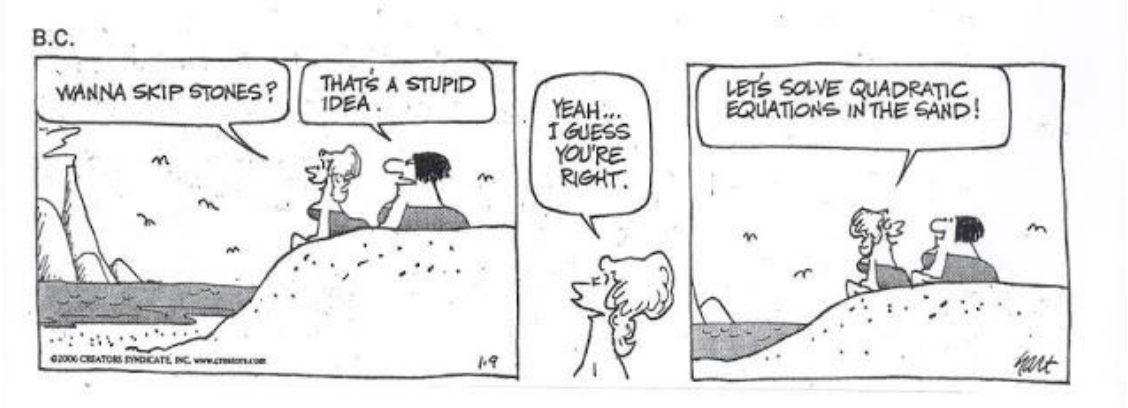
$$x^2 - 10x + 16 = 0$$
$$8^2 - 10(8) + 16 \stackrel{?}{=} 0$$
$$64 - 80 + 16 \stackrel{?}{=} 0$$
$$-16 + 16 \stackrel{?}{=} 0$$
$$0 = 0 \quad \checkmark$$

One equation with two solutions? Yep, that's what we have. This special kind of equation, where the variable is squared (and may very well have two solutions), has a special name: We call it a ***quadratic equation*** (from "*quadrus*," Latin for a four-sided *square*).

Homework

1. For each quadratic equation, verify that the two given solutions are really solutions:

a.	$x^2 + 3x - 10 = 0$	$x = -5;\ x = 2$
b.	$n^2 - 25 = 0$	$n = 5;\ n = -5$
c.	$a^2 + 7a = -12$	$a = -3;\ a = -4$
d.	$w^2 = 7w + 18$	$w = 9;\ w = -2$
e.	$2y^2 + 8y = 0$	$y = -4;\ y = 0$



❑ *QUADRATIC EQUATIONS*

Recall the equation from part d. of the homework:

$$w^2 = 7w + 18$$

Because the variable is squared, this is an example of a ***quadratic equation***. We can write this quadratic equation in **standard form** by subtracting $7w$ from each side of the equation and then subtracting 18 from each side of the equation:

$$w^2 - 7w - 18 = 0 \quad \longleftarrow \text{ A quadratic equation in } \textbf{\textit{standard form}}$$

Notice that the quadratic (squared) term is written first, the linear term (the one with the variable to the 1st power) comes next, and the constant term comes last; the right side of the equation is 0. We also note that the coefficient of the squared term is 1, the coefficient of the linear term is –7, and the constant term is –18.

> The ***coefficient*** tells us how many of a particular thing we have. If we have $9n$, then we have 9 n's, and so the coefficient of $9n$ is 9. The coefficient of the term $-7xy$ is –7, and the coefficient of $13w^2$ is 13.
>
> Sometimes the coefficient is "implied." For example, the coefficient of u^2 is 1 (since $u^2 = 1u^2$), and the coefficient of $-m$ is –1 (since $-m = -1m$).

We can solve some quadratic equations in our head. For instance, consider the quadratic equation

$$x^2 = 25$$

What number, when squared, equals 25? Recalling the homework above, we're not surprised to learn that there are two solutions to this equation: **5** and **–5**. After all, $5^2 = 25$, and $(-5)^2 = 25$. We conclude that the solutions of the quadratic equation $x^2 = 25$ are $x = \mathbf{5}$ and $x = \mathbf{-5}$. So, as we study this and future chapters, keep in mind that a quadratic equation will likely (but not necessarily) have two different solutions.

❑ SOLVING QUADRATIC EQUATIONS BY FACTORING

Suppose we came across the statement

$$AB = 0$$

This says that the product of A and B is 0. Can we deduce anything? The only way this can occur is if either $A = 0$ **or** $B = 0$. This is the basis of the method of this chapter.

So, if we're presented with, for instance, the quadratic equation

$$2w^2 - 31w + 84 = 0$$

we use the same logic as above, but we have to first convert the left side of the equation into a multiplication problem (like AB). And to convert an expression with three terms (containing addition and subtraction) into a multiplication problem, we *FACTOR!*

> If $AB = 0$
> then $A = 0$ or $B = 0$.

Our process in solving the equation is to **factor** the quadratic expression on the left side of the equation (noting that the right side is 0), set each factor to 0, and then solve each linear equation.

EXAMPLE 1: **Solve the quadratic equation $2w^2 - 31w + 84 = 0$.**

Solution:

$$2w^2 - 31w + 84 = 0 \qquad \text{(the original equation)}$$

$$\Rightarrow \quad (2w - 7)(w - 12) = 0 \qquad \text{(factor the quadratic)}$$

$$\Rightarrow \quad 2w - 7 = 0 \text{ or } w - 12 = 0 \qquad \text{(set each factor to 0)}$$

$$\Rightarrow \quad 2w = 7 \text{ or } w = 12 \qquad \text{(solve each equation)}$$

$$\Rightarrow \quad w = \frac{7}{2} \text{ or } w = 12$$

Thus, the solutions for w are

$$\boxed{\frac{7}{2},\ 12}$$

EXAMPLE 2: **Solve for x: $3x^2 - 7x - 40 = 0$**

Solution: This equation is in standard quadratic form, so it's all set to factor:

$3x^2 - 7x - 40 = 0$ (the original equation)

$\Rightarrow \quad (3x + 8)(x - 5) = 0$ (factor the left side)

$\Rightarrow \quad 3x + 8 = 0 \quad \text{or} \quad x - 5 = 0$ (set each factor to 0)

$\Rightarrow \quad x = \frac{-8}{3} = -\frac{8}{3} \quad \text{or} \quad x = 5$ (solve each equation)

Thus, the final solutions to the quadratic equation are $\boxed{5,\ -\frac{8}{3}}$

It's time to practice our equation-checking. Letting $x = 5$ in the original equation $3x^2 - 7x - 40 = 0$ gives

$$3(\mathbf{5})^2 - 7(\mathbf{5}) - 40 = 3(25) - 7(5) - 40 = 75 - 35 - 40 = 0 \quad \checkmark$$

Trying $x = -\frac{8}{3}$ in the original equation produces

$$3\left(-\frac{\mathbf{8}}{\mathbf{3}}\right)^2 - 7\left(-\frac{\mathbf{8}}{\mathbf{3}}\right) - 40 = 3\left(\frac{64}{9}\right) - 7\left(-\frac{8}{3}\right) - 40$$

$$= \frac{64}{3} + \frac{56}{3} - \frac{120}{3} = \frac{120}{3} - \frac{120}{3} = 0 \quad \checkmark$$

EXAMPLE 3: **Solve for y: $9y^2 - 16 = 0$**

Solution: It's a good-looking quadratic (even though the middle term, the y-term, is missing), so let's factor and set the factors to zero:

$9y^2 - 16 = 0$ (the original equation)

$\Rightarrow \quad (3y + 4)(3y - 4) = 0$ (factor the left side)

$\Rightarrow \quad 3y + 4 = 0 \quad \text{or} \quad 3y - 4 = 0$ (set each factor to 0)

$\Rightarrow \quad 3y = -4 \quad \text{or} \quad 3y = 4$ (remove the 4's)

$\Rightarrow \quad y = -\frac{4}{3} \quad \text{or} \quad y = \frac{4}{3}$ (divide both equations by 3)

Therefore, the solutions of the equation are

$\boxed{\frac{4}{3},\ -\frac{4}{3}}$ which can also be written $\pm\frac{4}{3}$.

EXAMPLE 4: **Solve for u: $9u^2 = 42u - 49$**

Solution: This quadratic equation is not in standard form, so the first two steps will be to transform it into standard form:

$9u^2 = 42u - 49$ (the original equation)

$\Rightarrow \quad 9u^2 - 42u = -49$ (subtract $42u$)

$\Rightarrow \quad 9u^2 - 42u + 49 = 0$ (add 49 — standard form)

$\Rightarrow \quad (3u - 7)(3u - 7) = 0$ (factor)

$\Rightarrow \quad 3u - 7 = 0 \quad \text{or} \quad 3u - 7 = 0$ (set each factor to 0)

$\Rightarrow \quad u = \frac{7}{3} \quad \text{or} \quad u = \frac{7}{3}$ (solve each equation)

We obtained two solutions, but they're equal to each other, so there's just one solution (or are there two solutions which are the same?):

$\boxed{\frac{7}{3}}$

Homework

2. Solve each quadratic equation by factoring:

a. $w^2 - 12w + 35 = 0$

b. $2x^2 - 13x + 15 = 0$

c. $n^2 - 25n + 150 = 0$

d. $6a^2 - 31a + 40 = 0$

e. $x^2 + 2x - 15 = 0$

f. $y^2 - 14y + 49 = 0$

g. $z^2 - 9 = 0$

h. $3h^2 - 17h + 10 = 0$

i. $16u^2 - 8u + 1 = 0$

j. $25w^2 - 4 = 0$

3. Solve each quadratic equation by factoring:

a. $2x^2 + 10x - 28 = 0$

b. $25y^2 = 4$

c. $2z^2 + 2 = -4z$

d. $4a^2 = 3 - 4a$

e. $6u^2 = 47u + 8$

f. $0 = 4t^2 + 8t + 3$

g. $x^2 + x - 42 = 0$

h. $2n^2 + n - 3 = 0$

i. $4x^2 = 1$

j. $a^2 + 12a + 36 = 0$

k. $9k^2 = 9k - 2$

l. $9n^2 - 30n + 25 = 0$

m. $6z^2 = 10z$

n. $49t^2 = 4$

o. $25h^2 = 30h - 9$

p. $6x^2 - 7x - 10 = 0$

q. $0 = 6y^2 + y - 2$

r. $4n^2 = 25n + 21$

s. $16q^2 - 24q + 9 = 0$

t. $64t^2 - 9 = 0$

u. $-n^2 + n + 56 = 0$ [Hint: multiply each side by –1]

v. $-2x^2 + x + 3 = 0$

w. $-14a^2 = 3 - 13a$

❑ *FINDING A QUADRATIC EQUATION FROM THE SOLUTIONS*

When solving a quadratic equation by factoring, we set the quadratic to zero, factor the quadratic, set each of the two linear factors to zero, and then solve the two linear equations. However, for the next example, we have to find an equation given that we know its solutions; to do this we simply reverse the process of solving the equation.

EXAMPLE 5: **Find a quadratic equation whose solutions are 3 and –12.**

Solution:

You can use any variable you like, but I'll use x:

$x = 3 \quad \text{OR} \quad x = -12$ (the given two solutions)

$\Rightarrow \quad x - 3 = 0 \quad \text{OR} \quad x + 12 = 0$ (rewrite so that each equation is 0)

$\Rightarrow \quad (x - 3)(x + 12) = 0$ (the product of the factors is 0)

$\Rightarrow \quad \boxed{x^2 + 9x - 36 = 0}$ (double distribute)

Note: The above example asked for "a" quadratic equation, not "the" quadratic equation. Why? Because there are an infinite number of quadratic equations which have 3 and –12 as their solutions.

For example, check out $10x^2 + 90x - 360 = 0$. If you solve it by the factoring method, you'd likely would start by factoring out the GCF of 10:

$$10x^2 + 90x - 360 = 0$$

$$\Rightarrow \quad \mathbf{10}(x^2 + 9x - 36) = 0$$

$$\Rightarrow \quad 10(x - 3)(x + 12) = 0.$$

Dividing each side of the equation by 10 gives the equation

$$(x - 3)(x + 12) = 0,$$

whose solutions are exactly those required by the problem, 3 and –12. Therefore, a perfectly fine answer to the question would be the quadratic equation $10x^2 + 90x - 360 = 0$. In fact, for any number $\boldsymbol{k} \neq 0$, the quadratic equation $\boldsymbol{k}(x^2 + 9x - 36) = 0$ would properly answer the question. Nevertheless, let's agree that the best answer is the simplest one.

Homework

Find a quadratic equation (in standard form) with the given solutions:

4. 2 and –7
5. $\frac{2}{3}$ and 1
6. $-\frac{1}{2}$ and $\frac{4}{5}$
7. 0 and 9
8. The only solution is 10
9. The only solution is $-\frac{2}{3}$

Review Problems

10. Solve by factoring: $25y^2 = 1$

11. Solve by factoring: $n^2 + 16n + 64 = 0$

12. Solve by factoring: $26y^2 = 5y$

13. Solve by factoring: $25t^2 + 30t + 9 = 0$

14. Solve by factoring: $24x^2 = 14x + 3$

15. Solve by factoring: $81h^2 = 25$

16. Find a quadratic equation (in standard form) with the given solutions:

    a. -3 and -5

    b. $\frac{3}{2}$ and -2

    c. $-\frac{3}{5}$ and $\frac{1}{2}$

    d. 0 and $\frac{5}{6}$

    e. The only solution is -8

    f. The only solution is $\frac{7}{4}$

❑ *TO ∞ AND BEYOND*

A. Find a quadratic equation (in standard form) with the given solutions:

1. $\pm\sqrt{7}$
2. $1+\sqrt{2}$ and $1-\sqrt{2}$
3. $-1+\sqrt{5}$ and $-1-\sqrt{5}$
4. 1 and 2 and 3

B. Solve for x: $x^3 + 3x^2 - 10x = 0$

Hint: There are three solutions.

C. Solve for n: $n^4 - 25n^2 + 144 = 0$

Hint: There are four solutions.

D. Solve for x: $\dfrac{x^2 - 5x + 6}{\sqrt[7]{\sin^2 x - \ln x + x^2 - \csc^3(e^x)}} = 0$

Solutions

1. Just plug in the proposed solutions and make sure they work.

2. a. 5, 7 b. $5, \frac{3}{2}$ c. 10, 15 d. $\frac{8}{3}, \frac{5}{2}$

e. 3, –5 f. 7 g. ± 3 h. $5, \frac{2}{3}$

i. $\frac{1}{4}$ j. $\pm\frac{2}{5}$

3. a. 2, –7 b. $\pm\frac{2}{5}$ c. –1 d. $\frac{1}{2}, -\frac{3}{2}$

e. $8, -\frac{1}{6}$ f. $-\frac{1}{2}, -\frac{3}{2}$ g. 6, –7 h. $1, -\frac{3}{2}$

i. $\pm\frac{1}{2}$ j. –6 k. $\frac{1}{3}, \frac{2}{3}$ l. $\frac{5}{3}$

m. $0, \frac{5}{3}$ n. $\pm\frac{2}{7}$ o. $\frac{3}{5}$ p. $2, -\frac{5}{6}$

q. $\frac{1}{2}, -\frac{2}{3}$ r. $7, -\frac{3}{4}$ s. $\frac{3}{4}$ t. $\pm\frac{3}{8}$

u. 8, –7 v. $-1, \frac{3}{2}$ w. $\frac{3}{7}, \frac{1}{2}$

4. $x = 2$ OR $x = -7 \Rightarrow x - 2 = 0$ OR $x + 7 = 0 \Rightarrow (x - 2)(x + 7) = 0$
$\Rightarrow x^2 + 5x - 14 = 0$

5. $x = \frac{2}{3}$ OR $x = -1 \Rightarrow x - \frac{2}{3} = 0$ OR $x + 1 = 0 \Rightarrow \left(x - \frac{2}{3}\right)(x + 1) = 0$
$\Rightarrow x^2 + x - \frac{2}{3}x - \frac{2}{3} = 0 \Rightarrow x^2 + \frac{1}{3}x - \frac{2}{3} = 0 \Rightarrow 3x^2 + x - 2 = 0$

6. $10x^2 - 3x - 4 = 0$

7. $x^2 - 9x = 0$

8. $(x-10)(x-10) = 0 \Rightarrow x^2 - 20x + 100 = 0$

9. $9x^2 + 12x + 4 = 0$

10. $\pm\frac{1}{5}$ **11**. -8 **12**. $0, \frac{5}{26}$

13. $-\frac{3}{5}$ **14**. $\frac{3}{4}, -\frac{1}{6}$ **15**. $\pm\frac{5}{9}$

16. a. $x^2 + 8x + 15 = 0$ b. $2x^2 + x - 6 = 0$

c. $10x^2 + x - 3 = 0$ d. $6x^2 - 5x = 0$

e. $x^2 + 16x + 64 = 0$ f. $16x^2 - 56x + 49 = 0$

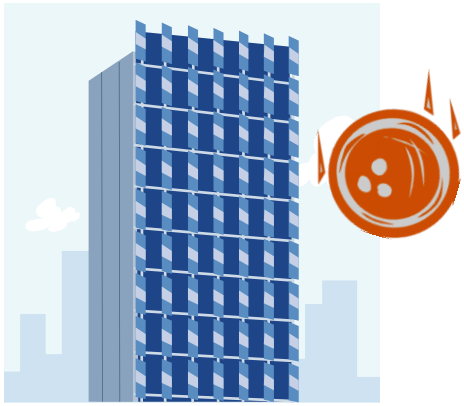
"Every job is a self-portrait
of the person who did it.
Autograph your work with excellence."

– Unknown

APPLICATIONS OF THE QUADRATIC FORMULA

❑ *FALLING OBJECTS (AND MATH TEACHERS)*

EXAMPLE 1: **A bowling ball is thrown upward from the ledge of a building and falls to the ground under the effects of just gravity (we neglect air resistance). Let h represent its height (in meters) above the ground after t seconds have gone by. Suppose the formula for h is given by:**



$$h = -2t^2 + 10t + 100$$

a. Prove that when the ball is thrown (that is, when $t = 0$), it is 100 m above the ground.

b. Calculate the height of the ball 4 seconds after the ball is thrown.

c. How many seconds after the bowling ball is thrown does it crash into the ground?

Solution: In the formula

$$h = -2t^2 + 10t + 100$$

t is the time and h is the height above the ground. So if we put a particular value of t into the formula, we should be able to calculate the height of the bowling ball at that moment in time. Conversely, we can put in a value of h, and then calculate the moment in time, t, when the ball was at that given height.

a. When the ball is first thrown, the value of t is 0 (like starting a stopwatch). So let's plug $t = 0$ into the formula and see what we get for h, the height:

$$h = -2t^2 + 10t + 100 \qquad \text{(the height formula)}$$
$$\Rightarrow \quad h = -2(\mathbf{0})^2 + 10(\mathbf{0}) + 100 \qquad \text{(set } t = 0\text{)}$$
$$\Rightarrow \quad h = -2(0) + 10(0) + 100 \qquad \text{(exponents first)}$$
$$\Rightarrow \quad h = 0 + 0 + 100 \qquad \text{(then multiplication)}$$
$$\Rightarrow \quad h = 100 \qquad \text{(and then addition)}$$

And so we've proved that the height when $t = 0$ is **100 m**.

b. This is the same idea as the first question — we'll just set $t = 4$ and determine the value of h:

$$h = -2t^2 + 10t + 100 \qquad \text{(the height formula)}$$
$$\Rightarrow \quad h = -2(\mathbf{4})^2 + 10(\mathbf{4}) + 100 \qquad \text{(set } t = 4\text{)}$$
$$\Rightarrow \quad h = -2(16) + 10(4) + 100 \qquad \text{(Order of Operations)}$$
$$\Rightarrow \quad h = -32 + 40 + 100$$
$$\Rightarrow \quad h = 8 + 100$$
$$\Rightarrow \quad h = 108$$

We conclude that the bowling ball is **108 m** above the ground after 4 seconds have transpired.

c. Now for the good part . . . I mean the hard part. In this problem the unknown is t, the time. But the height is not given, so what are we supposed to do? We ask ourselves,

What is the height of the bowling ball the moment it hits the ground? It's 0 meters. Thus, we place $h = 0$ into the formula to calculate the value (or values) of t. Are you ready?

$$h = -2t^2 + 10t + 100 \qquad \text{(the height formula)}$$

$$\downarrow$$

$$\mathbf{0} = -2t^2 + 10t + 100 \qquad \text{(hits the ground when } h = 0\text{)}$$

Notice that we have a quadratic equation. First, turn the whole thing around so that it's in standard form:

$$-2t^2 + 10t + 100 = 0$$

Let's multiply each side of the equation by –1 so that the leading coefficient (the value of a) is positive (not a necessary step, but a nice one):

$$2t^2 - 10t - 100 = 0$$

We should now be able to deduce that $a = 2$, $b = -10$, and $c = -100$. Placing these values into the Quadratic Formula, we can solve for t:

$$t = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(2)(-100)}}{2(2)}$$

$$t = \frac{10 \pm \sqrt{100 + 800}}{4} = \frac{10 \pm \sqrt{900}}{4} = \frac{10 \pm 30}{4}$$

Using the plus sign:

$$t = \frac{10 + 30}{4} = \frac{40}{4} = 10$$

Using the minus sign:

$$t = \frac{10 - 30}{4} = \frac{-20}{4} = -5$$

Like most quadratic equations, we've got two solutions at hand. But do they both work? Well, certainly they both work in the equation we solved, but let's look at the big

picture. The value $t = 0$ represented the beginning of the experiment, the moment the bowling ball was thrown from the ledge of the building. Therefore, the solution $t = -5$ makes no sense; that moment in time occurred 5 seconds before the ball was thrown -- that's absurd.

So we conclude that the ball hit the ground **10 seconds** after it was thrown.

Homework

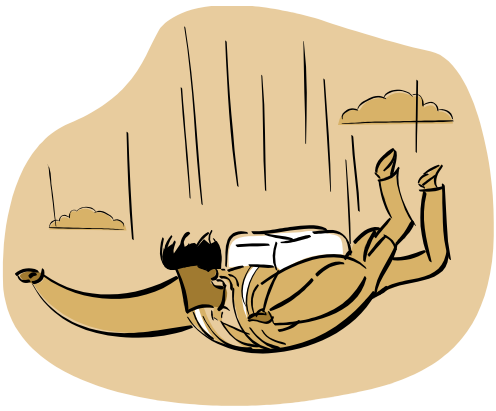
1. A math teacher is thrown upward from the ledge of a building. Let h represent his height (in meters) above the ground after t seconds have gone by. Suppose the formula for h is given by:

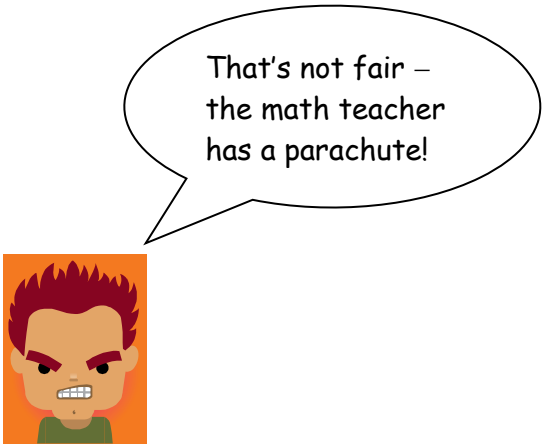
$$h = -3t^2 - 12t + 288$$

a. Prove that when the teacher is tossed, he is 288 m above the ground.

b. Calculate the height of the teacher 5 seconds after he is thrown.

c. How many seconds after the math teacher is thrown does he hit the ground?





2. An object is falling and its height (in meters), after t seconds have gone by, is given by:

$$h = -4t^2 + 121$$

a. At what height was the object released?

b. How many seconds until the object hits the ground?

3. A soccer ball is thrown from the top of a mountain and its height (in meters), after t seconds have gone by, is given by

$$h = -t^2 + 11t + 26$$

a. When the soccer ball is tossed, how far above the ground is it?

b. What is the ball's height 5 seconds after it's thrown?

c. How long will it take for the ball to hit the ground?

Solutions

1. a. Hint: let $t = 0$ b. 153 m c. 8 sec

2. a. 121 m b. 5.5 sec

3. a. 26 m b. 56 m c. 13 sec

DERIVING THE QUADRATIC FORMULA

This chapter assumes that you have finished solving quadratic equations by *Completing the Square*. You might understand at this point in your studies that the *factoring* method for solving quadratic equations seldom works, while Completing the Square *always* works.



And it works beautifully whether the solutions are integers, fractions (rational numbers), or radicals (irrational numbers). It even tells us when an equation has no solutions in $\mathbb{R}$, the system of real numbers (which occurs when we reach the square root of a negative number).


"I FOUND OUT TODAY IN SCHOOL THAT I'M QUADRATIC EQUATION CHALLENGED!"

But Completing the Square is quite boring and not the easiest thing in the world to do. There must be some way to complete the square on a "generic" quadratic equation, $ax^2 + bx + c = 0$, and end up with a formula that can be used to solve any quadratic equation more efficiently, whether it's factorable or not.

❑ THE QUADRATIC FORMULA

We will start with a generic quadratic equation in standard form, and then try to *complete the square* . . . here we go:

$$ax^2 + bx + c = 0$$ [We assume $a > 0$, because if it's not, we can multiply through by -1 and make $a > 0$.]

The technique of completing the square requires that we always have a leading coefficient of 1, and so the first step is to divide each side of the equation by a (which we know is not zero — otherwise, we wouldn't have a quadratic equation in the first place):

Divide each side of the equation by a:

$$\frac{ax^2 + bx + c}{a} = \frac{0}{a}$$

To simplify the left side of the equation, just be sure you divide <u>each</u> term by a:

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = \frac{0}{a}$$ (basically the reverse of adding fractions)

And then simplify each side:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \left(\text{since } \frac{bx}{a} = \frac{b}{a} \cdot \frac{x}{1} = \frac{b}{a} \cdot x = \frac{b}{a}x\right)$$

Now bring the constant to the right side of the equation; that is, subtract $\frac{c}{a}$ from each side of the equation (we could have done this step first):

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

It's now time to compute the "magic number," the quantity that will "complete the square." We start with the coefficient of the linear term, in this case $\frac{b}{a}$, take half of it, and then square that result:

$$\frac{b}{a} \times \frac{1}{2} = \frac{b}{2a}, \text{ and then } \left(\frac{b}{2a}\right)^2 = \frac{b^2}{(2a)^2} = \boxed{\frac{\boldsymbol{b^2}}{\boldsymbol{4a^2}}}$$ ← **The magic number**

It's time to add the magic number to each side of the equation:

$$x^2 + \frac{b}{a}x + \boxed{\frac{b^2}{4a^2}} = -\frac{c}{a} + \boxed{\frac{b^2}{4a^2}}$$

There are two things to do now: factor on the left and combine the fractions on the right. Let's factor first:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Notice that the variable x occurs just once in the entire equation (a good thing), buried amidst the parentheses on the left side of the equation.

Now we'll combine the fractions on the right side of the equation into a single fraction. Since the LCD of the denominators is $4a^2$, we need to multiply the top and the bottom of the first fraction by $4a$:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a}\left[\frac{\mathbf{4a}}{\mathbf{4a}}\right] + \frac{b^2}{4a^2} \qquad \left(\text{since } \frac{4a}{4a} = 1\right)$$

or, $$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac}{4a^2} + \frac{b^2}{4a^2}$$ (We now have a common denominator)

Adding the fractions yields the equation

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

Using the commutative property for addition, we can reverse the two terms in the numerator of the right side of the equation:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

The next step we need in order to isolate the x is to take the square root of each side of the equation, remembering that every non-negative number has <u>two</u> square roots, denoted $\pm\sqrt{\ }$:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{(2 square roots!)}$$

To isolate the x (which is the whole point of this endeavor), we'll bring the term $\frac{b}{2a}$ to the right side of the equation (and put it in front):

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

We've isolated the *x*. Yay! The rest is simplifying.

Now it's appropriate to split the radical into two separate radicals:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

Recall: $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$ (for $y > 0$)

Simplify the radical in the denominator:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Technically speaking, $\sqrt{4a^2} = 2|a|$. But since we assumed at the outset that $a > 0$, the radical becomes $2a$.

Finally, we combine the two fractions into a single fraction (can you believe the two fractions already have the same denominator? ☺):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In a single sentence: We've isolated the *x* in the quadratic equation $ax^2 + bx + c = 0$ by *completing the square*!

The solutions of the quadratic equation

$$\boldsymbol{ax^2 + bx + c = 0}$$

are given by **The Quadratic Formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Deriving the Quadratic Formula

❑ *NOTES ON THE QUADRATIC FORMULA*

- The Quadratic Formula can be used to solve <u>any</u> quadratic equation, whether or not it's factorable, and whether its solutions are integers (whole numbers and their opposites), fractions, or even radicals (like $\sqrt{2}$). And if the quadratic equation has no solutions at all in the real numbers (that is, they have imaginary solutions), the Quadratic Formula will tell us that, too.
- Because of the $\pm$ sign, there are potentially two solutions. In fact, the two solutions of the quadratic equation $ax^2 + bx + c = 0$ can be written separately if you prefer:

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad x = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

- The Quadratic Formula is a <u>single fraction</u>:

It is **<u>NOT</u>** $x = -b \pm \frac{\sqrt{b^2-4ac}}{2a}$

If you don't write it as a single fraction at every stage of your solution, either you or your instructor will misread it and you'll be the one to lose points on the problem.

Homework

1. Which symbol in the Quadratic Formula is responsible for possibly giving us two solutions to a quadratic equation?
2. How many solutions of the quadratic equation will there be if the quantity inside the square root sign (called the *radicand*) is 100?
3. How many solutions of the quadratic equation will there be if the radicand is 30?

4. How many solutions of the quadratic equation will there be if the radicand is 0?

5. How many solutions of the quadratic equation will there be if the radicand is –9?

6. Consider the possibility that the value of a is **0** in the quadratic equation $ax^2 + bx + c = 0$. Answer the question, Does the Quadratic Formula apply, in two ways:

   a. Consider what happens to the quadratic equation if $a = \mathbf{0}$.

   b. Now analyze what happens if we actually allow $a = \mathbf{0}$ to wriggle its way into the Quadratic Formula.

7. Consider the quantity "$\boldsymbol{b^2 - 4ac}$," the radicand inside the square root sign of the Quadratic Formula.

   a. If $b^2 - 4ac > 0$, then the equation has ____ solution(s).

   b. If $b^2 - 4ac = 0$, then the equation has ____ solution(s).

   c. If $b^2 - 4ac < 0$, then the equation has ____ solution(s).

   [The quantity $b^2 - 4ac$ is called the ***discriminant*** of the quadratic polynomial $ax^2 + bx + c$.]

Solutions

1. The plus/minus sign: ±

2. Since $\sqrt{100} = 10$, and since there's a plus/minus sign in front of the radical, there will be <u>two</u> solutions.

3. Even though $\sqrt{30}$ doesn't result in a whole number, there's still a plus/minus sign in front of it, so there will also be <u>two</u> solutions.

4. Since $\sqrt{0} = 0$, and since ± 0 is just the single number 0, the plus/minus sign basically disappears, leaving <u>one</u> solution.

5. Since $\sqrt{-9}$ does not exist in early algebra classes, the Quadratic Formula has no meaning, and so there are <u>no</u> solutions.

6. First, if $a = 0$, the quadratic equation $ax^2 + bx + c = 0$ becomes $bx + c = 0$, which is NOT quadratic anymore. Indeed, we learned months ago how to solve this equation:

$$bx + c = 0 \;\Rightarrow\; bx = -c \;\Rightarrow\; x = -\frac{c}{b}$$

Second, if we ignore the above fact and use $a = 0$ in the Quadratic Formula, the denominator becomes $2a = 2(0) = 0$. That is, we're dividing by zero, which is undefined. However you look at it, the Quadratic Formula with $a = 0$ simply makes no sense.

7. a. 2 b. 1 c. 0 (assuming we want to stay within the real numbers.)

Postscript:

The derivation of the Quadratic Formula in this chapter was the classic method, using the method of completing the square. This was a fine technique to use, since completing the square will be used in your later studies of circles, parabolas, and other shapes — and the more practice, the better.

But there's another way to derive the Quadratic Formula, perhaps a way that may be easier for you to follow. Let's do it now — beginning, of course, with our quadratic equation in standard form:

$$ax^2 + bx + c = 0$$

First, multiply each side of the equation by $4a$:

$$4a(ax^2 + bx + c) = 4a(0)$$

Distribute on the left and simplify on the right:

$$4a^2x^2 + 4abx + 4ac = 0$$

Bring the $4ac$ to the right side:

$$4a^2x^2 + 4abx = -4ac$$

Next, add b^2 to each side of the equation:

$$4a^2x^2 + 4abx + \boxed{b^2} = -4ac + \boxed{b^2}$$

Now factor the left side and flip the terms on the right side:

$$(2ax + b)^2 = b^2 - 4ac$$

Then remove the squaring by taking <u>both</u> square roots on the right:

$$2ax + b = \pm\sqrt{b^2 - 4ac}$$

Subtract b from each side of the equation:

$$2ax = -b \pm \sqrt{b^2 - 4ac}$$

And last, divide each side of the equation by $2a$:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

WE DID IT!!





THE QUADRATIC FORMULA – IRRATIONAL SOLUTIONS

❑ *INTRODUCTION*



Your first experience with the Quadratic Formula probably involved solving a quadratic equation like $6x^2 + 31x + 40 = 0$, whose solutions are $-\frac{5}{2}$ and $-\frac{8}{3}$. In fact, all of the answers back then were either positive or negative whole numbers, or fractions, or zero. [In other words, all the solutions were *rational numbers*.]

You may have realized that we obtained these kinds of answers because the square-root part of the Quadratic Formula $(\sqrt{b^2 - 4ac})$ always worked out to the square root of a perfect square, like $\sqrt{49}$ or $\sqrt{0}$.

But we know that there are square roots like $\sqrt{5}$. What would we do if we came up with a radical like that, assuming that no calculator is allowed? In this section, we'll see that there's really nothing special to worry about — just work the Quadratic Formula:

$$\text{If } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 1: **Solve for x: $2x^2 - 3x - 1 = 0$**

Solution: For this quadratic equation, $a = 2$, $b = -3$, and $c = -1$. Inserting these values into the Quadratic Formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9+8}}{4} = \boxed{\frac{3 \pm \sqrt{17}}{4}}$$

And that's it, because $\sqrt{17}$ cannot be simplified and the fraction cannot be reduced. If you'd like, the two solutions of the equation can be written separately:

$$x = \frac{3+\sqrt{17}}{4}, \frac{3-\sqrt{17}}{4}$$

Homework

1. Solve each quadratic equation:

a. $x^2 - 5x + 1 = 0$

b. $2n^2 + n - 7 = 0$

c. $3a^2 - 3a - 1 = 0$

d. $5t^2 + 7t + 1 = 0$

e. $5x^2 + 15x - 2 = 0$

f. $t^2 + t - 8 = 0$

g. $5h^2 - 13h = -5$

h. $7u^2 + u = 3$

i. $2k^2 + 7k = 8$

j. $2m^2 - 3 = -9m$

2. Solve for x: $-7x^2 + 5x + 1 = 0$

Hint: Although not necessary, it's traditional to always make a, the leading coefficient, positive. This way, your solutions will more likely match those provided by your teacher or the textbook. Since $a = -7$ in this quadratic equation, we can multiply (or divide) each side of the equation by -1, yielding $7x^2 - 5x - 1 = 0$. Now you do the rest.

❑ *MORE QUADRATIC EQUATIONS*

EXAMPLE 2: **Solve for x: $2x^2 - 8x + 3 = 0$**

Solution: The values of a, b, and c for use in the Quadratic Formula are

$a = 2 \qquad b = -8 \qquad c = 3$

So now we state the Quadratic Formula, insert the three values, and then do a lot of arithmetic and simplifying until the final answer is exact (no decimals) and as clean as possible.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{(The Quadratic Formula)}$$

$$= \frac{-(-\mathbf{8}) \pm \sqrt{(-\mathbf{8})^2 - 4(\mathbf{2})(\mathbf{3})}}{2(\mathbf{2})} \qquad (a = 2 \quad b = -8 \quad c = 3)$$

$$= \frac{8 \pm \sqrt{64 - 24}}{4} \qquad \text{(square and multiply)}$$

$$= \frac{8 \pm \sqrt{40}}{4} \qquad \text{(finish the arithmetic)}$$

$$= \frac{8 \pm \sqrt{4 \cdot 10}}{4} \qquad \text{(factor the radicand)}$$

$$= \frac{8 \pm 2\sqrt{10}}{4} \qquad \text{(simplify the radical)}$$

$$= \frac{\cancel{2}(4 \pm \sqrt{10})}{\cancel{2} \cdot 2}$$ (factor the numerator and reduce)

We write our two solutions as

$$\boxed{x = \frac{4 \pm \sqrt{10}}{2}}$$

EXAMPLE 3: **Solve for y: $3y^2 = 13$**

Solution: We begin by subtracting 13 from each side of the equation to convert the equation into standard quadratic form:

$$3y^2 - 13 = 0$$

At this point, we see that $a = 3$, $b = 0$, and $c = -13$. Applying the Quadratic Formula:

$$y = \frac{-\mathbf{0} \pm \sqrt{\mathbf{0}^2 - 4(\mathbf{3})(-\mathbf{13})}}{2(\mathbf{3})} = \frac{\pm\sqrt{0 + 156}}{6} = \frac{\pm\sqrt{156}}{6}$$

$$= \frac{\pm\sqrt{4 \cdot 39}}{6} = \frac{\pm\sqrt{4}\sqrt{39}}{6} = \frac{\pm 2\sqrt{39}}{6} = \frac{\pm 2\sqrt{39}}{2 \cdot 3} = \frac{\pm \cancel{2}\sqrt{39}}{\cancel{2} \cdot 3}$$

And we have our final answer (two of them!) in simplest form:

$$\boxed{y = \pm\frac{\sqrt{39}}{3}}$$

EXAMPLE 5: **Solve for n: $2n^2 = 3n - 4$**

Solution: If we convert to standard form, the equation becomes

$$2n^2 - 3n + 4 = 0$$

where $a = 2$, $b = -3$, and $c = 4$. The Quadratic Formula produces

$$n = \frac{-(-\mathbf{3}) \pm \sqrt{(-\mathbf{3})^2 - 4(\mathbf{2})(\mathbf{4})}}{2(\mathbf{2})} = \frac{3 \pm \sqrt{9-32}}{4} = \frac{3 \pm \sqrt{-23}}{4}$$

Remembering that $\sqrt{-23}$ is not a number in this Algebra course, we stop right here and declare that the given equation has

No Solution

Homework

3. Solve each quadratic equation:

a. $3x^2 - 6x - 1 = 0$

b. $2k^2 = 8k - 6$

c. $2n^2 - 10 = 0$

d. $y^2 = 49$

e. $14d^2 - 4d = 4$

f. $x^2 + 3x + 10 = 0$

g. $5a^2 + 7a - 2 = 0$

h. $u^2 = 40$

i. $3z^2 = 3z$

j. $5x^2 - x + 1 = 0$

k. $x(x - 3) = 7$

l. $y(y + 1) = y(y - 7) + 13$

4. Solve each quadratic equation:

a. $x^2 + 6x + 1 = 0$

b. $2y^2 + 7y = 2$

c. $-n^2 + 28 = 0$

d. $5d^2 = 7d + 1$

e. $x^2 = -4 - 5x$

f. $x^2 + 2x + 1 = 0$

g. $-3a^2 = 5 - 12a$

h. $2x^2 = 7$

i. $3z^2 = 12z$

j. $n^2 + 9 = 0$

k. $-4n^2 + 12n = 9$

l. $t^2 - 9 = 0$

Review Problems

5. Solve each quadratic equation:

a. $x^2 - x - 56 = 0$

b. $u^2 - 8u = 0$

c. $r^2 = 9$

d. $9k^2 + 49 = -42k$

e. $n^2 + 2n - 80 = 0$

f. $-3t^2 = 11t + 6$

g. $2m^2 - 7m = -6$

h. $8d^2 = 2$

i. $-3a^2 = 7a - 6$

j. $-8h^2 - 8h = 0$

k. $n^2 - 3n + 4 = 0$

l. $2p^2 - 3p - 5 = 0$

m. $-3m^2 - 8m + 3 = 0$

n. $-8z^2 + 2z = 0$

o. $-3w^2 + 3 = 0$

p. $5g^2 = 5g + 2$

q. $-3y^2 + 6y = -3$

r. $12t^2 - 8t - 16 = 0$

s. $3x^2 = -15x + 3$

t. $0 = 21g^2 + 9g - 15$

u. $-5z^2 + 4z = -7$

v. $8q^2 + 4q = 3$

w. $y^2 - 4y = -2$

x. $2x^2 - 2x - 3 = 0$

y. $0 = 14d^2 - 4d - 4$

z. $27x^2 - 6x - 15 = 0$

Solutions

1. a. $\frac{5 \pm \sqrt{21}}{2}$ b. $\frac{-1 \pm \sqrt{57}}{4}$ c. $\frac{3 \pm \sqrt{21}}{6}$ d. $\frac{-7 \pm \sqrt{29}}{10}$

e. $\frac{-15 \pm \sqrt{265}}{10}$ f. $\frac{-1 \pm \sqrt{33}}{2}$ g. $\frac{13 \pm \sqrt{69}}{10}$ h. $\frac{-1 \pm \sqrt{85}}{14}$

i. $\frac{-7 \pm \sqrt{113}}{4}$ j. $\frac{-9 \pm \sqrt{105}}{4}$

2. $\frac{5 \pm \sqrt{53}}{14}$

3. a. The quadratic equation $3x^2 - 6x - 1 = 0$ is in standard form. We can read the values $a = 3$, $b = -6$, and $c = -1$. Placing these values into the Quadratic Formula gives

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-1)}}{2(3)} = \frac{6 \pm \sqrt{36+12}}{6} = \frac{6 \pm \sqrt{48}}{6} = \frac{6 \pm \sqrt{16 \cdot 3}}{6}$$

$$= \frac{6 \pm 4\sqrt{3}}{6} = \frac{2(3 \pm 2\sqrt{3})}{2 \cdot 3} = \frac{\cancel{2}(3 \pm 2\sqrt{3})}{\cancel{2} \cdot 3} = \frac{3 \pm 2\sqrt{3}}{3}$$

b. 1, 3 c. $\pm\sqrt{5}$ d. ± 7

e. $\frac{1 \pm \sqrt{15}}{7}$ f. No solution g. $\frac{-7 \pm \sqrt{89}}{10}$ h. $\pm 2\sqrt{10}$

i. 0, 1 j. No solution k. $\frac{3 \pm \sqrt{37}}{2}$ l. $\frac{13}{8}$

4. a. $-3 \pm 2\sqrt{2}$ b. $\frac{-7 \pm \sqrt{65}}{4}$ c. $\pm 2\sqrt{7}$ d. $\frac{7 \pm \sqrt{69}}{10}$

e. $-1, -4$ f. -1 g. $\frac{6 \pm \sqrt{21}}{3}$ h. $\pm\frac{\sqrt{14}}{2}$

i. 0, 4 j. No solution k. $\frac{3}{2}$ l. ± 3

5. a. $-7, 8$ b. $8, 0$ c. ± 3 d. $-\frac{7}{3}$

e. $-10, 8$ f. $-3, -\frac{2}{3}$ g. $2, \frac{3}{2}$ h. $\frac{1}{2}, -\frac{1}{2}$

i. $-3, \frac{2}{3}$ j. $-1, 0$ k. No solution l. $\frac{5}{2}, -1$

m. $-3, \frac{1}{3}$ n. $0, \frac{1}{4}$ o. ± 1 p. $\frac{5 \pm \sqrt{65}}{10}$

q. $1 \pm \sqrt{2}$ r. $\frac{1 \pm \sqrt{13}}{3}$ s. $\frac{-5 \pm \sqrt{29}}{2}$ t. $\frac{-3 \pm \sqrt{149}}{14}$

u. $\frac{2 \pm \sqrt{39}}{5}$ v. $\frac{-1 \pm \sqrt{7}}{4}$ w. $2 \pm \sqrt{2}$ x. $\frac{1 \pm \sqrt{7}}{2}$

y. $\frac{1 \pm \sqrt{15}}{7}$ z. $\frac{1 \pm \sqrt{46}}{9}$

"Do, or do not.

There is no 'try'."

– Yoda (*The Empire Strikes Back*)



PREPARING FOR THE QUADRATIC FORMULA

❑ THE GENERAL QUADRATIC EQUATION

Consider a general quadratic equation in standard form:

$$ax^2 + bx + c = 0$$

where x represents the variable (the unknown) and a, b, and c are numbers. Notice that the squared term is first, the linear term is second, the constant term is third, and there's a 0 on the right side of the equation. The examples below show some quadratic equations along with their corresponding values of a, b, and c.

$3x^2 - 7x + 10 = 0$	⇨	$a = 3$	$b = -7$	$c = 10$
$y^2 - 9 = 0$	⇨	$a = 1$	$b = 0$	$c = -9$
$-4z^2 + 19z = 0$	⇨	$a = -4$	$b = 19$	$c = 0$
$2w^2 = 8 - 5w$	⇨	$a = 2$	$b = 5$	$c = -8$

To find the values of a, b, and c for this last equation, we need to rewrite the quadratic equation in standard form:
$2w^2 + 5w - 8 = 0$, from which the values of a, b, and c can be determined.

Homework

1. For each quadratic equation, first make sure that it's in standard form ($ax^2 + bx + c = 0$); then determine the values of a, b, and c:

a. $3x^2 + 9x + 17 = 0$

b. $2n^2 - 8n + 14 = 0$

c. $6y^2 + 2y - 2 = 0$

d. $5t^2 - 13t - 1 = 0$

e. $12a^2 + 13 = 0$

f. $x^2 - 13x = 0$

g. $u^2 - u + 1 = 0$

h. $-2w^2 - 19 = 0$

i. $-w^2 + 14w = 0$

j. $-z^2 - 99 = 0$

k. $2x^2 = 4x + 3$

l. $-3y^2 - 2y = -5$

m. $c^2 + 4 = 7c$

n. $6m^2 = 1 + m$

o. $18k^2 = 0$

p. $-3x^2 = -3x$

❑ *ORDER OF OPERATIONS*

Remember that exponents are done before multiplication, which itself comes before any addition or subtraction. For example,

	$12^2 - 4(2)(3)$	
=	$144 - 4(2)(3)$	(exponent first)
=	$144 - 24$	(multiplication second)
=	**120**	(subtraction last)

For a second example,

$$
\begin{aligned}
& (-9)^2 - 4(3)(-5) & \\
= \quad & 81 - 4(3)(-5) & \text{(exponent first)} \\
= \quad & 81 - (-60) & \text{(multiplication second)} \\
= \quad & 81 + 60 & \text{(subtracting a negative is adding)} \\
= \quad & \mathbf{141} & \text{(addition last)}
\end{aligned}
$$

❑ *SQUARE ROOTS*

$\sqrt{36} = 6$ $\qquad -\sqrt{144} = -12$

$\sqrt{1} = 1$ $\qquad \sqrt{0} = 0$

$\sqrt{225} = 15$ $\qquad \sqrt{50} \approx 7.0711$

$\sqrt{-9}$ does not exist (in our algebra class)

is approximately equal to

❑ *OPPOSITES*

Recall that the *opposite* of N is $-N$. So we note the following:

1) The opposite of a positive number is a negative number; for example, $-(+7) = -7$. Thus, if $b = 7$, then $-b = -7$.

2) The opposite of a negative number is a positive number; for example, $-(-12) = 12$. Thus, if $b = -12$, then $-b = 12$.

3) The opposite of a squared quantity (that isn't 0) is negative; for example, $-9^2 = -81$. On the other hand, don't forget that $(-9)^2 = 81$.

4) The opposite of 0 is 0.

❑ PLUS/MINUS

The symbol " ± " is read "***plus or minus***" and is just a short way of indicating two numbers at once. For example, if you want to indicate the two numbers 7 and –7, you may write just ±7. Another example might be $\pm\sqrt{81}$, which stands for the two numbers 9 and –9. But $\pm\sqrt{0}$ would only be 0, since +0 and –0 are really just two ways to express 0.

The equation $x^2 = 25$ has two solutions, namely that x could be 5 or –5, so we could write the solution as $x = \pm 5$.

Let's work out a specific problem containing the "plus or minus" sign.

Simplify: $\dfrac{10 \pm \sqrt{25}}{5}$

Taking the square root yields $\dfrac{10 \pm 5}{5}$.

Using the plus sign, we get $\dfrac{10+5}{5} = \dfrac{15}{5} = \mathbf{3}$,

while using the minus set yields $\dfrac{10-5}{5} = \dfrac{5}{5} = \mathbf{1}$.

Thus, the expression $\dfrac{10 \pm \sqrt{25}}{5}$ is merely a strange way to represent the numbers 3 and 1.

Homework

2. Evaluate each expression:

a. $13^2 - 4(3)(2)$

b. $0^2 - 4(2)(-1)$

c. $(-3)^2 - 4(-1)(-2)$

d. $(-5)^2 - 4(1)(0)$

e. $0^2 - 4(17)(0)$

f. $(-4)^2 - (-4)(-5)$

3. Evaluate each expression:

a. $\sqrt{49}$ b. $-\sqrt{6+3}$

c. $\sqrt{16}+\sqrt{9}$ d. $\sqrt{-25}$

e. $\sqrt{(-32)(-2)}$ f. $\sqrt{(-5)^2-4(1)(6)}$

g. $\sqrt{6^2-4(1)(9)}$ h. $\sqrt{(-10)^2-4(8)(2)}$

i. $\sqrt{1^2-4(1)(1)}$ j. $\sqrt{(-7)^2-4(4)(-2)}$

4. Evaluate each expression (the leading minus sign represents the *opposite* of the quantity which follows it):

a. $-(13)$ b. $-(-5)$ c. $-(-(-7))$

d. $-(+7)$ e. $-(-1)$ f. $-(0)$

5. What number or numbers are represented by each of the following?

a. ± 121 b. $\pm\sqrt{121}$ c. $\pm\sqrt{0}$ d. $\pm\sqrt{-1}$

6. Simplify each expression:

a. $100+\sqrt{49}$ b. $-5-\sqrt{121}$ c. $\dfrac{3+\sqrt{16}}{2}$ d. $\dfrac{-8-\sqrt{4}}{10}$

e. $7-\sqrt{49}$ f. $-10+\sqrt{100}$ g. $\dfrac{4\pm\sqrt{16}}{6}$ h. $\dfrac{-1\pm\sqrt{81}}{9}$

7. Evaluate the expression b^2-4ac for the given values of a, b, and c:

a. $a=-10,\ b=-5,\ c=3$ b. $a=4,\ b=6,\ c=-1$

c. $a=5,\ b=0,\ c=7$ d. $a=-2,\ b=-3,\ c=0$

8. Evaluate the expression $\pm\sqrt{b^2-4ac}$ for the given values of a, b, and c:

   a. $a = 2,\ b = -7,\ c = -15$ b. $a = 9,\ b = 30,\ c = 25$

   c. $a = 1,\ b = 0,\ c = -81$ d. $a = 2,\ b = -3,\ c = 0$

❑ *THE ULTIMATE EXAMPLE*

For our final example of this chapter, let's evaluate the expression

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad \text{for the values } a = 1,\ b = -5,\ c = 6$$

First change every variable to a pair of parentheses — this is not required, but it's a handy little trick to help us avoid errors:

$$= \frac{-(\ ) \pm \sqrt{(\ )^2 - 4(\ )(\ )}}{2(\ )}$$

$$= \frac{-(\mathbf{-5}) \pm \sqrt{(\mathbf{-5})^2 - 4(\mathbf{1})(\mathbf{6})}}{2(\mathbf{1})} \qquad \text{(substitute the given values)}$$

$$= \frac{5 \pm \sqrt{25-24}}{2} \qquad \text{(do exponents and multiplying)}$$

$$= \frac{5 \pm \sqrt{1}}{2} \qquad \text{(finish the inside of the radical)}$$

$$= \frac{5 \pm 1}{2} \qquad \text{(the positive square root of 1 is 1)}$$

$$= \frac{5+1}{2} \ \textit{or}\ \frac{5-1}{2} \qquad \text{(split the plus/minus sign)}$$

$$= \frac{6}{2} \ \textit{or}\ \frac{4}{2} \qquad \text{(simplify the numerators)}$$

$$= \boxed{3 \ \textit{or}\ 2} \qquad \text{(and finish up)}$$

There's no homework for this section, but don't be ☹ — there will be plenty of these problems when you study the Quadratic Formula.

Solutions

1. For each quadratic equation, substitute each "solution" (separately) into the original equation. Then work the arithmetic on each side of the equation separately. [Do NOT swap things back and forth across the equals sign.] In all cases, the two sides should balance at the end of the calculations.

2.

a.	3, 9, 17	b.	2, –8, 14	c.	6, 2, –2	d.	5, –13, –1
e.	12, 0, 13	f.	1, –13, 0	g.	1, –1, 1	h.	–2, 0, –19
i.	–1, 14, 0	j.	–1, 0, –99	k.	2, –4, –3	l.	–3, –2, 5
m.	1, –7, 4	n.	6, –1, –1	o.	18, 0, 0	p.	–3, 3, 0

3. a. 145 b. 8 c. 1 d. 25 e. 0 f. –4

4. a. 7 b. –3 c. 7 d. Does not exist e. 8
f. 1 g. 0 h. 6 i. Does not exist j. 9

5. a. –13 b. 5 c. –7 d. –7 e. 1 f. 0

6. a. 121, –121 b. 11, –11 c. 0 d. Does not exist

7. a. 107 b. –16 c. $\frac{7}{2}$ d. –1 e. 0

f. 0 g. $\frac{4}{3}, 0$ h. $\frac{8}{9}, -\frac{10}{9}$



8. a. $b^2 - 4ac$

$= \quad (\ )^2 - 4(\ )(\ )$

Converting each variable to a set of parentheses is a handy way to make sure everything is written properly.

$= \quad (-5)^2 - 4(-10)(3)$

Note: The parentheses around the –5 are required, because as we've learned, $(-5)^2 = 25$, while $-5^2 = -25$.

$= \quad 25 - (-120) \;=\; 25 + 120 \;=\; 145$

b. 52 c. –140 d. 9

9. a. $\pm\sqrt{b^2 - 4ac}$

$= \quad \pm\sqrt{(\ )^2 - 4(\ )(\ )}$

$= \quad \pm\sqrt{(-7)^2 - 4(2)(-15)}$

$= \quad \pm\sqrt{49 - (-120)}$

$= \quad \pm\sqrt{49 + 120}$

$= \quad \pm\sqrt{169}$

$= \quad \pm 13$

b. 0 c. ± 18 d. ± 3

> "*Education* is what you get when you read the fine print. *Experience* is what you get if you don't."
>
> – ***Pete Seeger***

THE QUADRATIC FORMULA – RATIONAL SOLUTIONS

We now have all the tools needed to solve quadratic equations, whether they are factorable or not. In this chapter, our solutions will always be rational numbers; that is, no square roots in the final answers.



❑ *THE QUADRATIC FORMULA*

Here's the formula we created a couple of chapters back:

The quadratic equation $ax^2 + bx + c = 0$

has the solutions: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

❑ *EXAMPLES OF USING THE QUADRATIC FORMULA*

EXAMPLE 1: **Solve for x: $x^2 - 9x - 10 = 0$**

Solution: We see that this is a quadratic equation in standard form, where $a = 1$, $b = -9$, and $c = -10$. By the Quadratic Formula, x has (possibly) two solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(-10)}}{2(1)}$$

$$\Rightarrow \quad x = \frac{9 \pm \sqrt{81 + 40}}{2}$$

$$\Rightarrow \quad x = \frac{9 \pm \sqrt{121}}{2}$$

$$\Rightarrow \quad x = \frac{9 \pm 11}{2}$$

Using the plus sign: $x = \frac{9+11}{2} = \frac{20}{2} = 10$

Using the minus sign: $x = \frac{9-11}{2} = \frac{-2}{2} = -1$

Final answer: $\boxed{x = 10, -1}$

Let's check these solutions:

Letting $x = 10$ in the original equation,

$$x^2 - 9x - 10 = \mathbf{10^2} - 9(\mathbf{10}) - 10 = 100 - 90 - 10 = 0 \quad ✓$$

Letting $x = -1$,

$$x^2 - 9x - 10 = (\mathbf{-1})^2 - 9(\mathbf{-1}) - 10 = 1 + 9 - 10 = 0 \quad ✓$$

EXAMPLE 2: **Solve for n: $6n^2 + 40 = -31n$**

Solution: The first step is to transform this quadratic equation into standard form. Adding $31n$ to each side (and putting the $31n$ between the $6n^2$ and the 40) produces

$$6n^2 + 31n + 40 = 0 \qquad \text{[It's now in standard form.]}$$

Noting that $a = 6$, $b = 31$, and $c = 40$, we're ready to apply the Quadratic Formula:

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad n = \frac{-31 \pm \sqrt{(31)^2 - 4(6)(40)}}{2(6)}$$

$$\Rightarrow \quad n = \frac{-31 \pm \sqrt{961 - 960}}{12}$$

$$\Rightarrow \quad n = \frac{-31 \pm \sqrt{1}}{12}$$

$$\Rightarrow \quad n = \frac{-31 \pm 1}{12}$$

The two solutions are given by

$$\frac{-31+1}{12} = \frac{-30}{12} = -\frac{5}{2}$$

and $$\frac{-31-1}{12} = \frac{-32}{12} = -\frac{8}{3}$$

$$\boxed{n = -\frac{5}{2}, -\frac{8}{3}}$$

EXAMPLE 3: **Solve for p: $2p^2 = 200$**

Solution: Bringing the 200 over to the left side of the equation gives us our quadratic equation in standard form:

$$2p^2 - 200 = 0$$

which, if it helps you, can be viewed as

$$2p^2 + \mathbf{0}p - 200 = 0$$

Notice that $a = 2$, $b = 0$, and $c = -200$. Substituting these values into the Quadratic Formula yields our two solutions for p:

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad p = \frac{-0 \pm \sqrt{(0)^2 - 4(2)(-200)}}{2(2)}$$

$$\Rightarrow \quad p = \frac{0 \pm \sqrt{0 - 4(2)(-200)}}{4}$$

$$\Rightarrow \quad p = \frac{\pm\sqrt{1600}}{4}$$

$$\Rightarrow \quad p = \frac{\pm 40}{4} = \pm 10$$

Thus, the solutions are $\boxed{p = \pm 10}$

Note: At or near the beginning of this problem, we could have divided each side of the equation by 2. The Quadratic Formula, although containing different values for *a* and *c*, would nevertheless have yielded the same answers, ±10. Try it.

EXAMPLE 4: **Solve for u: $5u^2 = 9u$**

Solution: First convert to standard form: $5u^2 - 9u = 0$, from which we deduce that $a = 5$, $b = -9$, and $c = 0$. Plugging these three values into the Quadratic Formula gives:

$$u = \frac{-(-\mathbf{9}) \pm \sqrt{(-\mathbf{9})^2 - 4(\mathbf{5})(\mathbf{0})}}{2(\mathbf{5})} = \frac{9 \pm \sqrt{81 - 0}}{10} = \frac{9 \pm 9}{10}$$

No radicals remain, so that issue is settled; using the plus and minus signs separately yields the two solutions:

$$\frac{9+9}{10} = \frac{18}{10} = \frac{9}{5} \quad \text{and} \quad \frac{9-9}{10} = \frac{0}{10} = 0$$

Final answer: $\boxed{u = \frac{9}{5},\ 0}$

EXAMPLE 5: **Solve for u: $28u - 4u^2 = 49$**

Solution: The standard quadratic form requires that the quadratic term (the squared variable) be in front, followed by the linear term, followed by the constant, all set equal to zero; that is, $ax^2 + bx + c = 0$. We achieve this goal with the following steps:

$$28u - 4u^2 = 49 \qquad \text{(the given equation)}$$

$$\Rightarrow \quad -4u^2 + 28u = 49 \qquad \text{(commutative property)}$$

$$\Rightarrow \quad -4u^2 + 28u - 49 = 0 \qquad \text{(subtract 49 from each side)}$$

At this point, it's in good form for the Quadratic Formula, but it's traditional to not allow the leading coefficient (the –4) to be negative. One way to change –4 into 4 is to multiply it by –1; but, of course, we will have to do this to each side of the equation.

$$\Rightarrow \quad \mathbf{-1}[-4u^2 + 28u - 49] = \mathbf{-1}[0] \qquad \text{(multiply each side by } -1\text{)}$$

$$\Rightarrow \quad 4u^2 - 28u + 49 = 0 \qquad \text{(distribute)}$$

When written this way,

$$4u^2 - 28u + 49 = 0,$$

and we deduce that $a = 4$, $b = -28$, and $c = 49$. On to the Quadratic Formula:

$$u = \frac{-(-28) \pm \sqrt{(-28)^2 - 4(4)(49)}}{2(4)}$$

$$= \frac{28 \pm \sqrt{784 - 784}}{8}$$

$$= \frac{28 \pm \sqrt{0}}{8} = \frac{28 \pm 0}{8} = \frac{28}{8} = \frac{7}{2}$$

And we're done: $\boxed{u = \frac{7}{2}}$

Notice that although all the previous quadratic equations had two solutions, this equation has just <u>one</u> solution. (Or does it have two solutions that are the same?)

<u>EXAMPLE 6:</u> **Solve for z: $z^2 + 3z + 3 = 0$**

<u>Solution:</u> This quadratic equation seems innocent enough, so let's take the values $a = 1$, $b = 3$, and $c = 3$ and place them in their proper places in the Quadratic Formula:

$$z = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(3)}}{2(1)} = \frac{-3 \pm \sqrt{9 - 12}}{2} = \frac{-3 \pm \sqrt{-3}}{2}$$

Houston . . . we have a problem. Under the assumption that you have not yet encountered imaginary numbers, we cannot consider the square root of a negative number at this point in your studies. If we try to take the square root of –3, the calculator will almost for sure indicate an error. We can't finish this calculation (at least not in the early and middle stages of Algebra), so we must agree that there is simply **NO SOLUTION** to the equation $z^2 + 3z + 3 = 0$.

Homework

1. Solve each quadratic equation:

   a. $x^2 - 3x - 10 = 0$
   b. $6n^2 + 19n - 7 = 0$
   c. $d^2 + 7d = 0$
   d. $2u^2 + u + 1 = 0$
   e. $z^2 - 49 = 0$
   f. $t^2 + 16 = 0$
   g. $n^2 - 10n = 0$
   h. $w^2 - 144 = 0$

2. Solve each quadratic equation:

   a. $n^2 + 10n = -25$
   b. $4x^2 = 4x - 1$
   c. $3m^2 = 8m$
   d. $x^2 = -5 + 3x$
   e. $-x^2 - 5x + 6 = 0$
   f. $2x^2 - 5x - 3 = 0$

3. Solve each quadratic equation:

   a. $2w^2 - 70w + 500 = 0$
   b. $w^2 + 2w + 5 = 0$
   c. $9w^2 - 24w + 16 = 0$
   d. $7n^2 = 10n$
   e. $2x^2 - 15x + 7 = 0$
   f. $15x^2 - 8 = 2x$
   g. $4a^2 + 35a = -24$
   h. $21t^2 = 20t + 9$

4. Solve each quadratic equation:

   a. $w^2 - 12w + 35 = 0$
   b. $2x^2 - 13x + 15 = 0$
   c. $n^2 - 20n + 150 = 0$
   d. $6a^2 - 31a + 40 = 0$
   e. $-2x^2 - 10x + 28 = 0$
   f. $25y^2 = 4$
   g. $2z^2 + 2 = -4z$
   h. $-4a^2 = 4a - 3$
   i. $6u^2 = 47u + 8$
   j. $0 = 4t^2 + 8t + 3$
   k. $-n^2 + n + 56 = 0$
   l. $-14x^2 = -40 + 46x$

Review Problems

5. Solve for y: $15y^2 + 2y - 8 = 0$
6. Solve for a: $-49a^2 - 9 = -42a$
7. Solve for x: $4x^2 = 1$
8. Solve for z: $10z^2 = 9z$
9. Solve for n: $3n^2 = 7n - 5$
10. Solve for w: $-2w^2 + 5w + 3 = 0$

❑ *To ∞ and Beyond!*

A. Solve for x: $\pi x^2 = \sqrt{2}\,x - \sqrt{\pi}$

B. Solve for x: $ax^2 + bx^2 + cx + dx + \pi^2 - \pi = 0$

Solutions

1. a. 5, –2 b. $\frac{1}{3}, -\frac{7}{2}$ c. 0, –7 d. No solution
e. ± 7 f. No solution g. 10, 0 h. ± 12

2. a. –5 b. $\frac{1}{2}$ c. $0, \frac{8}{3}$
d. No solution e. 1, –6 f. $3, -\frac{1}{2}$

3. a. 25, 10 b. No solution c. $\frac{4}{3}$ d. $0, \frac{10}{7}$

e. $7, \frac{1}{2}$ f. $-\frac{2}{3}, \frac{4}{5}$ g. $-8, -\frac{3}{4}$ h. $\frac{9}{7}, -\frac{1}{3}$

4. a. 5, 7 b. $5, \frac{3}{2}$ c. No solution d. $\frac{8}{3}, \frac{5}{2}$

e. 2, –7 f. $\pm\frac{2}{5}$ g. –1 h. $\frac{1}{2}, -\frac{3}{2}$

i. $8, -\frac{1}{6}$ j. $-\frac{1}{2}, -\frac{3}{2}$ k. 8, –7 l. $-4, \frac{5}{7}$

5. $\frac{2}{3}, -\frac{4}{5}$

6. $\frac{3}{7}$

7. $\pm\frac{1}{2}$

8. $0, \frac{9}{10}$

9. No solution

10. $3, -\frac{1}{2}$

"Education's purpose is to replace an empty mind with an open one."

Malcolm Forbes

SOLVING QUADRATICS BY TAKING SQUARE ROOTS

❑ *INTRODUCTION*



Let's look at the quadratic equation

$$x^2 = 100$$

One way you may have learned to solve this equation is by **factoring**:

$$x^2 = 100 \;\Rightarrow\; x^2 - 100 = 0 \;\Rightarrow\; (x + 10)(x - 10) = 0$$

$$\Rightarrow \quad x + 10 = 0 \text{ or } x - 10 = 0 \;\Rightarrow\; x = -10 \text{ or } x = 10$$

The solutions of the quadratic equation $x^2 = 100$ are simply $\boldsymbol{x = \pm 10}$.

But there's a second way to solve a quadratic equation like $x^2 = 100$: Just take the square root of each side of the equation, remembering that the number 100 has **TWO** square roots, namely 10 and –10. Therefore, $\boldsymbol{x = \pm 10}$, and no factoring is required for this simple quadratic equation.

For another example, let's solve the quadratic equation $n^2 = 30$. Remembering that 30 has two square roots, we calculate n to have two solutions, $\pm\sqrt{\mathbf{30}}$.

For a third example, where we will need to simplify the radical, consider the quadratic equation $y^2 = 72$. When we take the square root of each side of the equation — *and when we remember that* 72 *has two square roots* — we see that

$$y^2 = 72 \;\Rightarrow\; y = \pm\sqrt{72} = \pm\sqrt{36 \cdot 2} = \pm\sqrt{36} \cdot \sqrt{2} = \pm 6\sqrt{2}$$

In short, the solutions of $y^2 = 72$ are $\boldsymbol{\pm 6\sqrt{2}}$.

❑ *THE SQUARE ROOT THEOREM*

Now for the general statement:

> The solutions of the equation
> $x^2 = A$ are $x = \pm\sqrt{A}$

The Square Root Theorem

Notes:

1) The value of A in the Square Root Theorem is assumed to be zero or positive; that is, $A \geq 0$. Otherwise, the square root will be imaginary, which may or may not represent a solution in your Algebra course; in this chapter, the square root of a negative number will not count as a solution.

2) How do students usually mess up this kind of equation? By forgetting to include both square roots (that is, they forget the "±" sign). DON'T MESS UP!



EXAMPLE 1: **Solve the quadratic equation: $(x + 7)^2 = 81$**

Solution: According to the Square Root Theorem, we can remove the squaring by taking the square root of both sides of the equation, remembering that the number 81 has two square roots:

$(x + 7)^2 = 81$ (the original equation)

$\Rightarrow \quad x+7 = \pm\sqrt{81}$ (the Square Root Theorem)

$\Rightarrow \quad x+7 = \pm 9$ ($\sqrt{81} = 9$)

$\Rightarrow \quad x = -7 \pm 9$ (subtract 7 from each side)

Using the plus sign yields $x = -7 + 9 = 2$.

Using the minus sign yields $x = -7 - 9 = -16$.

$$\boxed{x = 2 \text{ or } -16}$$

EXAMPLE 2: **Solve for y: $(y-3)^2 = 32$**

Solution: First we need to remove, or undo, the squaring in this quadratic equation. This is where we apply the Square Root Theorem:

$y-3 = \pm\sqrt{32}$ (32 has two square roots)

To isolate the y, add 3 to both sides:

$y = 3 \pm \sqrt{32}$

Since $\sqrt{32} = \sqrt{16\times 2} = \sqrt{16}\times\sqrt{2} = 4\sqrt{2}$, we simplify our solution:

$$\boxed{y = 3 \pm 4\sqrt{2}}$$

Be sure it's clear to you that we have written two solutions to our quadratic equation:

$3+4\sqrt{2}$ and $3-4\sqrt{2}$

EXAMPLE 3: **Solve for n: $(n-3)^2 = -49$**

Solution: Applying the Square Root Theorem to remove the squaring gives us the equation

$$n - 3 = \pm\sqrt{-49}$$

We needn't go any further; after all, we're not dealing with imaginary numbers in this chapter. So we're done right here, and we conclude that the equation has

No Solution

Homework

1. Solve each equation by applying the Square Root Theorem:

a. $x^2 = 144$	b. $y^2 = 51$	c. $z^2 = 72$
d. $a^2 = 0$	e. $b^2 = -9$	f. $(x+1)^2 = 25$
g. $(n-3)^2 = 100$	h. $(u+10)^2 = 1$	i. $(a-5)^2 = 32$
j. $(b+7)^2 = 50$	k. $(w+13)^2 = -4$	l. $(m-3)^2 = 75$

❑ *SPLITTING RADICALS IN DIVISION*

Do you remember the rule about splitting the square root of a product: $\sqrt{ab} = \sqrt{a}\sqrt{b}$? The same kind of rule works for division. For example, we know that $\sqrt{\frac{9}{25}} = \frac{3}{5}$, since $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$. Now let's work it out by "splitting" the radical:

$$\underbrace{\sqrt{\frac{9}{25}} = \frac{\sqrt{9}}{\sqrt{25}}}_{\text{split the radical}} = \frac{3}{5}, \text{ the same answer!}$$

In short, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ [assuming that $a \geq 0$ and $b > 0$]

Homework

2. Calculate each square root by "splitting" the radical:

a. $\sqrt{\frac{1}{25}}$ b. $\sqrt{\frac{9}{49}}$ c. $\sqrt{\frac{16}{81}}$ d. $\sqrt{\frac{100}{121}}$ e. $\sqrt{\frac{144}{36}}$

We conclude this chapter with an example that requires us to "split" the square root of a fraction.

EXAMPLE 4: **Solve for z:** $\left(z-\frac{4}{5}\right)^2 = \frac{8}{25}$

Solution: Using the Square Root Theorem, we take the square root of each side of the equation, remembering that $\frac{8}{25}$ has two square roots:

$$z-\frac{4}{5} = \pm\sqrt{\frac{8}{25}} \qquad \text{(the Square Root Theorem)}$$

$$\Rightarrow \quad z = \frac{4}{5} \pm \sqrt{\frac{8}{25}} \qquad \text{(isolate the } z \text{ by adding 4/5)}$$

$$\Rightarrow \quad z = \frac{4}{5} \pm \frac{\sqrt{8}}{\sqrt{25}} \qquad \text{(split the radical)}$$

$$\Rightarrow \quad z = \frac{4}{5} \pm \frac{2\sqrt{2}}{5} \qquad \text{(simplify both square roots)}$$

$$\Rightarrow \quad \boxed{z = \frac{4 \pm 2\sqrt{2}}{5}} \qquad \text{(combine into a single fraction)}$$

Again, note that we have found two solutions. They may be ugly, but both of them satisfy the equation.

Homework

3. Solve each equation by applying the Square Root Theorem:

a. $\left(x-\frac{1}{2}\right)^2 = \frac{3}{4}$ b. $\left(t+\frac{2}{3}\right)^2 = \frac{1}{9}$ c. $\left(z-\frac{4}{5}\right)^2 = \frac{19}{25}$

d. $\left(x+\frac{3}{5}\right)^2 = \frac{12}{25}$ e. $\left(b-\frac{9}{10}\right)^2 = \frac{81}{100}$ f. $\left(g-\frac{3}{7}\right)^2 = \frac{24}{49}$

Review Problems

4. Solve each equation by applying The Square Root Theorem:

a. $x^2 = 121$ b. $y^2 = 50$ c. $z^2 = 0$

d. $n^2 = -25$ e. $t^2 = 288$ f. $a^2 - 14 = 0$

g. $(x+1)^2 = 75$ h. $(c-3)^2 = 10$ i. $(x+10)^2 = -1$

j. $\left(w+\frac{1}{2}\right)^2 = \frac{3}{4}$ k. $\left(u-\frac{4}{3}\right)^2 = \frac{26}{9}$ l. $\left(a+\frac{8}{5}\right)^2 = \frac{32}{25}$

Solutions

1. a. $x = \pm 12$ b. $y = \pm\sqrt{51}$ c. $z = \pm 6\sqrt{2}$

d. $a = 0$ e. No solution f. $x = 4, -6$

g. $n = 13, -7$ h. $u = -9, -11$ i. $a = 5 \pm 4\sqrt{2}$

j. $b = -7 \pm 5\sqrt{2}$ k. No solution l. $m = 3 \pm 5\sqrt{3}$

2. a. $\sqrt{\frac{1}{25}} = \frac{\sqrt{1}}{\sqrt{25}} = \frac{1}{5}$ b. $\frac{3}{7}$ c. $\frac{4}{9}$ d. $\frac{10}{11}$ e. 2

3. a. $\left(x - \frac{1}{2}\right)^2 = \frac{3}{4} \Rightarrow x - \frac{1}{2} = \pm\sqrt{\frac{3}{4}} \Rightarrow x = \frac{1}{2} \pm \frac{\sqrt{3}}{\sqrt{4}} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} = \frac{1 \pm \sqrt{3}}{2}$

b. $t = -\frac{1}{3}, -1$ c. $z = \frac{4 \pm \sqrt{19}}{5}$ d. $x = \frac{-3 \pm 2\sqrt{3}}{5}$

e. $b = \frac{9}{5}, 0$ f. $g = \frac{3 \pm 2\sqrt{6}}{7}$

4. a. $x = \pm 11$ b. $y = \pm 5\sqrt{2}$ c. $z = 0$

d. No solution e. $t = \pm 12\sqrt{2}$ f. $a = \pm\sqrt{14}$

g. $x = -1 \pm 5\sqrt{3}$ h. $c = 3 \pm \sqrt{10}$ i. No solution

j. $w = \frac{-1 \pm \sqrt{3}}{2}$ k. $u = \frac{4 \pm \sqrt{26}}{3}$ l. $a = \frac{-8 \pm 4\sqrt{2}}{5}$



"Learning is a treasure that will follow its owner everywhere."

Chinese Proverb

RADICALS – ADVANCED

EXAMPLE 2: **Simplify each radical expression:**

A. $\sqrt{50xy^2} = \sqrt{5^2 \cdot 2 \cdot x \cdot y^2} = 5y\sqrt{2x}$

B. $\sqrt{144a^{46}b^{19}} = \sqrt{12^2 \cdot (a^{23})^2 \cdot (b^9)^2 \cdot b} = 12a^{23}b^9\sqrt{b}$

C. $\sqrt[3]{16x^{48}w^{14}} = \sqrt[3]{2^3 \cdot 2 \cdot (x^{16})^3 \cdot (w^4)^3 \cdot w^2} = 2x^{16}w^4\sqrt[3]{2w^2}$

D. $\sqrt[4]{c^{20}d^{39}} = \sqrt[4]{(c^5)^4(d^9)^4 d^3} = c^5d^9\sqrt[4]{d^3}$

E. $\sqrt[5]{x^4y^{53}} = \sqrt[5]{x^4(y^{10})^5 y^3} = y^{10}\sqrt[5]{x^4y^3}$

F. $\sqrt[3]{54a^2b^{27}c^{17}} = \sqrt[3]{3^3 \cdot 2 \cdot a^2 \cdot (b^9)^3 \cdot (c^5)^3 \cdot c^2} = 3b^9c^5\sqrt[3]{2a^2c^2}$

G. $\sqrt[5]{64x^3y^5z^{25}w^{47}} = \sqrt[5]{2^5 \cdot 2 \cdot x^3 \cdot y^5 \cdot (z^5)^5 \cdot (w^9)^5 \cdot w^2}$
$= 2yz^5w^9\sqrt[5]{2x^3w^2}$

As we'll see in the following example, fractional exponents are indispensable for some radical problems.

EXAMPLE 3: **Simplify each radical expression:**

A. $\sqrt{\sqrt[3]{x}} = \left(x^{1/3}\right)^{1/2} = x^{1/6} = \sqrt[6]{x}$

B. $\sqrt[4]{\sqrt[3]{\sqrt{2}}} = \left[\left(2^{1/2}\right)^{1/3}\right]^{1/4} = 2^{1/24} = \sqrt[24]{2}$

C. $\sqrt[12]{a^9} = \left(a^9\right)^{1/12} = a^{9/12} = a^{3/4} = \sqrt[4]{a^3}$

NEW Problems:

Single Radical: $\sqrt{x}\sqrt[3]{y}$ $\sqrt[3]{ab^2}\sqrt[5]{a^2b}$

Single Term: $a\sqrt{x}+\sqrt{x^3}$ [Assume $x \geq 0$]

Homework

1. Simplify: a. $\sqrt{200xy^2z^3w^{16}v^{37}}$ b. $\sqrt{72a^{25}b^{12}}$

2. Simplify: a. $\sqrt[3]{16ab^2c^3d^{27}e^{50}}$ b. $\sqrt[3]{54x^{22}y^{32}}$

3. Simplify: a. $\sqrt[4]{32x^4y^{16}z^{39}}$ b. $\sqrt[4]{81a^4b^{196}}$

4. Simplify: a. $\sqrt[5]{64u^{32}v^{54}}$ b. $\sqrt[5]{486w^4z^{32}}$

5. Use your calculator to verify Example 3B.

6. DISprove the conjecture: $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$ [Hint: use numbers]

7. Simplify each nested radical:

a. $\sqrt{\sqrt[4]{7}}$ b. $\sqrt[3]{\sqrt{10}}$ c. $\sqrt[5]{\sqrt[7]{x}}$

d. $\sqrt[4]{\sqrt[4]{n}}$ e. $\sqrt[5]{\sqrt{\sqrt[3]{a}}}$ f. $\sqrt[4]{\sqrt[3]{\sqrt{x}}}$

8. Simplify:

a. $\sqrt[6]{x^2}$ b. $\sqrt[8]{n^6}$ c. $\sqrt[10]{x^5}$ d. $\sqrt[9]{z^3}$

e. $\sqrt[20]{a^{15}}$ f. $\sqrt[21]{n^{14}}$ g. $\sqrt[30]{m^{26}}$ h. $\sqrt[88]{x^{77}}$

9. Simplify $\sqrt{\sqrt[3]{\sqrt[4]{3}}}$. Use your calculator to verify your answer.

Solutions

1. a. $10yzw^8v^{18}\sqrt{2xzv}$ b. $6a^{12}b^6\sqrt{2a}$

2. a. $2cd^9e^{16}\sqrt[3]{2ab^2e^2}$ b. $3x^7y^{10}\sqrt[3]{2xy^2}$

3. a. $2xy^4z^9\sqrt[4]{2z^3}$ b. $3ab^{49}$

4. a. $2u^6v^{10}\sqrt[5]{2u^2v^4}$ b. $3z^6\sqrt[5]{2w^4z^2}$

5. Does your verification prove that the answer is right?

6. For instance, $\sqrt{9+16} = \sqrt{25} = 5$, whereas $\sqrt{9}+\sqrt{16} = 3+4 = 7$. Note that these numbers constitute a *counterexample*.

7. a. $\left(7^{1/4}\right)^{1/2} = 7^{1/8} = \sqrt[8]{7}$ b. $\sqrt[6]{10}$

c. $\left(x^{1/7}\right)^{1/5} = x^{1/35} = \sqrt[35]{x}$ d. $\sqrt[16]{n}$

e. $\left(\left(a^{1/3}\right)^{1/2}\right)^{1/5} = a^{\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{5}} = a^{1/30} = \sqrt[30]{a}$ f. $\sqrt[24]{x}$

8. a. $\sqrt[6]{x^2} = x^{2/6} = x^{1/3} = \sqrt[3]{x}$ b. $\sqrt[8]{n^6} = n^{6/8} = n^{3/4} = \sqrt[4]{n^3}$

c. $\sqrt{x}$ d. $\sqrt[3]{z}$ e. $\sqrt[4]{a^3}$ f. $\sqrt[3]{n^2}$ g. $\sqrt[15]{m^{13}}$ h. $\sqrt[8]{x^7}$

9. a. $\sqrt[24]{3}$; the problem and the answer are each about 1.046839.

BEYOND SQUARE ROOTS

❑ *SQUARE ROOTS REVIEW*

The number 9 has <u>two</u> ***square roots***, 3 and –3. This is because $3^2 = 9$ <u>and</u> $(-3)^2 = 9$. The positive square root of 9 (the 3) is denoted $\sqrt{9}$, and the negative square root of 9 (the –3) is written $-\sqrt{9}$. In other words, $\sqrt{9} = 3$, and <u>only</u> 3, while $-\sqrt{9} = -3$.

In analyzing $\sqrt{-25}$, the "principle" square root of –25, we discover that we cannot find an answer for this problem, since the square of a real number can <u>never</u> be negative. If there is an answer to $\sqrt{-25}$, it lies outside $\mathbb{R}$, the set of real numbers.

Sometimes the square root radical is written $\sqrt[2]{x}$, with a little 2 cradled in the radical sign. The 2 refers to the fact that taking the square root is essentially the opposite of <u>squaring</u>.

❑ *HIGHER ROOTS*

Consider the number 8. Since two cubed is eight, $2^3 = 8$, we can say that 2 is a ***cube root*** of 8. In fact, it's the only cube root of 8, simply because there's no other real number whose cube is 8.
We write $\sqrt[3]{8} = 2$. Perhaps a little surprising is that we can calculate the cube root of a negative number without leaving the real numbers, $\mathbb{R}$. For example, $\sqrt[3]{-27}$ equals –3, since

$$(-3)^3 = -27 \qquad [-3 \cdot -3 \cdot -3 = -27]$$

The number 16 has <u>two</u> ***fourth roots***. The positive fourth root is $\sqrt[4]{16} = 2$, and the negative fourth root is $-\sqrt[4]{16} = -2$. After all, both 2 and –2 raised to the fourth power result in 16. However, just like square roots, $\sqrt[4]{-1}$ is not a real number.

The ***fifth root*** of 32 is 2; that is, $\sqrt[5]{32} = 2$. This is because $2^5 = 32$. Like cube roots, we can calculate the fifth root of a negative number. For example, $\sqrt[5]{-243}$ equals –3, since $(-3)^5 = -243$.

Homework

1. Find the square root(s) of

   a. 100 b. 15 c. 0 d. –36 e. 1

2. Find the cube root(s) of

   a. 64 b. –125 c. 0 d. 20 e. 1

3. Find the fourth root(s) of

   a. 81 b. 0 c. –625 d. 25 e. 1

4. Find the fifth root(s) of

   a. 1 b. 0 c. –243 d. 29 e. 32

5. Evaluate each radical:

   a. $\sqrt{169}$ b. $\sqrt{225}$ c. $\sqrt[3]{8}$ d. $\sqrt[3]{27}$ e. $\sqrt[3]{-125}$

   f. $\sqrt[4]{625}$ g. $\sqrt[4]{1}$ h. $\sqrt[4]{-16}$ i. $\sqrt[5]{-32}$ j. $\sqrt[5]{0}$

   k. $\sqrt[3]{64}$ l. $\sqrt[3]{216}$ m. $\sqrt[3]{-64}$ n. $-\sqrt[5]{-1}$ o. $\sqrt[4]{16}$

   p. $-\sqrt[4]{81}$ q. $\sqrt[3]{-1}$ r. $-\sqrt[4]{-1}$ s. $\sqrt[5]{243}$ t. $\sqrt{0} + \sqrt[3]{0}$

❑ *SIMPLIFYING MORE ROOTS*

Assume that x and y represent non-negative numbers. Then

$$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$$

The n*th* root of a product
is the product of the n*th* roots

Another useful rule for radicals is the following. If $\boldsymbol{x}$ represents a non-negative number ($\boldsymbol{x} \geq 0$), then

$$\sqrt[n]{x^n} = x$$

The *nth* root cancels
the *nth* power

[The special case of simplifying $\sqrt{x^2}$, where x has negative value, will not be dealt with in this chapter.]

EXAMPLES: **Simplify each radical expression:**

A. $\sqrt{200} = \sqrt{100 \times 2} = \sqrt{100} \times \sqrt{2} = 10\sqrt{2}$

B. $\sqrt[3]{81} = \sqrt[3]{27 \times 3} = \sqrt[3]{27} \times \sqrt[3]{3} = 3\sqrt[3]{3}$

C. $\sqrt[4]{1250} = \sqrt[4]{625 \cdot 2} = \sqrt[4]{625} \cdot \sqrt[4]{2} = 5\sqrt[4]{2}$

D. $\sqrt[3]{2250}$ Sometimes the radicand (the 2250) is too big to easily see if there's a perfect cube in it. So let's try a slightly different approach. We factor the 2250 into primes to get

$$2250 = 2 \cdot 3^2 \cdot 5^3$$

Clearly we can take the cube root of 5^3 (it's 5), but there are not enough of the other factors to take their cube roots. So we can write

$$\sqrt[3]{2250} = \sqrt[3]{2 \cdot 3^2 \cdot 5^3} = \sqrt[3]{5^3} \cdot \sqrt[3]{2 \cdot 3^2} = 5\sqrt[3]{18}$$

Homework

6. Simplify each radical:

a. $\sqrt{288}$	b. $\sqrt[3]{54}$	c. $\sqrt[3]{16}$	d. $\sqrt[3]{250}$
e. $\sqrt[4]{32}$	f. $\sqrt[4]{243}$	g. $\sqrt[4]{162}$	h. $\sqrt[4]{1}$
i. $\sqrt[3]{-54}$	j. $\sqrt[4]{-16}$	k. $\sqrt[5]{64}$	l. $\sqrt[5]{486}$
m. $\sqrt[3]{135}$	n. $\sqrt[4]{162}$	o. $\sqrt[3]{189}$	p. $\sqrt[5]{96}$
q. $\sqrt[3]{128}$	r. $\sqrt[4]{1250}$	s. $\sqrt[3]{250}$	t. $\sqrt[3]{432}$
u. $\sqrt[5]{320}$	v. $\sqrt[3]{48}$	w. $\sqrt[4]{648}$	x. $\sqrt[5]{2673}$

Review Problems

7. Simplify each radical:

a. $\sqrt[3]{108}$	b. $\sqrt[4]{405}$	c. $\sqrt[5]{192}$	d. $\sqrt[3]{-500}$
e. $\sqrt[4]{-32}$	f. $\sqrt[5]{-486}$	g. $\sqrt[3]{3000}$	h. $\sqrt[4]{567}$
i. $\sqrt[3]{648}$	j. $\sqrt[5]{320}$	k. $\sqrt[3]{81}$	l. $\sqrt[3]{250}$
m. $\sqrt[3]{-56}$	n. $\sqrt[4]{48}$	o. $\sqrt[4]{-405}$	p. $\sqrt[4]{768}$
q. $\sqrt[5]{128}$	r. $\sqrt[5]{1215}$		

Solutions

1.	a. ± 10	b. $\pm\sqrt{15}$	c. 0	d. Not real	e. ± 1
2.	a. 4	b. -5	c. 0	d. $\sqrt[3]{20}$	e. 1
3.	a. ± 3	b. 0	c. Not real	d. $\pm\sqrt[4]{25}$	e. ± 1
4.	a. 1	b. 0	c. -3	d. $\sqrt[5]{29}$	e. 2
5.	a. 13	b. 15	c. 2	d. 3	e. -5
	f. 5	g. 1	h. Not real	i. -2	j. 0
	k. 4	l. 6	m. -4	n. 1	o. 2
	p. -3	q. -1	r. Not real	s. 3	t. 0

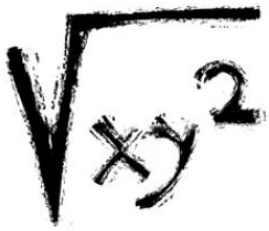
6. a. $12\sqrt{2}$ b. $3\sqrt[3]{2}$ c. $2\sqrt[3]{2}$ d. $5\sqrt[3]{2}$ e. $2\sqrt[4]{2}$ f. $3\sqrt[4]{3}$
g. $3\sqrt[4]{2}$ h. 1 i. $-3\sqrt[3]{2}$ j. Not real k. $2\sqrt[5]{2}$ l. $3\sqrt[5]{2}$
m. $3\sqrt[3]{5}$ n. $3\sqrt[4]{2}$ o. $3\sqrt[3]{7}$ p. $2\sqrt[5]{3}$ q. $4\sqrt[3]{2}$ r. $5\sqrt[4]{2}$
s. $5\sqrt[3]{2}$ t. $6\sqrt[3]{2}$ u. $2\sqrt[5]{10}$ v. $2\sqrt[3]{6}$ w. $3\sqrt[4]{8}$ x. $3\sqrt[5]{11}$

7. a. $3\sqrt[3]{4}$ b. $3\sqrt[4]{5}$ c. $2\sqrt[5]{6}$ d. $-5\sqrt[3]{4}$ e. Not real
f. $-3\sqrt[5]{2}$ g. $10\sqrt[3]{3}$ h. $3\sqrt[4]{7}$ i. $6\sqrt[3]{3}$ j. $2\sqrt[5]{10}$
k. $3\sqrt[3]{3}$ l. $5\sqrt[3]{2}$ m. $-2\sqrt[3]{7}$ n. $2\sqrt[4]{3}$ o. Not real
p. $4\sqrt[4]{3}$ q. $2\sqrt[5]{4}$ r. $3\sqrt[5]{5}$

> *"An educational system isn't worth a great deal if it teaches young people how to make a living– but doesn't teach them how to make a life."*
>
> – Unknown

MORE OPERATIONS ON RADICALS

The chapter on fractional exponents showed us that x to the power $1/n$ is equal to the nth root of x:



$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

This way of expressing radicals using exponents allows us to prove a pair of rules we previously took for granted.

❑ *PROVING THE THEOREMS ABOUT RADICALS*

THEOREM: Assume that x and y represent non-negative numbers. Then

$$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$$

The root of a product
is the product of the roots.

Proof: Our technique will be to convert the left side of the formula to exponent form, use a law of exponents, and then convert back to radical form to arrive at the right side of the formula.

$\sqrt[n]{xy}$

$= (xy)^{1/n}$ (definition of fractional exponent)

$= \quad x^{1/n}\, y^{1/n}$ (law of exponents)

$= \quad \sqrt[n]{x}\,\sqrt[n]{y}$ (definition of fractional exponent)

THEOREM: If x represents a non-negative number (that is, if $\boldsymbol{x \geq 0}$), then

$$\sqrt[n]{x^n} = x$$

The *n*th root cancels the *n*th power.

Proof: Our method of proof is similar to the previous theorem.

$$\sqrt[n]{x^n} = (x^n)^{1/n} = x^{n \cdot \frac{1}{n}} = x^1 = x$$

❑ *ADDING AND SUBTRACTING RADICALS*

We have a simple radical multiplication rule which states that $\sqrt{a}\sqrt{b} = \sqrt{ab}$. How about a rule like $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$? Seems reasonable; let's perform an experiment. Let a = 16 and b = 9.

Left side: $\sqrt{a} + \sqrt{b} = \sqrt{16} + \sqrt{9} = 4 + 3 = \mathbf{7}$

Right side: $\sqrt{a+b} = \sqrt{16+9} = \sqrt{25} = \mathbf{5}$ ☹

We got two different results; we thus conclude that $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$. [Of course, if a = 0 or b = 0, then it does work; but if it doesn't always work, it's not a rule. It's considered a false statement.]

So maybe we can't add $\sqrt{7}$ and $\sqrt{11}$ together, but we can add $2\sqrt{7}$ and $3\sqrt{7}$ together. This is simply the *combining of like terms*. After all, we may not be very comfortable with a number like $\sqrt{7}$, but certainly 2 of them plus 3 more of them should total 5 of them: $2\sqrt{7} + 3\sqrt{7} = 5\sqrt{7}$.

Alternatively, we can apply the distributive property to the sum $2\sqrt{7}+3\sqrt{7}$ by factoring out the GCF: $\sqrt{7}(2+3)$, again giving $5\sqrt{7}$.

Thus, to add (or subtract) radicals, we can use either the concept of like terms or the idea of factoring.

EXAMPLE 1: **Simplify each radical expression:**

A. $3\sqrt{11}+5\sqrt{11} = 8\sqrt{11}$

They're like terms, so we can add them up, just like $3x + 5x = 8x$.
Or, factor out the GCF: $\sqrt{11}\left(3+5\right) = 8\sqrt{11}$

B. $18\sqrt[3]{10}-20\sqrt[3]{10} = -2\sqrt[3]{10}$

C. $\sqrt{3}+\sqrt{5}$ cannot be worked out. They're unlike terms, and no amount of simplification will turn them into like terms.

D. $3\sqrt{20}+10\sqrt{45}$ It might appear that these terms cannot be combined, but don't jump to conclusions — a little simplification first will actually yield an answer:

$$\begin{aligned} & 3\sqrt{20}+10\sqrt{45} \\ = \; & 3\sqrt{4\cdot 5}+10\sqrt{9\cdot 5} \\ = \; & 3\cdot 2\sqrt{5}+10\cdot 3\sqrt{5} \\ = \; & 6\sqrt{5}+30\sqrt{5} \qquad \text{[Like Terms!]} \\ = \; & 36\sqrt{5} \end{aligned}$$

E. $\sqrt{7}+\sqrt[3]{7}$ can't be simplified. Even though the radicands (the 7's) are the same, the roots are different, and therefore they are unlike terms and cannot be added together.

Homework

1. Simplify each radical expression:

a. $2\sqrt{x}+10\sqrt{x}$ b. $3\sqrt{7}-2\sqrt{2}$ c. $10\sqrt[3]{n}+3\sqrt[3]{n}$

d. $6\sqrt{x}-2\sqrt[3]{x}$ e. $9\sqrt{8}+10\sqrt{50}$ f. $\sqrt[3]{24}+\sqrt[3]{81}$

g. $3\sqrt{80}-5\sqrt{45}$ h. $10\sqrt[3]{189}+\sqrt[3]{56}$ i. $\sqrt[3]{z^2}+2\sqrt[3]{z^2}$

j. $\sqrt{12}+2\sqrt{75}$ k. $2\sqrt{20}+2\sqrt{45}$ l. $2\sqrt{275}-3\sqrt{99}$

❑ *MULTIPLYING RADICALS*

Using the formula $\sqrt[n]{x}\cdot\sqrt[n]{y}=\sqrt[n]{xy}$, we can multiply "like roots."

EXAMPLE 2: **Simplify each radical expression:**

A. $\sqrt[5]{3}\cdot\sqrt[5]{7}=\mathbf{\sqrt[5]{21}}$

B. $\sqrt{7}\sqrt{7}=\sqrt{49}=\mathbf{7}$

C. $\sqrt[3]{17}\times\sqrt[3]{3}=\mathbf{\sqrt[3]{51}}$

D. $\sqrt{2}\sqrt{8}=\sqrt{16}=\mathbf{4}$

Alternate Method: $\sqrt{2}\sqrt{8}=\sqrt{2}\left(2\sqrt{2}\right)=2\left(\sqrt{2}\right)^2=2(2)=4$

E. $\sqrt{14}\sqrt{2}=\sqrt{28}=\sqrt{4\cdot7}=\sqrt{4}\sqrt{7}=\mathbf{2\sqrt{7}}$

EXAMPLE 3: **Simplify each radical expression:**

A. $\sqrt{2}\left(1+\sqrt{2}\right)=\sqrt{2}+\sqrt{2}\sqrt{2}=\mathbf{\sqrt{2}+2}$

B. $\sqrt{6}\left(\sqrt{10}-\sqrt{6}\right) = \sqrt{6}\sqrt{10}-\sqrt{6}\sqrt{6} = \sqrt{60}-6 = \mathbf{2\sqrt{15}-6}$

C. $\left(\sqrt{3}+1\right)\left(\sqrt{3}-5\right) = \sqrt{3}\sqrt{3}-5\sqrt{3}+1\sqrt{3}-5$

$= 3-4\sqrt{3}-5 = \mathbf{-2-4\sqrt{3}}$

D. $\left(1+\sqrt{7}\right)^2 = \left(1+\sqrt{7}\right)\left(1+\sqrt{7}\right) = 1+\sqrt{7}+\sqrt{7}+7 = \mathbf{8+2\sqrt{7}}$

E. $\left(3+\sqrt{10}\right)\left(3-\sqrt{10}\right) = 9-3\sqrt{10}+3\sqrt{10}-\sqrt{10}\sqrt{10}$

$= 9-10 = \mathbf{-1}$

Homework

2. Find the product:

a. $\sqrt{14}\sqrt{6}$ b. $\sqrt[3]{3}\times\sqrt[3]{9}$ c. $\sqrt{2}\sqrt{7}$

d. $\sqrt{171}\sqrt{171}$ e. $\sqrt{14}\sqrt{2}$ f. $\sqrt[3]{10}\sqrt[3]{12.5}$

g. $\sqrt[4]{2}\sqrt[4]{8}$ h. $\sqrt{x}\sqrt{yz}$ i. $\sqrt[3]{2}\sqrt[3]{4}$

3. Find the product:

a. $\sqrt{n}\left(1+\sqrt{n}\right)$ b. $\left(x+\sqrt{y}\right)\left(x-\sqrt{y}\right)$ c. $\left(2+\sqrt{3}\right)^2$

d. $\left(\sqrt{p}-q\right)^2$ e. $\left(\sqrt{7}-2\right)\left(\sqrt{7}+2\right)$ f. $\left(x-\sqrt{n}\right)^2$

g. $\left(\sqrt{a}+\sqrt{c}\right)\left(\sqrt{a}-\sqrt{c}\right)$ h. $2\left(\sqrt{3}+\sqrt{4}\right)$ i. $\left(\sqrt{x}+\sqrt{x}\right)^2$

❑ *DIVIDING RADICALS*

Just as the square root of a product is the product of the square roots

($\sqrt{ab} = \sqrt{a}\sqrt{b}$), the square root of a quotient is the quotient of the square roots.

THEOREM: Assume a and b are positive real numbers. Then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Proof:

$$\sqrt{\frac{a}{b}}$$

$$= \left(\frac{a}{b}\right)^{1/2}$$ (convert radical to fractional exponent)

$$= \frac{a^{1/2}}{b^{1/2}}$$ (a law of exponents)

$$= \frac{\sqrt{a}}{\sqrt{b}}$$ (convert back to radical form)

Before we get to the heart of dividing radicals, let's recall the little formula $\left(\sqrt[n]{x}\right)^n = x$. Check out the following examples:

$$\left(\sqrt{u}\right)^2 = u \qquad \left(\sqrt[3]{w}\right)^3 = w \qquad \left(\sqrt[4]{xy}\right)^4 = xy$$

$$\sqrt{c}\sqrt{c} = \left(\sqrt{c}\right)^2 = c \qquad \sqrt{17} \times \sqrt{17} = \left(\sqrt{17}\right)^2 = 17$$

Now consider the number $\frac{1}{\sqrt{2}}$. It's just a fraction, and a calculator gives an *approximate* value of 0.707106781. But now consider the scenario where you have no calculator, but you need a decimal version of the fraction. Even if you could look up the positive square root of 2 somehow — and find that it's about 1.414213562 — you'd still be stuck with the following long division problem:

$$1.414213562 \overline{)\ 1.000000000}$$

This is a killer long division problem — it would take a really long time to get just a few digits of the answer.

But now watch this trick: I'm going to take the fraction $\frac{1}{\sqrt{2}}$ and multiply it by the fraction $\frac{\sqrt{2}}{\sqrt{2}}$ (which, of course, equals 1). Here we go:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \underbrace{\left[\frac{\sqrt{2}}{\sqrt{2}}\right]}_{=1} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{2}$$

We've shown that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. [Double-check with your calculator.]

What's the purpose of all this? Consider the long division problem created by the second fraction, $\frac{\sqrt{2}}{2}$:

$$2\overline{)1.414213562}$$

This is <u>much</u> simpler to carry out than the previous long division problem, and yet it yields the same answer. This is the historical reason why students, especially in the sciences, would convert $\frac{1}{\sqrt{2}}$ into $\frac{\sqrt{2}}{2}$. It made calculations much, much simpler.

Are you going to perform this kind of maneuver — removing radicals from the denominator of a fraction — in your Algebra course? Yes, and for two reasons. One is historical; it's been done like this for centuries. As Tevya from *Fiddler on the Roof* might say, "We do it because it's our tradition!" A second reason: It's a lot easier when we look up answers in the "back of the book" if we can all agree on what form to leave our answers in.

I'll let you in on a little secret: If you take Calculus, there will be problems where you will <u>want</u> to leave radicals in the denominator. Indeed, you will actually remove radicals from the <u>numerator</u>.

We thus have a basic agreement in math: <u>Do not leave radicals in the denominator of a fraction</u>. Converting a fraction <u>with</u> a radical in the denominator into an equivalent

Some trig books will leave answers like $\frac{1}{\sqrt{3}}$, so be prepared.

fraction without a radical in the denominator is called ***rationalizing the denominator***.

EXAMPLE 4:

A. Rationalize the denominator, which is a **monomial**:

$$\frac{3}{\sqrt{x}} = \frac{3}{\sqrt{x}}\left[\frac{\sqrt{x}}{\sqrt{x}}\right] = \frac{3\sqrt{x}}{\sqrt{x}\sqrt{x}} = \frac{3\sqrt{x}}{x}$$

$$= 1$$

B. Rationalize the denominator, which is a **binomial**:

$$\frac{\sqrt{2}}{5-\sqrt{10}}$$

For this example, the trick is in figuring out what, exactly, we should multiply the top and the bottom of the fraction by.

$$\frac{\sqrt{2}}{5-\sqrt{10}} = \frac{\sqrt{2}}{5-\sqrt{10}}\left[\frac{5+\sqrt{10}}{5+\sqrt{10}}\right] = \frac{5\sqrt{2}+\sqrt{20}}{25-10} = \frac{5\sqrt{2}+2\sqrt{5}}{15}$$

$$= 1$$

Notes: Many students try multiplying top and bottom by $\sqrt{10}$, figuring that it would remove the $\sqrt{10}$. Try it and you'll see this procedure fail.

Other students think they're more clever and multiply top and bottom by $5-\sqrt{10}$. Go for it; you'll still have radicals in the bottom.

The quantity $5+\sqrt{10}$, which does eradicate the radical from the bottom, is called the ***conjugate*** of $5-\sqrt{10}$.

Homework

4. Simplify so that there are no radicals in the denominator:

a. $\frac{2}{\sqrt{8}}$ b. $\frac{x^2}{\sqrt{x}}$ c. $\frac{3}{5+\sqrt{2}}$ d. $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

e. $\frac{1}{\sqrt{3}}$ f. $\frac{6}{\sqrt{9}}$ g. $\frac{8}{\sqrt{2}}$ h. $\frac{\sqrt{x}}{\sqrt{y}}$

i. $\frac{5}{1+\sqrt{5}}$ j. $\frac{1+\sqrt{2}}{1-\sqrt{2}}$ k. $\frac{7}{3\sqrt{2}}$ l. $\frac{x}{n\sqrt{x}}$

m. $\frac{7}{1+\sqrt{7}}$ n. $\frac{a+\sqrt{b}}{a-\sqrt{b}}$ o. $\frac{a}{b\sqrt{c}}$ p. $\frac{7+\sqrt{103}}{7+\sqrt{103}}$

5. Two of the following are valid, while two of them are bogus. Which are which?

a. $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$ b. $\sqrt{a-b} = \sqrt{a}-\sqrt{b}$

c. $\sqrt{ab} = \sqrt{a}\sqrt{b}$ d. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Review Problems

6. a. Find the two fourth roots of 256.

b. Find all the cube roots of –1000.

c. Find all the square roots of –121.

7. Simplify each expression:

a. $\sqrt{7}\sqrt{5}$ b. $\sqrt{7}+\sqrt{5}$ c. $\frac{\sqrt{7}}{\sqrt{5}}$ d. $\sqrt{7}-\sqrt{5}$

e. $\sqrt{2}+\sqrt[3]{2}$ f. $\sqrt{28}+\sqrt{175}$ g. $\left(\sqrt{h}+7\right)^2$ h. $\left(3-\sqrt{t}\right)^2$

8. Simplify so that there are no radicals in the denominator:

a. $\dfrac{4}{\sqrt{35}-5}$ b. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ c. $\dfrac{12}{\sqrt{9}+3}$

9. Simplify each expression:

a. $\sqrt{18}\sqrt{8}$ b. $\sqrt{7}-\sqrt{7}$ c. $\dfrac{1+\sqrt{7}}{\sqrt{7}}$ d. $\sqrt{50}-\sqrt{8}$

e. $\dfrac{3}{\sqrt{28}+5}$ f. $\dfrac{\sqrt{a}+b}{\sqrt{a}-b}$ g. $\dfrac{3-\sqrt{25}}{\sqrt{25}-3}$

10. Let $r_1 = \dfrac{3+\sqrt{11}}{4}$ and $r_2 = \dfrac{3-\sqrt{11}}{4}$.

Prove a. $r_1 + r_2 = \dfrac{3}{2}$ b. $r_1 \times r_2 = -\dfrac{1}{8}$

11. True/False:

a. 36 has two square roots.

b. $\sqrt{49} = \pm 7$

c. If $x \le 0$, then $\sqrt{x}$ is not a real number.

d. 64 has exactly one cube root.

e. $-\sqrt[4]{20}$ is a real number.

f. $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$ Assume $a, b \ge 0$

g. $\sqrt{ab} = \sqrt{a}\sqrt{b}$ Assume $a, b \ge 0$

h. $\sqrt[3]{18} = 3\sqrt{2}$

i. $\sqrt{20}+\sqrt{5} = 5$

j. $\sqrt{20}+\sqrt{5} = 3\sqrt{5}$

k. $\left(\sqrt{a}+\sqrt{b}\right)^2 = a+b+2\sqrt{ab}$

l. $\dfrac{7}{\sqrt{n}} = 7\sqrt{n}$

"The wisest mind has something yet to learn."

George Santayana (1863 - 1952)

m. $\dfrac{1}{\sqrt{n}+1} = \dfrac{\sqrt{n}-1}{n-1}$

n. $\sqrt[3]{x^2}+\sqrt[3]{x} = x$

❑ *To ∞ and Beyond*

A. Show that $\dfrac{\sqrt{6}+\sqrt{2}}{4} = \dfrac{\sqrt{2+\sqrt{3}}}{2}$.

Note: A calculator gives good evidence, but it's not a proof.

B. Show that $2\dfrac{r}{\sqrt{2}}\sqrt{r^2-\dfrac{r^2}{2}} = r^2$

Solutions

1. a. $12\sqrt{x}$ b. As is c. $13\sqrt[3]{n}$ d. As is e. $68\sqrt{2}$
f. $5\sqrt[3]{3}$ g. $-3\sqrt{5}$ h. $32\sqrt[3]{7}$ i. $3\sqrt[3]{z^2}$ j. $12\sqrt{3}$
k. $10\sqrt{5}$ l. $\sqrt{11}$

2. a. $\sqrt{84} = 2\sqrt{21}$ b. $\sqrt[3]{27} = 3$ c. $\sqrt{14}$
d. 171 e. $\sqrt{28} = 2\sqrt{7}$ f. $\sqrt[3]{125} = 5$
g. $\sqrt[4]{16} = 2$ h. $\sqrt{xyz}$ i. $\sqrt[3]{8} = 2$

3. a. $\sqrt{n}+n$ b. $x^2 - x\sqrt{y} + x\sqrt{y} - y = x^2 - y$
c. $4+4\sqrt{3}+3 = 7+4\sqrt{3}$ d. $\left(\sqrt{p}-q\right)\left(\sqrt{p}-q\right) = p-2q\sqrt{p}+q^2$
e. $7-4 = 3$ f. $x^2-2x\sqrt{n}+n$ g. $a-c$
h. $2\left(\sqrt{3}+2\right) = 2\sqrt{3}+4$ i. $\left(2\sqrt{x}\right)^2 = 4x$

4. a. $\frac{2}{\sqrt{8}}\left[\frac{\sqrt{8}}{\sqrt{8}}\right] = \frac{2\sqrt{8}}{8} = \frac{4\sqrt{2}}{8} = \frac{\sqrt{2}}{2}$ b. $x\sqrt{x}$

c. $\frac{3}{5+\sqrt{2}}\left[\frac{5-\sqrt{2}}{5-\sqrt{2}}\right] = \frac{15-3\sqrt{2}}{25-2} = \frac{15-3\sqrt{2}}{23}$ d. $\frac{x+2\sqrt{xy}+y}{x-y}$

e. $\frac{\sqrt{3}}{3}$ f. 2 g. $4\sqrt{2}$ h. $\frac{\sqrt{xy}}{y}$

i. $\frac{5}{1+\sqrt{5}}\left[\frac{1-\sqrt{5}}{1-\sqrt{5}}\right] = \frac{5-5\sqrt{5}}{1-5} = \frac{5-5\sqrt{5}}{-4} = \frac{5-5\sqrt{5}}{-4}\left[\frac{-1}{-1}\right] = \frac{5\sqrt{5}-5}{4}$

The last step is optional. It was done just to remove the negative sign from the bottom of the fraction.

j. $-3-2\sqrt{2}$ k. $\frac{7}{3\sqrt{2}}\left[\frac{\sqrt{2}}{\sqrt{2}}\right] = \frac{7\sqrt{2}}{3\cdot 2} = \frac{7\sqrt{2}}{6}$ l. $\frac{\sqrt{x}}{n}$

m. $\frac{7\sqrt{7}-7}{6}$ n. $\frac{a^2+2a\sqrt{b}+b}{a^2-b}$ o. $\frac{a\sqrt{c}}{bc}$ p. 1

5. c. and d. are valid — a. and b. are bogus.

6. a. ± 4 b. -10 c. None

7. a. $\sqrt{35}$ b. As is c. $\frac{\sqrt{35}}{5}$ d. As is

e. As is f. $7\sqrt{7}$ g. $h+14\sqrt{h}+49$ h. $9-6\sqrt{t}+t$

8. a. $\frac{2\sqrt{35}+10}{5}$ b. $\frac{a-2\sqrt{ab}+b}{a-b}$ c. 2

9. a. 12 b. 0 c. $\frac{\sqrt{7}+7}{7}$ d. $3\sqrt{2}$

e. $2\sqrt{7}-5$ f. $\frac{a+2b\sqrt{a}+b^2}{a-b^2}$ g. -1

10. Do the arithmetic and it should work out.

11. a. T b. F c. F d. T e. T f. F g. T h. F
i. F j. T k. T l. F m. T n. F

CH 9 – RADICALS

We've learned that there is a positive real number whose square is 3, and that this number is called the positive square root of 3. We also have seen that this number is written with a 3 inside a radical sign. This chapter discusses the various operations involving these "radicals."

$\sqrt{3}$

❑ All Kinds of Roots

The number 9 has two ***square roots***, 3 and –3. This is because 3 squared is 9 and –3 squared is 9. The positive square root of 9 (the 3) is denoted $\sqrt{9}$, and the negative square root of 9 (the –3) is written $-\sqrt{9}$. In other words, $\sqrt{9} = 3$, and only 3, while $-\sqrt{9} = -3$.

In analyzing $\sqrt{-25}$, the positive square root of –25, we discover that we cannot find an answer for this problem, since the square of a real number can never be negative. If there is an answer to $\sqrt{-25}$, it lies outside $\mathbb{R}$.

Consider the number 8. Since $2^3 = 8$, we can say that 2 is a ***cube root*** of 8. In fact, it's the only cube root of 8, simply because there's no other real number whose cube is 8. Perhaps a little surprising is that we can calculate the cube root of a negative number without leaving $\mathbb{R}$. For example, $\sqrt[3]{-27} = -3$, since $(-3)^3 = -27$.

The number 16 has two ***fourth roots***. The positive fourth root is $\sqrt[4]{16} = 2$, and the negative fourth root is $-\sqrt[4]{16} = -2$. After all, both 2 and –2 raised to the fourth power result in 16. However, just like square roots, $\sqrt[4]{-1}$ is not a real number.

The ***fifth root*** of 32 is 2; that is, $\sqrt[5]{32} = 2$ -- this is because $2^5 = 32$. Like cube roots, we can calculate the fifth root of a negative number. For example, $\sqrt[5]{-243}$ equals –3, since $(-3)^5 = -243$.

Consider the expression $\sqrt[3]{ab}$. The entire expression can be called a **radical**. The symbol $\sqrt{}$ is called the **radical sign**, and the quantity ab is called the **radicand**.

A Final Comment: The number 23 has two square roots, $\sqrt{23}$ and $-\sqrt{23}$. Not only do these square roots not result in an integer, they actually result in irrational numbers. Similarly, each of the following is an irrational number: $\sqrt[3]{17}$, $\sqrt[4]{123}$, and $\sqrt[5]{-1001}$ -- each of them is an infinite, non-repeating decimal.

Homework

1. Find the square root(s) of

   a. 100 b. 15 c. 0 d. –36 e. 1

2. Find the cube root(s) of

   a. 64 b. –125 c. 0 d. 20 e. 1

3. Find the fourth root(s) of

   a. 81 b. 0 c. –625 d. 25 e. 1

4. Find the fifth root(s) of

   a. 1 b. 0 c. –243 d. 29 e. 32

❑ Simplifying Radicals

During our discussion of radicals, we will assume that all variables are non-negative (positive or zero). Therefore, even though $\sqrt{x^2} = |x|$ when x is an arbitrary real number, our restriction to non-negative values implies that we can dispense with the absolute value signs: If $x \geq 0$, then $\sqrt{x^2} = x$.

The basic rule regarding radicals is contained in the following rule.

THEOREM: Assume that x and y represent non-negative numbers. Then

$$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$$

The root of a product
is the product of the roots.

Proof: Our technique will be to convert the left side of the formula to exponent form, use a law of exponents, and then convert back to radical form to arrive at the right side of the formula.

	$\sqrt[n]{xy}$	
=	$(xy)^{1/n}$	(definition of fractional exponent)
=	$x^{1/n}\,y^{1/n}$	(law of exponents)
=	$\sqrt[n]{x}\,\sqrt[n]{y}$	(definition of fractional exponent)

Q.E.D.

THEOREM: If x represents a non-negative number, then

$$\sqrt[n]{x^n} = x$$

The *n*th root cancels the *n*th power.

Proof: Our method of proof is similar to the previous theorem.

$$\sqrt[n]{x^n} = (x^n)^{1/n} = x^{n\cdot\frac{1}{n}} = x^1 = x$$ Q.E.D.

EXAMPLE 1: **Simplify each radical expression:**

A. $\sqrt{200} = \sqrt{100\times 2} = \sqrt{100}\times\sqrt{2} = 10\sqrt{2}$

B. $\sqrt[3]{81} = \sqrt[3]{27\times 3} = \sqrt[3]{27}\times\sqrt[3]{3} = 3\sqrt[3]{3}$

C. $\sqrt[4]{1250} = \sqrt[4]{625\cdot 2} = \sqrt[4]{625}\sqrt[4]{2} = 5\sqrt[4]{2}$

D. $\sqrt[3]{2250}$ Sometimes the radicand (the 2250) is too big to easily see if there's a perfect cube in it. So let's try a slightly different approach. We factor the 2250 into primes to get

$$2250 = 2\cdot 3^2\cdot 5^3.$$

Clearly we can take the cube root of 5^3 (it's 5), but there are not enough of the other factors to take their cube roots. So we can write

$$\sqrt[3]{2250} = \sqrt[3]{2\cdot 3^2\cdot 5^3} = 5\sqrt[3]{2\cdot 3^2} = 5\sqrt[3]{18}$$

EXAMPLE 2: **Simplify each radical expression:**

A. $\sqrt{50xy^2} = \sqrt{5^2 \cdot 2 \cdot x \cdot y^2} = 5y\sqrt{2x}$

B. $\sqrt{144a^{46}b^{19}} = \sqrt{12^2 \cdot (a^{23})^2 \cdot (b^9)^2 \cdot b} = 12a^{23}b^9\sqrt{b}$

C. $\sqrt[3]{16x^{48}w^{14}} = \sqrt[3]{2^3 \cdot 2 \cdot (x^{16})^3 \cdot (w^4)^3 \cdot w^2} = 2x^{16}w^4\sqrt[3]{2w^2}$

D. $\sqrt[4]{c^{20}d^{39}} = \sqrt[4]{(c^5)^4(d^9)^4 d^3} = c^5d^9\sqrt[4]{d^3}$

E. $\sqrt[5]{x^4y^{53}} = \sqrt[5]{x^4(y^{10})^5 y^3} = y^{10}\sqrt[5]{x^4y^3}$

F. $\sqrt[3]{54a^2b^{27}c^{17}} = \sqrt[3]{3^3 \cdot 2 \cdot a^2 \cdot (b^9)^3 \cdot (c^5)^3 \cdot c^2} = 3b^9c^5\sqrt[3]{2a^2c^2}$

G. $\sqrt[5]{64x^3y^5z^{25}w^{47}} = \sqrt[5]{2^5 \cdot 2 \cdot x^3 \cdot y^5 \cdot (z^5)^5 \cdot (w^9)^5 \cdot w^2}$
$= 2yz^5w^9\sqrt[5]{2x^3w^2}$

As we'll see in the following example, fractional exponents are indispensable for some radical problems.

EXAMPLE 3: **Simplify each radical expression:**

A. $\sqrt{\sqrt[3]{x}} = \left(x^{1/3}\right)^{1/2} = x^{1/6} = \sqrt[6]{x}$

B. $\sqrt[4]{\sqrt[3]{\sqrt{2}}} = \left[\left(2^{1/2}\right)^{1/3}\right]^{1/4} = 2^{1/24} = \sqrt[24]{2}$

C. $\sqrt[12]{a^9} = \left(a^9\right)^{1/12} = a^{9/12} = a^{3/4} = \sqrt[4]{a^3}$

Homework

5. Simplify: a. $\sqrt{200xy^2z^3w^{16}v^{37}}$ b. $\sqrt{72a^{25}b^{12}}$

6. Simplify: a. $\sqrt[3]{16ab^2c^3d^{27}e^{50}}$ b. $\sqrt[3]{54x^{22}y^{32}}$

7. Simplify: a. $\sqrt[4]{32x^4y^{16}z^{39}}$ b. $\sqrt[4]{81a^4b^{196}}$

8. Simplify: a. $\sqrt[5]{64u^{32}v^{54}}$ b. $\sqrt[5]{486w^4z^{32}}$

9. Use your calculator to verify Example 3B.

10. DISprove the conjecture: $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$ [Hint: use numbers]

11. Simplify each nested radical:

a. $\sqrt{\sqrt[4]{7}}$ b. $\sqrt[3]{\sqrt{10}}$ c. $\sqrt[5]{\sqrt[7]{x}}$

d. $\sqrt[4]{\sqrt[4]{n}}$ e. $\sqrt[5]{\sqrt[3]{\sqrt{a}}}$ f. $\sqrt[4]{\sqrt[3]{\sqrt{x}}}$

12. Simplify:

a. $\sqrt[6]{x^2}$ b. $\sqrt[8]{n^6}$ c. $\sqrt[10]{x^5}$ d. $\sqrt[9]{z^3}$

e. $\sqrt[20]{a^{15}}$ f. $\sqrt[21]{n^{14}}$ g. $\sqrt[30]{m^{26}}$ h. $\sqrt[88]{x^{77}}$

13. Simplify $\sqrt{\sqrt[3]{\sqrt[4]{3}}}$. Use your calculator to verify your answer.

❑ Adding and Subtracting Radicals

To add (or subtract) radicals, we can use either the concept of like terms or the idea of factoring.

EXAMPLE 4: **Simplify each radical expression:**

A. $3\sqrt{7}+5\sqrt{7} = 8\sqrt{7}$

They're like terms, so we can add them up, just like $3x + 5x = 8x$.
Or -- factor out the GCF: $\sqrt{7}(3+5) = 8\sqrt{7}$

B. $18\sqrt[3]{10}-20\sqrt[3]{10} = -2\sqrt[3]{10}$

C. $\sqrt{3}+\sqrt{5}$ cannot be worked out. They're unlike terms, and no amount of simplification will turn them into like terms.

D. $3\sqrt{20}+10\sqrt{45}$ It might appear that these terms cannot be combined, but don't jump to conclusions -- a little simplification first will yield an answer:

$$3\sqrt{20}+10\sqrt{45} = 3\sqrt{4\cdot 5}+10\sqrt{9\cdot 5} = 3\cdot 2\sqrt{5}+10\cdot 3\sqrt{5}$$
$$= 6\sqrt{5}+30\sqrt{5} = 36\sqrt{5}$$

E. $\sqrt{7}+\sqrt[3]{7}$ can't be simplified. Even though the radicands (the 7's) are the same, the roots are different, and therefore they are unlike terms and cannot be added together.

Homework

14. Simplify each radical expression:

a. $2\sqrt{x}+10\sqrt{x}$ b. $3\sqrt{7}-2\sqrt{2}$ c. $10\sqrt[3]{n}+3\sqrt[3]{n}$

d. $6\sqrt{x}-2\sqrt[3]{x}$ e. $9\sqrt{8}+10\sqrt{50}$ f. $\sqrt[3]{24}+\sqrt[3]{81}$

g. $3\sqrt{80}-5\sqrt{45}$ h. $10\sqrt[3]{189}+\sqrt[3]{56}$ i. $\sqrt[3]{z^2}+2\sqrt[3]{z^2}$

❑ Multiplying Radicals

Using the formula $\sqrt[n]{x}\sqrt[n]{y} = \sqrt[n]{xy}$, we can multiply "like roots." For example:

$$\sqrt[5]{3}\sqrt[5]{7} = \sqrt[5]{21}$$

Here's another example, with variables and simplifying at the end:

$$\sqrt[7]{x^5}\sqrt[7]{x^4} = \sqrt[7]{x^9} = \sqrt[7]{x^7x^2} = x\sqrt[7]{x^2}$$

Homework

15. Find the product:

a. $\sqrt{14}\sqrt{6}$ b. $\sqrt[3]{3}\sqrt[3]{9}$ c. $\sqrt{x^3}\sqrt{x^5}$

d. $\sqrt[3]{x^2y^5}\sqrt[3]{xy^8}$ e. $\sqrt{14}\sqrt{2}$ f. $\sqrt[3]{10}\sqrt[3]{12.5}$

g. $\sqrt[4]{2}\sqrt[4]{8}$ h. $\sqrt{x}\sqrt{yz}$ i. $\sqrt[3]{2}\sqrt[3]{4}$

16. Find the product:

a. $\sqrt{n}\left(1+\sqrt{n}\right)$ b. $\left(x+\sqrt{y}\right)\left(x-\sqrt{y}\right)$ c. $\left(2+\sqrt{3}\right)^2$

d. $\left(\sqrt{p}-q\right)^2$ e. $\left(\sqrt{7}-2\right)\left(\sqrt{7}+2\right)$ f. $\left(x-\sqrt{n}\right)^2$

g. $\left(\sqrt{a}+\sqrt{c}\right)\left(\sqrt{a}-\sqrt{c}\right)$ h. $2\left(\sqrt{3}+\sqrt{4}\right)$ i. $\left(\sqrt{x}+\sqrt{x}\right)^2$

❑ Dividing Radicals

First we state and prove a theorem that we'll need later when we derive the Quadratic Formula.

THEOREM: Assume a and b are positive real numbers. Then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

PROOF:

$$\sqrt{\frac{a}{b}}$$

$$= \left(\frac{a}{b}\right)^{1/2} \quad \text{(convert radical to fractional exponent)}$$

$$= \frac{a^{1/2}}{b^{1/2}} \quad \text{(a law of exponents)}$$

$$= \frac{\sqrt{a}}{\sqrt{b}} \quad \text{(convert back to radical form)} \quad \text{Q.E.D.}$$

Recall from Intermediate Algebra the following two procedures for removing a radical from the denominator of a fraction, called ***rationalizing the denominator***.

EXAMPLE 5:

A. $$\frac{3}{\sqrt{x}} = \frac{3}{\sqrt{x}} \underbrace{\left[\frac{\sqrt{x}}{\sqrt{x}}\right]}_{=1} = \frac{3\sqrt{x}}{\sqrt{x}\sqrt{x}} = \frac{3\sqrt{x}}{x}$$

B. $$\frac{\sqrt{2}}{5-\sqrt{10}} = \frac{\sqrt{2}}{5-\sqrt{10}} \underbrace{\left[\frac{5+\sqrt{10}}{5+\sqrt{10}}\right]}_{=1} = \frac{5\sqrt{2}+\sqrt{20}}{25-10} = \frac{5\sqrt{2}+2\sqrt{5}}{15}$$

[The quantity $5+\sqrt{10}$ is called the ***conjugate*** of $5-\sqrt{10}$.]

Homework

17. Simplify each radical expression:

a. $\frac{2}{\sqrt{8}}$ b. $\frac{x^2}{\sqrt{x}}$ c. $\frac{3}{5+\sqrt{2}}$ d. $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

e. $\frac{1}{\sqrt{3}}$ f. $\frac{6}{\sqrt{9}}$ g. $\frac{8}{\sqrt{2}}$ h. $\frac{\sqrt{x}}{\sqrt{y}}$

i. $\frac{5}{1+\sqrt{5}}$ j. $\frac{1+\sqrt{2}}{1-\sqrt{2}}$ k. $\frac{7}{3\sqrt{2}}$ l. $\frac{x}{n\sqrt{x}}$

m. $\frac{7}{1+\sqrt{7}}$ n. $\frac{a+\sqrt{b}}{a-\sqrt{b}}$ o. $\frac{a}{b\sqrt{c}}$ p. $\frac{7+\sqrt{103}}{7+\sqrt{103}}$

18. Two of the following are valid, while two of them are bogus. Which are which?

a. $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$ b. $\sqrt{a-b} = \sqrt{a}-\sqrt{b}$

c. $\sqrt{ab} = \sqrt{a}\sqrt{b}$ d. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Practice Problems

19. a. Find the two fourth roots of 256.
b. Find all the cube roots of –1000.
c. Find all the square roots of –121.

20. Simplify each expression:

a. $\sqrt{7}\sqrt{5}$ b. $\sqrt{7}+\sqrt{5}$ c. $\dfrac{\sqrt{7}}{\sqrt{5}}$ d. $\sqrt{7}-\sqrt{5}$

e. $\sqrt{2}+\sqrt[3]{2}$ f. $\sqrt{28}+\sqrt{175}$ g. $\left(\sqrt{h}+7\right)^2$ h. $\left(3-\sqrt{t}\right)^2$

21. Simplify: $\sqrt[3]{108x^7y^{27}z^{50}}$

22. Write as a single radical: a. $\sqrt[7]{\sqrt{\sqrt[6]{x}}}$ b. $\sqrt{\sqrt[3]{\sqrt{x}}}$

23. Rationalize the denominator:

a. $\dfrac{4}{\sqrt{35}-5}$ b. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ c. $\dfrac{12}{\sqrt{9}+3}$

24. Simplify each expression:

a. $\sqrt{18}\sqrt{8}$ b. $\sqrt{7}-\sqrt{7}$ c. $\dfrac{1+\sqrt{7}}{\sqrt{7}}$ d. $\sqrt{50}-\sqrt{8}$

25. Simplify: $\sqrt[4]{64x^7y^{27}z^{50}}$

26. Write as a single radical: $\sqrt[4]{\sqrt{\sqrt[3]{x}}}$

27. Simplify: $\sqrt[10]{x^5}\ \sqrt[14]{x^7}$

28. Rationalize the denominator:

a. $\dfrac{3}{\sqrt{28}+5}$ b. $\dfrac{\sqrt{a}+b}{\sqrt{a}-b}$ c. $\dfrac{3-\sqrt{25}}{\sqrt{25}-3}$

29. Let $r_1 = \dfrac{3+\sqrt{11}}{4}$ and $r_2 = \dfrac{3-\sqrt{11}}{4}$.

Prove a. $r_1+r_2 = \dfrac{3}{2}$ b. $r_1r_2 = -\dfrac{1}{8}$

30. True/False:

a. 36 has two square roots.

b. $\sqrt{49} = \pm 7$

c. If $x \le 0$, then $\sqrt{x}$ is not a real number.

d. 64 has exactly one cube root.

e. $-\sqrt[4]{20}$ is a real number.

f. $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

g. $\sqrt{ab} = \sqrt{a}\sqrt{b}$

h. For any real number x, $\sqrt{x^2} = x$.

i. $\sqrt[3]{18} = 3\sqrt{2}$

j. $\sqrt[5]{32n^{32}} = 2n^2$

k. $\sqrt{\sqrt[5]{\sqrt{a}}} = \sqrt[9]{a}$

l. $\sqrt[14]{x^{21}} = x\sqrt{x}$

m. $\sqrt{20} + \sqrt{5} = 5$

n. $\sqrt{20} + \sqrt{5} = 3\sqrt{5}$

o. $\left(\sqrt{a} + \sqrt{b}\right)^2 = a + b + 2\sqrt{ab}$

p. $\dfrac{7}{\sqrt{n}} = 7\sqrt{n}$

q. $\dfrac{1}{\sqrt{n}+1} = \dfrac{\sqrt{n}-1}{n-1}$

r. $\sqrt[3]{x^2} + \sqrt[3]{x} = x$

Solutions

1. a. ± 10 b. $\pm\sqrt{15}$ c. 0 d. $\varnothing$ e. ± 1

2. a. 4 b. -5 c. 0 d. $\sqrt[3]{20}$ e. 1

3. a. ± 3 b. 0 c. $\varnothing$ d. $\pm\sqrt[4]{25}$ e. ± 1

4. a. 1 b. 0 c. -3 d. $\sqrt[5]{29}$ e. 2

5. a. $10yzw^8v^{18}\sqrt{2xzv}$ b. $6a^{12}b^6\sqrt{2a}$

6. a. $2cd^9e^{16}\sqrt[3]{2ab^2e^2}$ b. $3x^7y^{10}\sqrt[3]{2xy^2}$

7. a. $2xy^4z^9\sqrt[4]{2z^3}$ b. $3ab^{49}$

8. a. $2u^6v^{10}\sqrt[5]{2u^2v^4}$ b. $3z^6\sqrt[5]{2w^4z^2}$

9. Does your verification prove that the answer is right?

10. For instance, $\sqrt{9+16} = \sqrt{25} = 5$, whereas $\sqrt{9}+\sqrt{16} = 3+4 = 7$. Note that these numbers constitute a *counterexample*.

11. a. $\left(7^{1/4}\right)^{1/2} = 7^{1/8} = \sqrt[8]{7}$ b. $\sqrt[6]{10}$

c. $\left(x^{1/7}\right)^{1/5} = x^{1/35} = \sqrt[35]{x}$ d. $\sqrt[16]{n}$

e. $\left(\left(a^{1/3}\right)^{1/2}\right)^{1/5} = a^{\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{5}} = a^{1/30} = \sqrt[30]{a}$ f. $\sqrt[24]{x}$

12. a. $\sqrt[6]{x^2} = x^{2/6} = x^{1/3} = \sqrt[3]{x}$ b. $\sqrt[8]{n^6} = n^{6/8} = n^{3/4} = \sqrt[4]{n^3}$

c. $\sqrt{x}$ d. $\sqrt[3]{z}$ e. $\sqrt[4]{a^3}$ f. $\sqrt[3]{n^2}$ g. $\sqrt[15]{m^{13}}$ h. $\sqrt[8]{x^7}$

13. a. $\sqrt[24]{3}$; the problem and the answer are each about 1.046839.

14. a. $12\sqrt{x}$ b. As is c. $13\sqrt[3]{n}$ d. As is e. $68\sqrt{2}$

f. $5\sqrt[3]{3}$ g. $-3\sqrt{5}$ h. $32\sqrt[3]{7}$ i. $3\sqrt[3]{z^2}$

15. a. $\sqrt{84} = 2\sqrt{21}$ b. $\sqrt[3]{27} = 3$ c. $\sqrt{x^8} = x^4$

d. $\sqrt[3]{x^3y^{13}} = xy^4\sqrt[3]{y}$ e. $\sqrt{28} = 2\sqrt{7}$ f. $\sqrt[3]{125} = 5$

g. $\sqrt[4]{16} = 2$ h. $\sqrt{xyz}$ i. $\sqrt[3]{8} = 2$

16. a. $\sqrt{n}+n$ b. $x^2 - x\sqrt{y} + x\sqrt{y} - y = x^2 - y$

c. $4+4\sqrt{3}+3 = 7+4\sqrt{3}$ d. $\left(\sqrt{p}-q\right)\left(\sqrt{p}-q\right) = p - 2q\sqrt{p} + q^2$

e. $7 - 4 = 3$ f. $x^2 - 2x\sqrt{n} + n$ g. $a - c$

h. $2\left(\sqrt{3}+2\right) = 2\sqrt{3}+4$ i. $\left(2\sqrt{x}\right)^2 = 4x$

17. a. $\dfrac{2}{\sqrt{8}}\left[\dfrac{\sqrt{8}}{\sqrt{8}}\right] = \dfrac{2\sqrt{8}}{8} = \dfrac{4\sqrt{2}}{8} = \dfrac{\sqrt{2}}{2}$ b. $x\sqrt{x}$

c. $\dfrac{3}{5+\sqrt{2}}\left[\dfrac{5-\sqrt{2}}{5-\sqrt{2}}\right] = \dfrac{15-3\sqrt{2}}{25-2} = \dfrac{15-3\sqrt{2}}{23}$ d. $\dfrac{x+2\sqrt{xy}+y}{x-y}$

e. $\dfrac{\sqrt{3}}{3}$ f. 2 g. $4\sqrt{2}$ h. $\dfrac{\sqrt{xy}}{y}$

i. $\dfrac{5}{1+\sqrt{5}}\left[\dfrac{1-\sqrt{5}}{1-\sqrt{5}}\right] = \dfrac{5-5\sqrt{5}}{1-5} = \dfrac{5-5\sqrt{5}}{-4} = \dfrac{5-5\sqrt{5}}{-4}\left[\dfrac{-1}{-1}\right] = \dfrac{5\sqrt{5}-5}{4}$

The last step is optional -- it was done just to remove the negative sign from the bottom of the fraction.

j. $-3-2\sqrt{2}$ k. $\dfrac{7}{3\sqrt{2}}\left[\dfrac{\sqrt{2}}{\sqrt{2}}\right] = \dfrac{7\sqrt{2}}{3\cdot 2} = \dfrac{7\sqrt{2}}{6}$

l. $\frac{\sqrt{x}}{n}$ m. $\frac{7\sqrt{7}-7}{6}$ n. $\frac{a^2+2a\sqrt{b}+b}{a^2-b}$ o. $\frac{a\sqrt{c}}{bc}$ p. 1

18. c. and d. are valid -- a. and b. are bogus.

19. a. ± 4 b. -10 c. None

20. a. $\sqrt{35}$ b. $\sqrt{7}+\sqrt{5}$ c. $\frac{\sqrt{35}}{5}$ d. $\sqrt{7}-\sqrt{5}$
e. $\sqrt{2}+\sqrt[3]{2}$ f. $7\sqrt{7}$ g. $h+14\sqrt{h}+49$ h. $9-6\sqrt{t}+t$

21. $3x^2y^9z^{16}\sqrt[3]{4xz^2}$ **22**. a. $\sqrt[8]{x^4}$ b. $\sqrt[12]{x}$

23. a. $\frac{2\sqrt{35}+10}{5}$ b. $\frac{a-2\sqrt{ab}+b}{a-b}$ c. 2

24. a. 12 b. 0 c. $\frac{\sqrt{7}+7}{7}$ d. $3\sqrt{2}$

25. $2xy^6z^{12}\sqrt[4]{4x^3y^3z^2}$ **26**. $\sqrt[24]{x}$

27. x **28**. a. $2\sqrt{7}-5$ b. $\frac{a+2b\sqrt{a}+b^2}{a-b^2}$ c. -1

29. Do the arithmetic and it should work out.

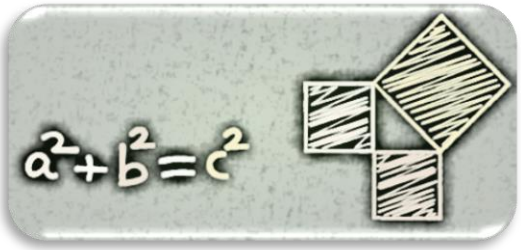
30. a. T b. F c. F d. T e. T f. F g. T h. F i. F
j. F k. F l. T m. F n. T o. T p. F q. T r. F

The important thing
is to not stop questioning.

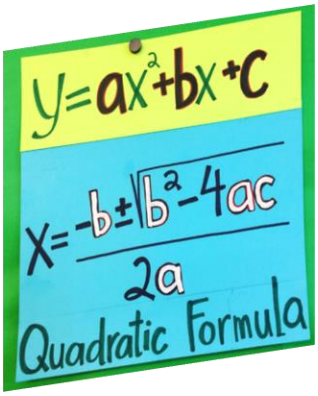
Albert Einstein

SQUARE ROOTS

You might remember **square roots** if you have studied the Pythagorean Theorem. Perhaps they were hidden, but when the end of a right-triangle problem resulted in an equation like $c^2 = 25$, you used the notion of *square root* when you determined that c had to be 5. (Another solution to this equation is $c = -5$, but it would be discarded in a triangle problem.)



If you're familiar with The Quadratic Formula, you explicitly saw square roots when you looked at the part of the formula containing the radical sign: $\sqrt{\ }$.



This chapter delves into more detail on square roots, including their precise meaning and how to simplify them. We'll need these skills to continue our study of the Pythagorean Theorem and the Quadratic Formula (which I'm *sure* you're looking forward to).

❑ *THE MEANING OF SQUARE ROOTS*

The number **81** has <u>two</u> square roots, 9 and –9. This is because $9 \times 9 = 81$, and also $-9 \times -9 = 81$.

Because both 11^2 and $(-11)^2$ equal 121, the two square roots of **121** are 11 and –11.

The number **1** has two square roots, namely 1 and –1. The reason: If either 1 or –1 is squared, the result is 1.

Every positive number has <u>two</u> square roots.

Zero has just one square root, 0, since $0^2 = 0$, and no other number squared would result in 0.

The number **–16** has <u>no</u> square roots. After all, what number squared equals –16? 4 can't work, since $4^2 = +16$. Nor can –4 work, since $(-4)^2$ is also +16. In fact, any number squared produces an answer that is <u>at least zero</u>. That is, no number can be squared and result in –16. Thus, –16 has no square root (until a later course, perhaps).

ANY number squared is greater than or equal to zero:

For any *n*,

$n^2 \geq 0$

Consider the fraction $\frac{1}{4}$. It has two square roots, $\frac{1}{2}$ and $-\frac{1}{2}$. Why? I'll tell you why. It's because $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ and also $\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$.

We can encapsulate this section by stating that

a is a ***square root*** of ***b*** if $a^2 = b$

Homework

1. a. Find as many numbers as you can whose square is 144.
   b. Find as many numbers as you can whose square is –144.
   c. How many square roots does 25 have? What are they? Why are they square roots of 25?
   d. How many square roots does 196 have? What are they? Why are they square roots of 196?
   e. How many square roots does 1 have? What are they?
   f. How many square roots does 0 have? What are they?
   g. How many square roots does –9 have? What are they?
   h. What are the two square roots of $\frac{4}{9}$?

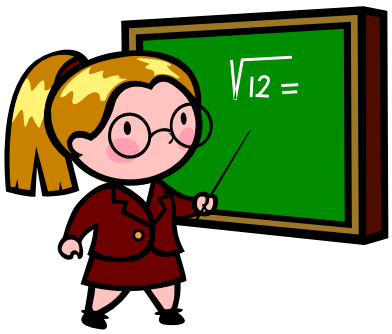
❑ *NOTATION*

The previous section showed us that, for example, the number 81 actually has two square roots, 9 and –9. How do we denote these two square roots of 81 without calculating their actual values? Notice that 81 has a positive square root (the 9) and a negative square root (the –9). We write

$\sqrt{81} = 9$ The positive square root of 81 is 9.

and $-\sqrt{81} = -9$ The negative square root of 81 is –9.

That is, the positive square root is written with a radical sign, while the negative square root is written with a radical sign preceded by a minus sign. The number 121 has two square roots, which we can write as $\sqrt{121} = 11$ and $-\sqrt{121} = -11$.



Thus, if someone asks you for the square root of 100, you must answer 10 and –10, because each of these is a square root of 100. However, if you're presented with " $\sqrt{100}$ ", you answer just 10, not –10, because we agreed that the radical sign refers to only the non-negative square root. Also, if someone asks you for the "non-negative" or ***principal square root*** of 100, then you must answer just 10, not –10.

The square root symbol $\sqrt{\ }$ is called the ***radical sign***, and the quantity inside the radical sign is called the ***radicand***. Why do we need a radical sign at all? After all, the two square roots of 49 are 7 and –7; it's just that simple. But what about the number 17? It also has two square roots. As you may recall, they don't result in nice numbers (they're infinite, non-repeating decimals), so we must have some notation to denote the two square roots of 17, even though we might not care to work them out. So, we use the radical sign and state that

There are two square roots of 17: $\sqrt{17}$ and $-\sqrt{17}$.

The vital thing to note is that $\left(\sqrt{17}\right)^2 = 17$ and $\left(-\sqrt{17}\right)^2 = 17$.

We state the following four conclusions regarding square roots:

- Every positive number has two square roots, one positive and one negative. The two square roots of n are denoted $\sqrt{n}$ and $-\sqrt{n}$.
- Zero has one square root (namely itself) — $\sqrt{0} = 0$.
- Negative numbers don't have any square roots at all (at least in this course). For example, $\sqrt{-9}$ does not exist in the real numbers, although it does exist if we extend the system of real numbers to include *imaginary numbers*.

- The most central issue in this whole discussion is that assuming x is positive (which means neither negative nor zero):

> The symbol $\sqrt{x}$ means the positive square root only!

Homework

2. a. Name the two square roots of 49. Explain.
   b. Name the two square roots of 29. Explain.
   c. Name the two square roots of 51. Explain.

> In computers, the *square root* of *n* might be written
>
> **sqrt(n)**

3. Use your calculator to give some credence to the statement that $(\sqrt{13})^2 = 13$.

4. Evaluate each expression:

a. $\sqrt{144}$	b. $-\sqrt{225}$	c. $\sqrt{-25}$	d. $-\sqrt{-36}$
e. $\sqrt{196}$	f. $\sqrt{1}$	g. $\sqrt{0}$	h. $\sqrt{256}$
i. $-\sqrt{64}$	j. $\sqrt{169}$	k. $\sqrt{-100}$	l. $\sqrt{289}$

5. Simplify each expression:

a. $(\sqrt{64})^2$	b. $(\sqrt{93})^2$	c. $(\sqrt{a})^2$	d. $(\sqrt{m})^2$

6. Consider the following examples:

i) $\sqrt{9^2} = \sqrt{81} = 9$ ii) $\sqrt{(-6)^2} = \sqrt{36} = 6$

Now work the following problems (without a calculator):

a. $\sqrt{10^2}$ b. $\sqrt{12{,}456^2}$ c. $\sqrt{(-7)^2}$ d. $\sqrt{(-3124)^2}$

7. It's easy to confuse the phrases "the negative square root of a number" and "the square root of a negative number." Give examples that clarify the difference between these phrases.

❑ *SIMPLIFYING SQUARE ROOTS*

Let's do an experiment. Our goal is to see if there's a way we can simplify square roots (kind of like we reduce fractions). Specifically, we'll see if we can split a square root into two separate square roots so that one of the square roots will end up being a whole number. That would make the number in the radical smaller than it was, and thus simpler to work with.

For example, let's see what we can do with $\sqrt{36}$. [We know the answer is 6 (and just positive 6, right?), so we'll see if we've accomplished anything useful in our experiment.]

We start with:	$\sqrt{36}$
Factor 36 into 9 times 4:	$\sqrt{9 \cdot 4}$
Now try splitting the radical down the middle:	$\sqrt{9} \cdot \sqrt{4}$
Now calculate the square roots:	$3 \cdot 2$
And finish the arithmetic:	6 ✓

Can you see that writing 36 as $3 \cdot 12$ wouldn't have done us any good?

Voilà! It worked. We may have just calculated $\sqrt{36}$ in a crazy long way, but now we have a method to simplify the square root of numbers like 50, which will not come out as a whole number. Let's try it.

EXAMPLE 1: **Simplify: $\sqrt{50}$**

Solution:

We start with: $\sqrt{50}$

Factor 50 into 25 times 2: $\sqrt{25 \cdot 2}$

Split the radical down the middle: $\sqrt{25} \cdot \sqrt{2}$

Now calculate the square root: $5 \cdot \sqrt{2}$

Remove the dot, and we're finished: $\boxed{5\sqrt{2}}$

> You might be wondering how we knew to factor 50 as $25 \cdot 2$, rather than $10 \cdot 5$, perhaps. Well, we could have, but it wouldn't have done us any good because neither 10 nor 5 has a nice square root, whereas 25 does. We say that 25 is a **perfect square** because it has a whole-number square root.
>
> The following is a list of some perfect squares:
>
> 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289
>
> These are the kinds of numbers we seek when we factor the radicand.

We can summarize the success of our experiment by saying that

The square root of a product is equal to the product of their square roots.

Thus, assuming that a and b are both *non-negative* (that means at least zero),

$$\sqrt{ab} = \sqrt{a}\,\sqrt{b}$$

The square root of a product is equal to the product of their square roots.

EXAMPLE 2: **Simplify each square root:**

A. $\sqrt{144} = \sqrt{36 \cdot 4} = \sqrt{36} \cdot \sqrt{4} = 6 \cdot 2 = \mathbf{12}$

(Of course, the steps aren't needed, since $\sqrt{144}$ is easily seen to be 12, but it proves that the method works.)

B. $\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = \mathbf{5\sqrt{3}}$ (25 is a perfect square)

C. $\sqrt{40} = \sqrt{4 \cdot 10} = \sqrt{4} \cdot \sqrt{10} = \mathbf{2\sqrt{10}}$ (4 is a perfect square)

D. $\sqrt{45} = \sqrt{9 \cdot 5} = \sqrt{9} \cdot \sqrt{5} = \mathbf{3\sqrt{5}}$ (9 is a perfect square)

E. $-\sqrt{288} = -\sqrt{144 \cdot 2} = -\sqrt{144} \cdot \sqrt{2} = -\mathbf{12\sqrt{2}}$

(144 is a perfect square)

F. $\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36} \cdot \sqrt{2} = \mathbf{6\sqrt{2}}$ (36 is a perfect square)

This problem could have been done in more steps; you are encouraged to use as many steps as you see fit (unless your teacher thinks otherwise). For example,

$$\sqrt{72} = \sqrt{9 \cdot 8} = \sqrt{9} \cdot \sqrt{8} = 3\sqrt{8} = 3\sqrt{4 \cdot 2}$$

$= \quad 3 \cdot \sqrt{4} \cdot \sqrt{2} \; = \; 3 \cdot 2 \cdot \sqrt{2} \; = \; \mathbf{6\sqrt{2}}$, the same answer

It's important to understand that if you had stopped at the step $3\sqrt{8}$, you would not have the right answer, even though you simplified the radical somewhat. This is because even more factors could be brought out of the square root sign.

G. $\sqrt{30}$ cannot be simplified. 30 does not contain any perfect-square factors.

H. $\sqrt{-64}$ does not exist in the set of real numbers.

Homework

8. In part F of the above example, we concluded that $\sqrt{72} \; = \; 6\sqrt{2}$. Use your calculator to "verify" this fact.

9. Simplify each square root:

a. $\sqrt{8}$	b. $\sqrt{24}$	c. $\sqrt{-75}$	d. $\sqrt{27}$	e. $\sqrt{98}$
f. $\sqrt{54}$	g. $\sqrt{150}$	h. $\sqrt{172}$	i. $\sqrt{105}$	j. $\sqrt{-20}$
k. $\sqrt{175}$	l. $\sqrt{200}$	m. $\sqrt{242}$	n. $\sqrt{46}$	o. $\sqrt{-1}$
p. $\sqrt{1}$	q. $\sqrt{0}$	r. $\sqrt{88}$	s. $\sqrt{147}$	t. $\sqrt{361}$
u. $\sqrt{338}$	v. $\sqrt{605}$	w. $\sqrt{250}$	x. $\sqrt{184}$	y. $\sqrt{189}$

10. Simplify each square root:

a. $\sqrt{18}$	b. $-\sqrt{72}$	c. $\sqrt{12}$	d. $\sqrt{32}$	e. $-\sqrt{52}$
f. $\sqrt{162}$	g. $\sqrt{125}$	h. $\sqrt{128}$	i. $\sqrt{256}$	j. $\sqrt{28}$

k. $\sqrt{-9}$ l. $-\sqrt{500}$ m. $\sqrt{289}$ n. $\sqrt{625}$ o. $\sqrt{800}$

p. $\sqrt{600}$ q. $\sqrt{-4}$ r. $\sqrt{448}$ s. $-\sqrt{121}$ t. $\sqrt{450}$

u. $\sqrt{363}$ v. $\sqrt{117}$ w. $-\sqrt{350}$ x. $\sqrt{490}$ y. $\sqrt{396}$

11. We have a pretty powerful rule, $\sqrt{ab} = \sqrt{a}\sqrt{b}$, that we've been using to simplify square roots. A natural question might be: Does it also work for addition? That is, is it always true that $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$? Test this theory yourself using the values $a = 16$ and $b = 9$.

Review Problems

12. a. How many square roots does 169 have? What are they? Why are they square roots of 169?

b. How many square roots does 29 have? What are they? Why are they square roots of 29?

c. Name the only number with exactly one square root.

d. Give a number that has no square root in the set of real numbers.

e. What is the smallest possible value of the expression $\sqrt{x}$?

f. Consider the number $-\sqrt{-36}$. Your uncle tells you the answer is 6, since the negatives cancel out. Why is your uncle ignorant of math? Should you tell him that he is?

g. Simplify: $\left(\sqrt{n}\right)^2$ (presuming n is a non-negative number.)

h. Calculate $\sqrt{123{,}987{,}005^2}$ without a calculator.

i. Calculate $\sqrt{(-1{,}273)^2}$ without a calculator.

13. Simplify each square root:

a. $\sqrt{250}$ b. $\sqrt{56}$ c. $\sqrt{112}$ d. $\sqrt{400}$ e. $\sqrt{144}$

f. $\sqrt{76}$ g. $-\sqrt{288}$ h. $\sqrt{8}$ i. $\sqrt{4}$ j. $-\sqrt{54}$

k. $\sqrt{300}$ l. $\sqrt{9}$ m. $\sqrt{49}$ n. $\sqrt{196}$ o. $\sqrt{475}$

p. $\sqrt{72}$ q. $\sqrt{240}$ r. $\sqrt{100}$ s. $\sqrt{52}$ t. $\sqrt{96}$

u. $\sqrt{98}$ v. $-\sqrt{2}$ w. $\sqrt{648}$ x. $\sqrt{500}$ y. $\sqrt{-3}$

14. Simplify each square root:

a. $\sqrt{275}$ b. $\sqrt{931}$ c. $\sqrt{208}$ d. $\sqrt{343}$ e. $-\sqrt{63}$

f. $-\sqrt{175}$ g. $\sqrt{360}$ h. $\sqrt{44}$ i. $\sqrt{375}$ j. $\sqrt{68}$

k. $\sqrt{810}$ l. $\sqrt{80}$ m. $\sqrt{99}$ n. $\sqrt{252}$ o. $\sqrt{160}$

p. $\sqrt{153}$ q. $-\sqrt{180}$ r. $\sqrt{588}$ s. $\sqrt{135}$ t. $\sqrt{192}$

u. $\sqrt{425}$ v. $\sqrt{176}$ w. $\sqrt{92}$ x. $\sqrt{720}$ y. $\sqrt{-60}$

Solutions

1. a. 12 and –12 each have squares equal to 144.

b. There's no number (in this class) whose square is –144.

c. 25 has two square roots, 5 and –5 (which we could write as ± 5).

5 is a square root of 25 because $5^2 = 25$.

–5 is a square root of 25 because $(-5)^2 = 25$.

d. 196 has two square roots, 14 and –14.

14 is a square root of 196 because $14^2 = 196$.

–14 is a square root of 196 because $(-14)^2 = 196$.

e. 1 has two square roots, 1 and –1.

f. 0 has only one square root, 0.

g. –9 has no square root at all (in the set of real numbers).

h. The two square roots of $\frac{4}{9}$ are $\frac{2}{3}$ and $-\frac{2}{3}$.

2. a. $\sqrt{49}$ and $-\sqrt{49}$, OR 7 and –7 because $7^2 = 49$ and $(-7)^2 = 49$

b. $\sqrt{29}$ and $-\sqrt{29}$ because $(\sqrt{29})^2 = 29$ and $(-\sqrt{29})^2 = 29$

c. $\sqrt{51}$ and $-\sqrt{51}$ because $(\sqrt{51})^2 = 51$ and $(-\sqrt{51})^2 = 51$

3. $(\sqrt{13})^2 \approx (3.605551275)^2 \approx 13$

4.

a. 12	b. –15	c. Does not exist
d. Does not exist	e. 14	f. 1
g. 0	h. 16	i. –8
j. 13	k. Does not exist	l. 17

5. a. 64 b. 93 c. a d. m

6. a. 10 b. 12,456 c. 7 d. 3124

7. An example of "*the negative square root of a number*" is $-\sqrt{100}$, which comes out to be –10, a fine answer.
As for "*the square root of a negative number*," an example would be $\sqrt{-1}$, which we've learned is not in the set of real numbers (all decimals).

8. Depending on your calculator, you'll get exactly the same digits or really close. Even if they don't match exactly, that's pretty good evidence that $\sqrt{72}$ truly equals $6\sqrt{2}$. But it's only evidence; perhaps a more powerful calculator might show a discrepancy in the 1000th digit.

9. a. $2\sqrt{2}$ b. $2\sqrt{6}$ c. Does not exist d. $3\sqrt{3}$ e. $7\sqrt{2}$
f. $3\sqrt{6}$ g. $5\sqrt{6}$ h. $2\sqrt{43}$ i. $\sqrt{105}$ j. Does not exist
k. $5\sqrt{7}$ l. $10\sqrt{2}$ m. $11\sqrt{2}$ n. $\sqrt{46}$ o. Does not exist
p. 1 q. 0 r. $2\sqrt{22}$ s. $7\sqrt{3}$ t. 19
u. $13\sqrt{2}$ v. $11\sqrt{5}$ w. $5\sqrt{10}$ x. $2\sqrt{46}$ y. $3\sqrt{21}$

10. a. $3\sqrt{2}$ b. $-6\sqrt{2}$ c. $2\sqrt{3}$ d. $4\sqrt{2}$ e. $-2\sqrt{13}$
f. $9\sqrt{2}$ g. $5\sqrt{5}$ h. $8\sqrt{2}$ i. 16 j. $2\sqrt{7}$
k. Does not exist l. $-10\sqrt{5}$ m. 17 n. 25 o. $20\sqrt{2}$
p. $10\sqrt{6}$ q. Does not exist r. $8\sqrt{7}$ s. –11 t. $15\sqrt{2}$
u. $11\sqrt{3}$ v. $3\sqrt{13}$ w. $-5\sqrt{14}$ x. $7\sqrt{10}$ y. $6\sqrt{11}$

11. Using $a = 16$ and $b = 9$ as test numbers:

$$\sqrt{a+b} = \sqrt{16+9} = \sqrt{25} = 5$$

$$\sqrt{a}+\sqrt{b} = \sqrt{16}+\sqrt{9} = 4+3 = 7$$

Not the same result! Our theory that $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$ is bogus (we have found a counterexample). It's also false if we use subtraction, but it's true if we use division (as long as we never divide by 0).

12. a. 169 has two square roots, 13 and –13 (or ± 13). They are square roots of 169 because $13^2 = 169$ and $(-13)^2 = 169$.

b. 29 has two square roots, $\sqrt{29}$ and $-\sqrt{29}$ (or $\pm\sqrt{29}$). They are square roots of 29 because $\left(\sqrt{29}\right)^2 = 29$ and $\left(-\sqrt{29}\right)^2 = 29$.

c. 0; its only square root is 0.

d. –49, for instance, has no square root (in this class). In fact, no negative number has a square root in this class.

e. The smallest possible value is 0, and that occurs when $x = 0$.

f. The negatives do <u>not</u> cancel out; they're not right next to each other. The Order of Operations requires that we start with the square root, which doesn't exist because the radicand is negative. Should you inform your uncle that he's ignorant of math? Not if you hope to be in his will.

g. n h. 123,987,005 i. 1,273

13.

a. $5\sqrt{10}$	b. $2\sqrt{14}$	c. $4\sqrt{7}$	d. 20	e. 12
f. $2\sqrt{19}$	g. $-12\sqrt{2}$	h. $2\sqrt{2}$	i. 2	j. $-3\sqrt{6}$
k. $10\sqrt{3}$	l. 3	m. 7	n. 14	o. $5\sqrt{19}$
p. $6\sqrt{2}$	q. $4\sqrt{15}$	r. 10	s. $2\sqrt{13}$	t. $4\sqrt{6}$
u. $7\sqrt{2}$	v. $-\sqrt{2}$	w. $18\sqrt{2}$	x. $10\sqrt{5}$	y. Does not exist

14.

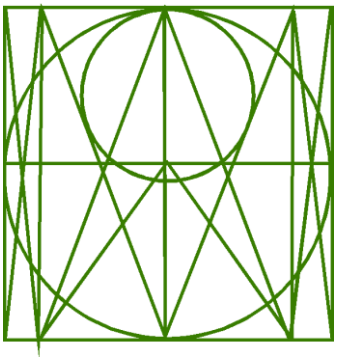
a. $5\sqrt{11}$	b. $7\sqrt{19}$	c. $4\sqrt{13}$	d. $7\sqrt{7}$	e. $-3\sqrt{7}$
f. $-5\sqrt{7}$	g. $6\sqrt{10}$	h. $2\sqrt{11}$	i. $5\sqrt{15}$	j. $2\sqrt{17}$
k. $9\sqrt{10}$	l. $4\sqrt{5}$	m. $3\sqrt{11}$	n. $6\sqrt{7}$	o. $4\sqrt{10}$
p. $3\sqrt{17}$	q. $-6\sqrt{5}$	r. $14\sqrt{3}$	s. $3\sqrt{15}$	t. $8\sqrt{3}$
u. $5\sqrt{17}$	v. $4\sqrt{11}$	w. $2\sqrt{23}$	x. $12\sqrt{5}$	y. Does not exist



"A man does what he must – in spite of personal consequences, in spite of obstacles and dangers and pressures – and that is the basis of all human morality."

John F. Kennedy (1917–1963)

RATIONAL FUNCTIONS



Remember what we call a number like $\frac{2}{7}$? This is called a *rational number* because it is the *ratio* of two integers. In a like manner, a *rational function* is the ratio of two special functions called ***polynomial functions***. Since a rational function is essentially a fraction, we will have to avoid dividing by zero, which means the domain of a such a function may not be all real numbers.

❑ *POLYNOMIAL FUNCTIONS*

Each of the following is a polynomial function:

$y = 7$	(a *linear* function – it's a horizontal line)
$y = -3x + \sqrt{2}$	(a *linear* function – it's a line with slope = –3)
$y = 2x^2 - x + 9$	(a *quadratic* function – it's a parabola)
$f(t) = \sqrt[4]{2}\,t^3 - t^2$	(a *cubic* function)
$P(w) = -\pi w^4 + 5w^2 + 8$	(a *quartic* function)
$Q(a) = \frac{2}{3}a^5 - 4a + 1$	(a *quintic* function)

The key to any ***polynomial function*** is that all the exponents on the input variable come from the set of whole numbers: {0, 1, 2, 3, . . .}. The coefficients (the numbers in front of the variables), on the other hand, can come from anywhere in $\mathbb{R}$, the set of real numbers.

Consider the quartic (4th degree) polynomial function

$$y = -2\pi x^4 - \tfrac{9}{10}x^3 + 9.2x + \sqrt{2}$$

First look at the exponents; they are all whole numbers. Notice that the x in the term $9.2x$ has an understood exponent of 1. Even the last term, $\sqrt{2}$, can be written as $\sqrt{2}\,x^0$, and so even the exponent on this last term is a whole number.

Thus, all the exponents on the x's (4, 3, 1, and 0) come from the whole numbers, while all the coefficients (-2π, $-\frac{9}{10}$, 9.2, $\sqrt{2}$) come from $\mathbb{R}$. Considering the definition of polynomial function, the given function is indeed a polynomial function.

Each of the following is <u>not</u> a polynomial function:

$y = \frac{1}{x}$	($\frac{1}{x} = x^{-1}$ and -1 is not a whole number)
$y = \sqrt{x}$	($\sqrt{x} = x^{1/2}$ and $\frac{1}{2}$ is not a whole number)
$f(x) = \frac{1}{\sqrt[3]{x}}$	($\frac{1}{\sqrt[3]{x}} = x^{-1/3}$ and $-\frac{1}{3}$ is not a whole number)
$g(x) = \lvert x-1 \rvert$	(no absolute values allowed around the x)
$E(x) = 2^x$	(since x is in the exponent, it can be any number)
$T(x) = \sin x$	(it's on your calculator, but it's not a polynomial function)
$y = \log x$	(a log function can never be a polynomial function)
$x^2 + y^2 = 25$	(it's a circle – it's not a function of any kind)

Homework

1. Explain why $y = \pi x^5 - \sqrt{2}\,x^3 + \frac{1}{4}x - 17.5$ <u>is</u> a polynomial function.

2. Explain why $h(x) = \sqrt[3]{5}x^4 - \sqrt{2x} + \frac{1}{2}$ is <u>not</u> a polynomial function.

3. The highest exponent on the variable in a polynomial function is called its ***degree***. Find the degree of each polynomial function:

   a. $y = \pi$ b. $y = x^3 - 17x^2 + 8$

   c. $y = \sqrt{2}x^8 - \frac{\pi}{2}x^{10}$ d. $y = 7x + 5$

4. a. What is the domain of any polynomial function?
   b. T/F: Every parabola is a polynomial function.
   c. T/F: Every non-vertical line is a polynomial function.
   d. T/F: Every line is a polynomial function.

❑ *RATIONAL FUNCTIONS*

A ***rational function*** is defined to be the *ratio* of two *polynomial functions*. If P and Q are polynomial functions, then $R = \frac{P}{Q}$ is a rational function. A typical example of a rational function is $y = \frac{3.9x^2 + 7x - 9}{x + 8}$ (a quadratic function over a linear function).

<u>EXAMPLE 1:</u> **Graph:** $\boldsymbol{y = \frac{1}{x - 2}}$

<u>Solution:</u> Since y is the ratio of a constant polynomial function and a linear polynomial function, we know that y is a rational function.

We begin our analysis of this rational function by determining the **domain**. In order that the fraction be defined, we must <u>not</u> divide by zero. What value of x makes the denominator zero? The value $x = 2$ will. (Just set $x - 2 = 0$ and solve for x.)

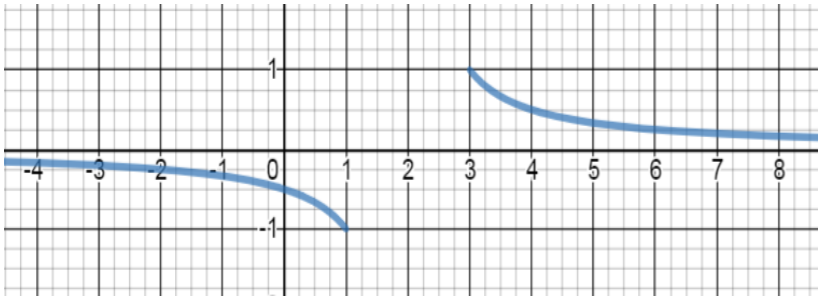
Therefore, the domain is all real numbers except 2; that is, the domain is $\mathbb{R} - \{2\}$.

Intercepts come next. If $x = 0$, then $y = \frac{1}{0-2} = -\frac{1}{2}$. Thus, $\mathbf{(0, -\frac{1}{2})}$ is the y-intercept. To find an x intercept, set $y = 0$. This gives $0 = \frac{1}{x-2} \Rightarrow 0(x-2) = \frac{1}{x-2}(x-2) \Rightarrow 0 = 1$, which has no solution. Thus, there are **no** x-intercepts.

Now for some ordered pairs that satisfy the formula $y = \frac{1}{x-2}$:

x	y
–3	$-\frac{1}{5}$
–2	$-\frac{1}{4}$
–1	$-\frac{1}{3}$
0	$-\frac{1}{2}$
1	–1
2	Und.
3	1
4	$\frac{1}{2}$
5	$\frac{1}{3}$
6	$\frac{1}{4}$
7	$\frac{1}{5}$

If we plot these points and connect them with a smooth curve, we would get the following graph:

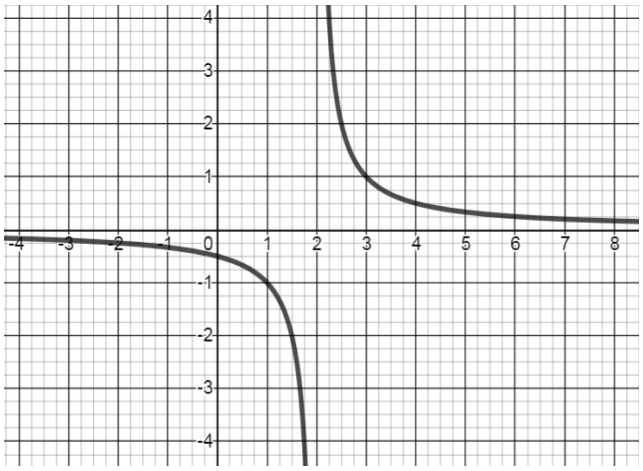


What some students do at this point is to lazily connect the points (3, 1) and (1, –1) with a straight line. Talk about jumping to conclusions! Our domain of $\mathbb{R} - \{2\}$ implies that x cannot be 2 in this function; the straight-line trick won't work. So we agree that a major chunk of the graph is missing. How do we get a better picture of the graph? We try some x-values that are near 2:

$(1\frac{1}{2}, -2)$ $(1\frac{3}{4}, -4)$ $(1\frac{7}{8}, -8)$

$(2\frac{1}{2}, 2)$ $(2\frac{1}{4}, 4)$ $(2\frac{1}{8}, 8)$

Adding these points to our previous attempt at a graph gives us a much better picture:



This graph has some real cool **limits**. Suppose we let x approach ∞. The y-values are positive (the curve is above the x-axis), but are getting smaller and smaller, approaching zero. Thus, **as $x \to \infty$, $y \to 0$**. [This can be read: "*As x grows infinitely large, y is getting closer and closer to 0.*]

Now let x approach $-\infty$. The y-values are negative but are rising toward zero. Therefore, as $\boldsymbol{x \to -\infty, y \to 0}$.

The number 2 seems to be an interesting x-value. Although x can never be 2 in this function, it looks like the curve is getting closer and closer to the vertical line $x = 2$. In fact, if we let x approach 2 from the right, the curve is growing taller and taller, and so we have the limit: **As $x \to 2$ (from the right), $y \to \infty$**. [This can be read: "*As x gets closer and closer to 2, approaching 2 from the right (meaning values larger than 2), y is growing infinitely large.*]

Now let x approach 2 from the left. This time the curve is dropping rapidly, toward negative infinity. This observation yields the limit: **As $x \to 2$ (from the left), $y \to -\infty$**.

Let's summarize the four limits we've deduced:

- As $x \to \infty$, $y \to 0$. $\lim_{x \to \infty} y = 0$
- As $x \to -\infty$, $y \to 0$. $\lim_{x \to -\infty} y = 0$
- As $x \to 2$ (from the right), $y \to \infty$. $\lim_{x \to 2^+} y = \infty$
- As $x \to 2$ (from the left), $y \to -\infty$. $\lim_{x \to 2^-} y = -\infty$

Do you see that as you move far to the right or far to the left, the curve gets closer and closer to the x-axis? We say that the line **$y = 0$** (which is the x-axis) is a ***horizontal asymptote***.

Now look at the region of the graph near $x = 2$. The curve gets closer and closer to the vertical line $x = 2$ (in fact, on both sides of the vertical line). We call the line **$x = 2$** a ***vertical asymptote***.

EXAMPLE 2: **Graph:** $y = \dfrac{2x-1}{x+2}$

Solution: First we find the **domain**. Recall that this function will be undefined when the denominator is zero, which occurs when $x = -2$. Thus, the domain is $\mathbb{R} - \{-2\}$.

Now let's explore the **intercepts**:

If $x = 0$, then $y = \dfrac{2(0)-1}{0+2} = -\dfrac{1}{2}$. There's a y-intercept at $\mathbf{(0, -\frac{1}{2})}$.

If $y = 0$, then $0 = \dfrac{2x-1}{x+2} \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$. So $\mathbf{(\frac{1}{2}, 0)}$ is an x-intercept.

It's time for some more ordered pairs for this function. Use your calculator to verify each of the following:

(–1, –3) (1, 0.33) (3, 1) (5, 1.29) (10, 1.58)

(15, 1.71) (20, 1.77) (100, 1.95) (1000, 1.995)

What's happening as x grows very large? It appears that y is approaching 2. That is, **as $x \to \infty$, $y \to 2$.**

Now we'll let x go the other direction:

(–3, 7) (–5, 3.67) (–10, 2.63) (–20, 2.28)

(–100, 2.05) (–1000, 2.01)

These points show that as $x \to -\infty$, $y \to 2$.

Finally, here are some ordered pairs for x's near –2 (the only real number not in the domain):

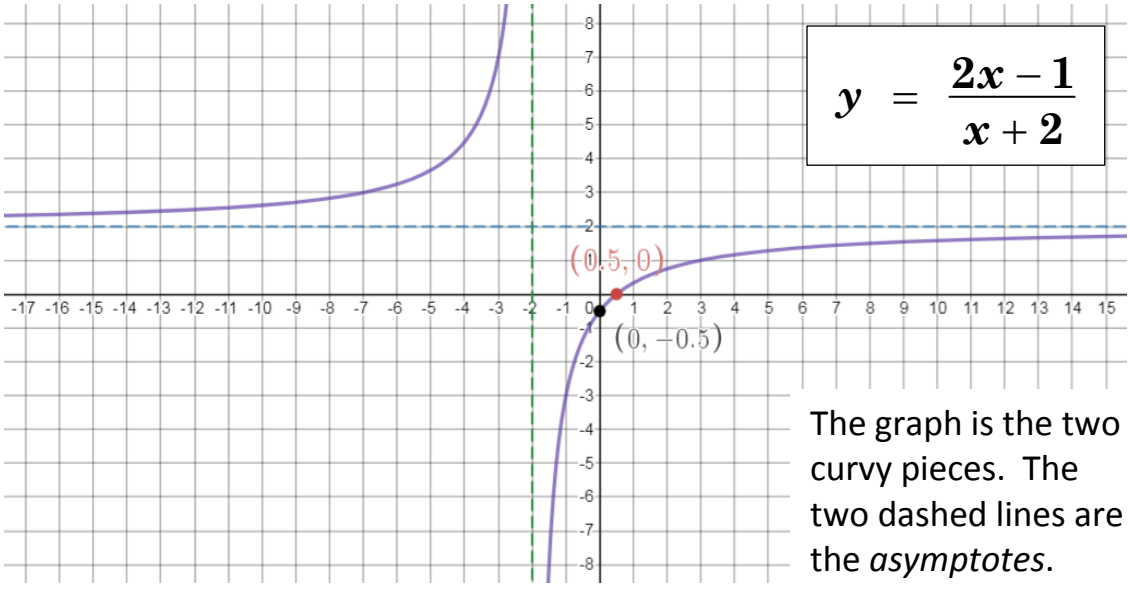
(–1.5, –8) (–1.9, –48) (–1.99, –498)

Thus, **as $x \to -2$ (from the right), $y \to -\infty$.**

(–2.5, 12) (–2.1, 52) (–2.01, 502)

So, as $x \to -2$ (from the left), $y \to \infty$.

Plotting as many of the calculated points as possible, including the two intercepts, and considering the four limits we've found, the following graph emerges:



The graph is the two curvy pieces. The two dashed lines are the *asymptotes*.

We can now be reasonably sure of the **asymptotes** (denoted by the dashed lines). Either by recalling the limits described above or by looking at the graph, we conclude that there's **a vertical asymptote at $x = -2$** and **a horizontal asymptote at $y = 2$**.

EXAMPLE 3: **Graph:** $y = \dfrac{4}{1+x^2}$

Solution: Why is this function rational? Because it's the *ratio* $\dfrac{P}{Q}$ of two polynomial functions: the constant polynomial $P(x) = 4$ and the quadratic polynomial $Q(x) = 1 + x^2$.

To find the **domain**, set the denominator to zero to see what's not in the domain: $1 + x^2 = 0$. This equation has no solution in $\mathbb{R}$, since solving it leads to $x = \pm\sqrt{-1}$, which are not real numbers. In fact, for any value of x, the quantity $1 + x^2$ is at least 1 (why?), so it certainly can't be zero. Since the denominator can never be zero, there's nothing to be excluded from the domain, and therefore the domain is $\mathbb{R}$. We can also figure that the graph will not have a **vertical asymptote**, since the denominator can never be 0.

Notice that if we put in some positive value of x, we'll get a certain y-value. Now look at what will happen if we put $-x$ (the *opposite* of x) into the formula. Since $(-x)^2$ is equal to x^2, we will get the same y-value. This implies that the left side of the graph will be the mirror image of the right side. We say that the graph has ***y-axis symmetry*** (or is *symmetric with respect to the y-axis*).

Now we seek the **intercepts**. Set $x = 0$ to get $y = 4$, and so the y-intercept is **(0, 4)**. Now set $y = 0$, giving

$$0 = \frac{4}{1+x^2} \quad \Rightarrow \quad 0(1+x^2) = \frac{4}{1+x^2}(1+x^2) \quad \Rightarrow \quad 0 = 4$$

This absurd result indicates that the equation has no solution; hence, there are **no** x-intercepts.

It's time for some additional ordered pairs for this function:

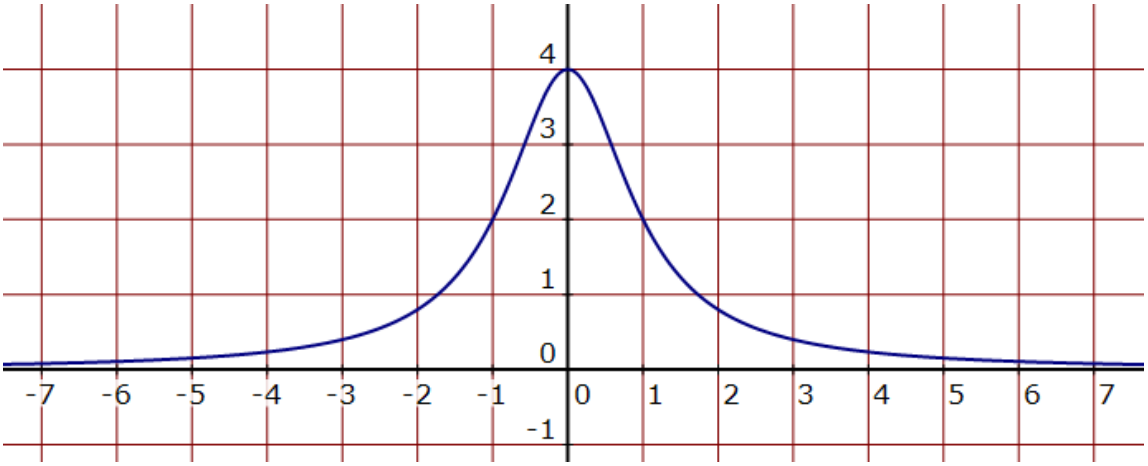
(1, 2) (2, 0.8) (3, 0.4) (4, 0.24) (10, 0.04) (200, 0.0001)

These points suggest the limit: **As $x \to \infty$, $y \to 0$**. This implies that **$y = 0$ is a horizontal asymptote**.

Here are some more ordered pairs, designed to see what happens as we approach the y-axis from the right:

(0.75, 2.56) (0.5, 3.2) (0.25, 3.76) (0.1, 3.96) (0.02, 3.998)

If we plot all the points calculated so far, and if we recall the y-axis symmetry, we get the following graph:



We determined at the outset that the *domain* of this rational function is $\mathbb{R}$. Is it clear from the graph that this is indeed the case?

If you happen to know what the **range** of a function means, you can see that the range of this function is all real numbers that are greater than 0 but less than or equal to 4: **Range = (0, 4]**

Homework

5. Consider the rational function in Example 2: $y = \frac{2x-1}{x+2}$.

   Without referring to the graph, prove that y <u>can</u> have the value 2.01, but y can <u>never</u> have the value 2.

6. Find the domain:

   a. $y = \frac{2x+7}{9}$ b. $f(x) = \frac{2x-3}{4+x}$

   c. $g(x) = \frac{3x}{2x-10}$ d. $R(x) = \frac{x-1}{-7x+4}$

   e. $f(x) = \frac{x^2-9}{x^2-100}$ f. $g(x) = \frac{8x-16}{x^2+25}$

7. Find the intercepts:

   a. $f(x) = \frac{x-4}{x-2}$ b. $y = \frac{3}{5x-15}$

   c. $y = \frac{2x+1}{x-3}$ d. $g(x) = \frac{5-x}{6x+1}$

8. Find the asymptotes:

   a. $R(x) = \frac{8x+1}{4x-4}$ b. $y = \frac{2x-3}{2x+1}$

   c. $y = \frac{3x-7}{x+2}$ d. $h(x) = \frac{2x+7}{4x-4}$

9. Find the domain, the intercepts, and the asymptotes of

   $$y = \frac{1}{4+x^2}$$

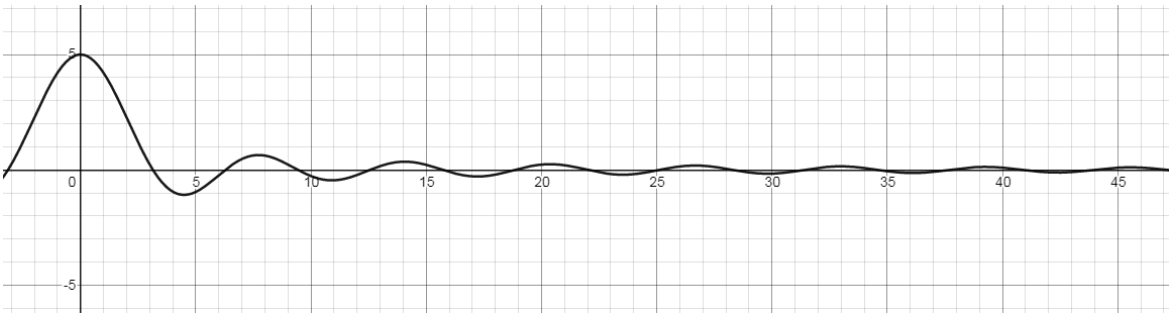
10. Perform a complete analysis of the function $y = \frac{2}{x-3}$.

11. Perform a complete analysis of the function $y = \dfrac{3x-5}{x-2}$.

12. Perform a complete analysis of the function $y = \dfrac{2}{2+x^2}$.

13. Perform a complete analysis of the function $y = \dfrac{-1}{x+1}$.

14. Consider the graph



Explain why the horizontal line $y = 0$ (that is, the x-axis) is a horizontal asymptote for the curve.

Review Problems

15. a. Explain why $f(x) = \sqrt{7}x^{10} + \pi x^7 - 6x - 1$ <u>is</u> a polynomial function. What is its degree?

b. Explain why $y = 3x^5 - \sqrt{x} + \pi$ is <u>not</u> a polynomial function.

16. a. A horizontal line (is, is not) a polynomial function.

b. The function $y = \dfrac{1}{x}$ (is, is not) a polynomial function.

c. What is the degree of the polynomial function $y = 7x - \pi$?

d. Is a circle a polynomial function?

17. Consider the rational function $y = \frac{7}{2x-8}$.

a. Find the domain.
b. Find all the intercepts.
c. Find all the asymptotes.
d. Calculate y if $x = 4.1$.

18. Find all the intercepts and asymptotes of $r(x) = \frac{8x+6}{2x-3}$, and graph.

19. Graph $y = \frac{-2x-2}{x-1}$.

As $x \to 1$ (from the right), $y \to$ ___.
As $x \to 1$ (from the left), $y \to$ ___.
As $x \to \infty$, $y \to$ ___.
As $x \to -\infty$, $y \to$ ___.

20. Graph $y = \frac{5}{2+x^2}$. Discuss domain, symmetry, and asymptotes.

As $x \to \infty$, $y \to$ ___.
As $x \to -\infty$, $y \to$ ___.
As $x \to 0$ (from the right), $y \to$ ___.
As $x \to 0$ (from the left), $y \to$ ___.

21. True/False:

a. $y = \sqrt[3]{7}x^{10} - \pi x^3 + \sqrt{2}$ is a polynomial function.

b. $y = \frac{1}{x^5}$ is a polynomial function.

c. The graph of $f(x) = \frac{1}{2x+10}$ has a vertical asymptote at $x = -5$.

d. The graph of $g(x) = \frac{10x+9}{5x-11}$ has a horizontal asymptote at $y = 10$.

e. The domain of the function $y = \frac{6}{1+x^2}$ is $\mathbb{R} - \{\pm 1\}$.

f. For the graph of $y = \frac{3x+1}{x-\pi}$, as $x \to \infty$, $y \to 3$.

Solutions

1. All coefficients are from $\mathbb{R}$, and all exponents are from $\mathbb{W}$ (the whole numbers).

2. The middle term is $\sqrt{2}\,x^{1/2}$, and $\frac{1}{2} \notin \mathbb{W}$.

3. a. 0 b. 3 c. 10 d. 1

4. a. $\mathbb{R}$ b. False c. True d. False

5. $y = \frac{2x-1}{x+2} \Rightarrow 2.01 = \frac{2x-1}{x+2} \Rightarrow 2.01x + 4.02 = 2x - 1 \Rightarrow x = -502$.
So, (–502, 2.01) is on the graph, and indeed y can be 2.01.

Now let's pretend that y could be 2; then
$2 = \frac{2x-1}{x+2} \Rightarrow 2x + 4 = 2x - 1 \Rightarrow 4 = -1 \Rightarrow$ No solution. Thus, there is no x which will make $y = 2$.

6. a. $\mathbb{R}$ b. $\mathbb{R} - \{-4\}$ c. $\mathbb{R} - \{5\}$ d. $\mathbb{R} - \left\{\frac{4}{7}\right\}$ e. $\mathbb{R} - \{\pm 10\}$ f. $\mathbb{R}$

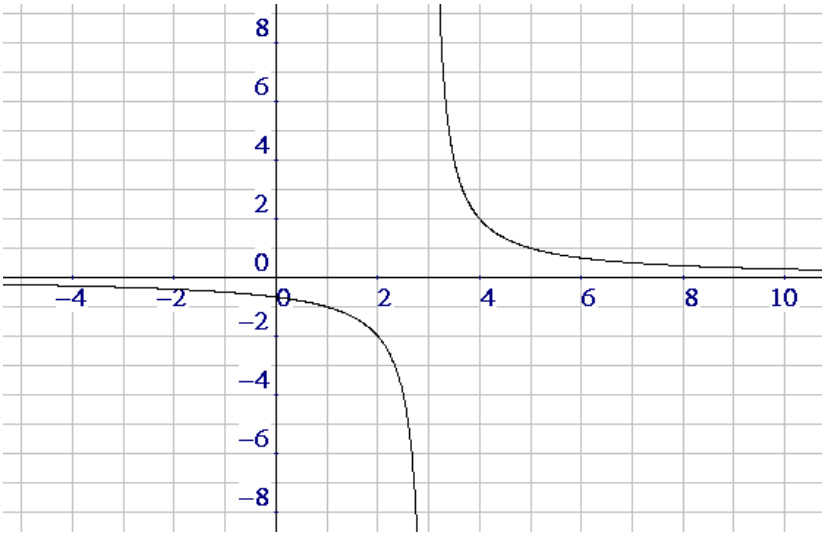
7. a. (4, 0) (0, 2) b. $(0, -\frac{1}{5})$ c. $(-\frac{1}{2}, 0)$ $(0, -\frac{1}{3})$ d. (5, 0) (0, 5)

8. a. $x = 1$ $y = 2$ b. $x = -\frac{1}{2}$ $y = 1$ c. $x = -2$ $y = 3$ d. $x = 1$ $y = \frac{1}{2}$

9. Since the only way the formula can be messed up is by dividing by 0, and since the denominator can never be zero (verify this yourself), the domain is $\mathbb{R}$.

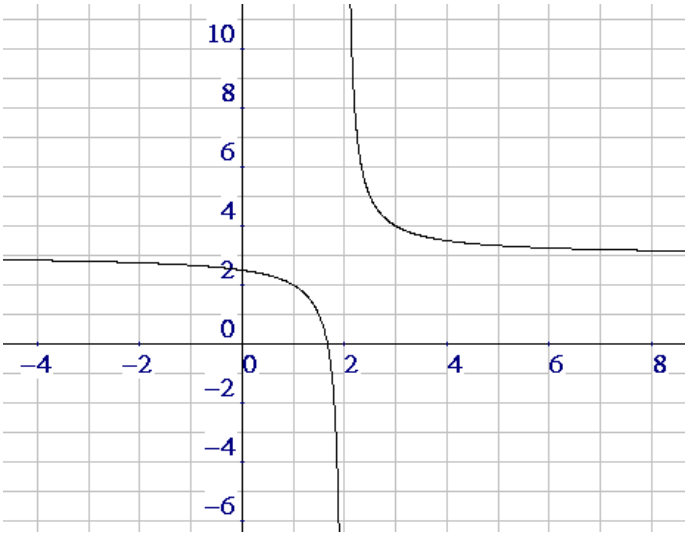
Setting $x = 0$ gives a y-value of 1/4, so the y-intercept is $(0, \frac{1}{4})$. If you set $y = 0$, you'll get no solution for y. Thus, there is no x-intercept.
There are no vertical asymptotes, since the denominator is never zero.
Letting x approach either ∞ or $-\infty$, y approaches 0. Thus, a horizontal asymptote is $y = 0$ (the x-axis).

10.



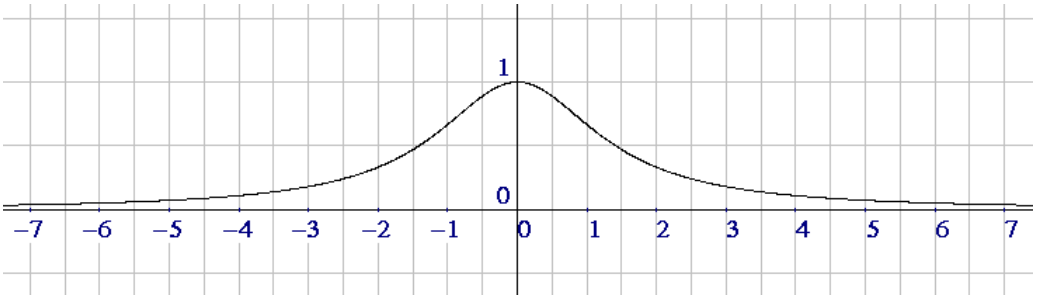
Domain = $\mathbb{R} - \{3\}$
x-int: none
y-int: $(0, -\frac{2}{3})$
As $x \to 3$ (from the right), $y \to \infty$
As $x \to 3$ (from the left), $y \to -\infty$
As $x \to \infty$, $y \to 0$
As $x \to -\infty$, $y \to 0$
vert asy: $x = 3$
horiz asy: $y = 0$

11.



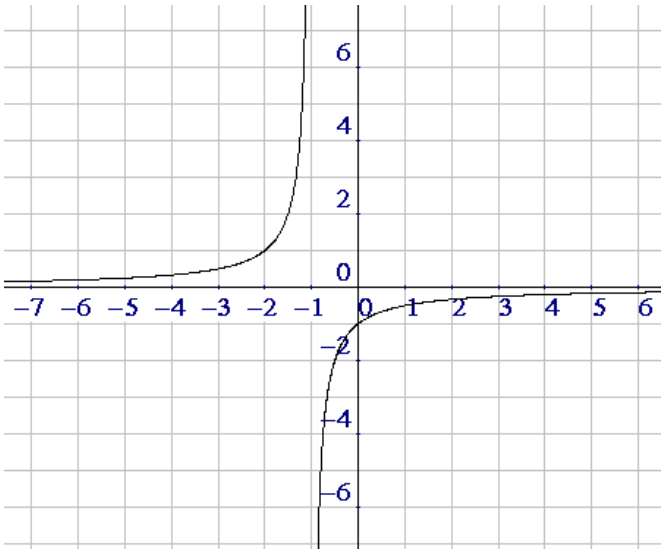
Domain = $\mathbb{R} - \{2\}$
x-int: $(\frac{5}{3}, 0)$
y-int: $(0, \frac{5}{2})$
As $x \to 2$ (from the right), $y \to \infty$
As $x \to 2$ (from the left), $y \to -\infty$
As $x \to \infty$, $y \to 3$
As $x \to -\infty$, $y \to 3$
vert asy: $x = 2$
horiz asy: $y = 3$

12.



Domain = $\mathbb{R}$ | x-int: none | y-int: (0, 1)
As $x \to \infty$, $y \to 0$ | As $x \to -\infty$, $y \to 0$
vert asy: none | horiz asy: $y = 0$
maximum point at (0, 1)

13.



Domain = $\mathbb{R} - \{-1\}$
x-int: none
y-int: (0, −1)
As $x \to -1$ (from the right), $y \to -\infty$
As $x \to -1$ (from the left), $y \to \infty$
As $x \to \infty$, $y \to 0$
As $x \to -\infty$, $y \to 0$
vert asy: $x = -1$
horiz asy: $y = 0$

14. Because of the limit: As $x \to \infty$, $y \to 0$. Even though the graph intersects its own horizontal asymptote infinitely often, the curve nevertheless continues to get closer and closer to the x-axis (the line $y = 0$), and this is ultimately what is meant by a horizontal asymptote.

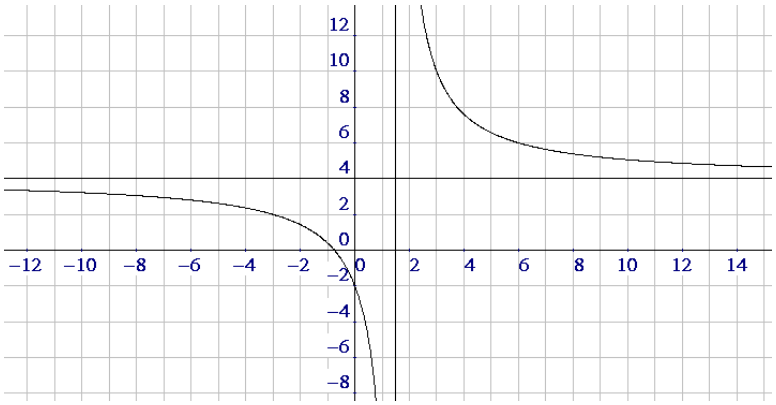
15. a. f is a polynomial because the coefficients are real numbers and the exponents (the 10, 7 and 1) are whole numbers. Its degree is 10.

b. Look at the middle term; it can be written as $x^{1/2}$, a term whose exponent is <u>not</u> from the whole numbers.

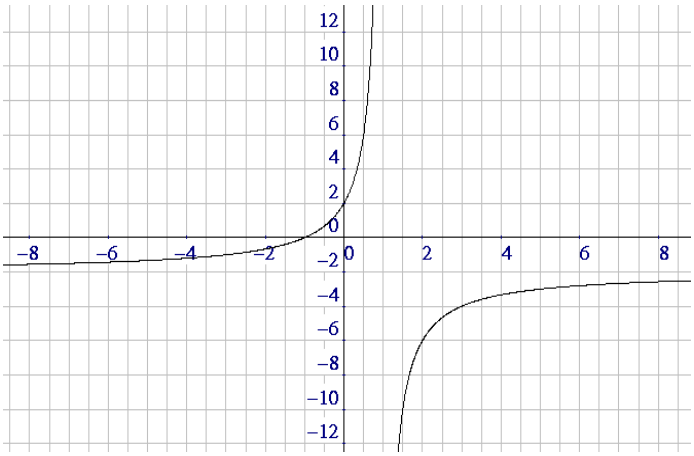
16. a. is b. is not ($1/x = x^{-1}$) c. 1 d. It's not even a function, let alone the special function called a polynomial.

17. a. $\mathbb{R} - \{4\}$ b. $(0, -7/8)$ c. $x = 4$ and $y = 0$ d. 35

18. Intercepts: $(0, -2)$ and $(-\frac{3}{4}, 0)$; vert asy: $x = \frac{3}{2}$; horiz asy: $y = 4$

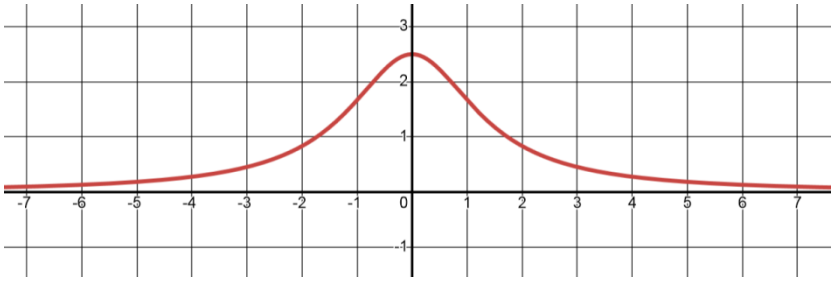


19.



Limits: $-\infty$; ∞; -2; -2

20.



Domain = $\mathbb{R}$

y-axis symmetry

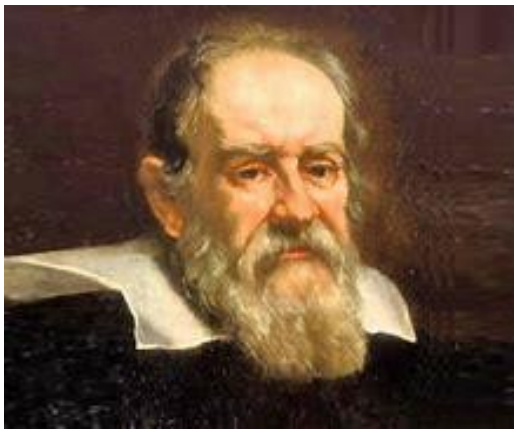
No vert asy

Horiz asy: $y = 0$

Limits: 0; 0; $\frac{5}{2}$; $\frac{5}{2}$

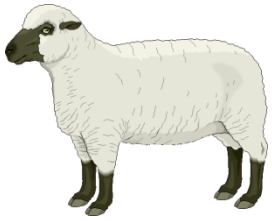
21. a. T b. F c. T d. F e. F f. T

" The universe cannot be read until we have learned the language in which it is written. It is written in mathematical language."



Galileo Galiliei

THE RATIONAL NUMBERS





From counting the sheep returning to the cave at day's end, to compressing music and video for transfer across the Internet, mathematics is primarily the study of various sets of numbers.

❑ THE NATURAL NUMBERS

The ***natural numbers*** is the set $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. The three dots is a punctuation symbol called an "ellipsis," and means "so on and so on." This set is clearly the fundamental number system in math. All the other numbers systems we shall visit in this course are built upon the natural numbers.

Homework

If you don't know what $\cup$ and $\subseteq$ mean, see the chapter entitled **Sets**.

1. a. Is there a smallest element in $\mathbb{N}$?
   b. Is there a largest element in $\mathbb{N}$?

2. How many elements does $\mathbb{N}$ have?

3. Consider the set of *whole numbers*: $\mathbb{W} = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$.
   a. Do $\mathbb{N}$ and $\mathbb{W}$ have the same number of elements?
   b. Which statement is true, $\mathbb{N} \subseteq \mathbb{W}$ or $\mathbb{W} \subseteq \mathbb{N}$?

4. T/F:
   a. The sum of two natural numbers is a natural number.
   b. The difference of two natural numbers is a natural number.
   c. The product of two natural numbers is a natural number.
   d. The quotient of two natural numbers is a natural number.

5. Does the equation $7x - 8 = 27$ have a solution in $\mathbb{N}$?

6. Does the equation $3x + 9 = 3$ have a solution in $\mathbb{N}$?

7. Prove each of the following theorems about natural numbers (don't panic — I'll teach you how):
   a. The sum of two even numbers is even.
   b. The product of two even numbers is even.
   c. The sum of two odd numbers is even.
   d. The product of two odd numbers is odd.
   e. The sum of an even and an odd is odd.
   f. The product of an even and an odd is even.
   g. If n is odd, then n^2 is odd.
   h. If n^2 is even, then n is even. This statement follows directly from part g. and one of the concepts from the chapter Logic.

8. a. Find the smallest prime number larger than 50.
   b. The prime factorization of 120 is $2^3 \cdot 3 \cdot 5$. Find the prime factorization of 504.

"The Natural Numbers were given by God. All the rest were created by man."

–Kronecker

❑ THE INTEGERS

The ***integers*** (int' e jers) is the set obtained when the natural numbers are combined with zero and the negatives of the natural numbers:

$$\mathbb{Z} = \{1, 2, 3, \ldots\} \cup \{0\} \cup \{\ldots, -3, -2, -1\}$$

In other words,

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

The symbol $\mathbb{Z}$ is used because the German word for number is Zahl.

Recall the homework problem, "Does the equation $3x + 9 = 3$ have a solution in $\mathbb{N}$?" There may be no solution to this equation in $\mathbb{N}$, but there is certainly a solution in the integers $\mathbb{Z}$, namely -2. This is one of the reasons we have the set of integers — we can solve equations that we couldn't otherwise solve.

Homework

9. Write a subset statement involving $\mathbb{N}$ and $\mathbb{Z}$.

10. List the numbers that are in $\mathbb{Z}$ but not in $\mathbb{N}$.

11. T/F:
    a. The sum of two integers is an integer.
    b. The difference of two integers is an integer.
    c. The product of two integers is an integer.
    d. The quotient of two integers is an integer. (You may assume that you're not dividing by zero.)

12. Consider the conjecture:

    If a and b are elements of $\mathbb{Z}$ with $a < b$,
    then there exists a z in $\mathbb{Z}$ such that $a < z < b$.

    First find an example which would make the conjecture true.
    Now give a counterexample to show that the conjecture is false.

13. Does the equation $3x + 9 = 3$ have a solution in $\mathbb{Z}$?

14. Does the equation $2x - 1 = 6$ have a solution in $\mathbb{Z}$?

❑ THE RATIONAL NUMBERS

The equation $x + 2 = 0$ has no solution in $\mathbb{N}$. We remedied that defect by creating $\mathbb{Z}$. But the equation $3x = 4$ has no solution even in $\mathbb{Z}$. We can fix this problem by expanding the number systems even further.

When we write a *ratio* of integers, $\frac{a}{b}$, where $b \neq 0$, we have what's called a ***rational number***. In other words, any number which can be written as a ratio of two integers is called a rational number. Another term for ratio is *quotient*, so we use the letter $\mathbb{Q}$ to represent the set of rational numbers.

$$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a,b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$$

A number is *rational* if it can be written as an integer over an integer, where the bottom, of course, isn't zero. Note that a rational number can be written in an infinite number of ways; for example, the rational number 0.5 can be written as $\frac{1}{2}$, or $\frac{-2}{-4}$, or $\frac{13}{26}$, etc.

EXAMPLES OF RATIONAL NUMBERS:

A. $\frac{7}{8} \in \mathbb{Q}$, since 7 and 8 are integers and $8 \neq 0$.

B. $0 \in \mathbb{Q}$, since 0 can be written $\frac{0}{1}$, among other ways.

C. $-\frac{9}{4} \in \mathbb{Q}$, since $-\frac{9}{4} = \frac{-9}{4}$, which is a ratio of integers.

D. $1.234 \in \mathbb{Q}$, since $1.234 = 1\frac{234}{1000} = \frac{1234}{1000}$, a ratio of integers.

E. $259 \in \mathbb{Q}$ since 259 can be written as the ratio $\frac{259}{1}$.

F. $0.737373\ldots \in \mathbb{Q}$ (recall the three dots mean that the block "73" repeats forever). You will prove that this number is a ratio of integers in the homework.

G. $0.7388888\ldots \in \mathbb{Q}$. The proof is similar to the previous example, but a little trickier.

H. $\sqrt{2}$ is <u>not</u> in $\mathbb{Q}$ (i.e., $\sqrt{2} \notin \mathbb{Q}$). In other words, $\sqrt{2}$ can <u>never</u> be written as the ratio of two integers — it's simply not a fraction consisting of integers. This will be a homework problem in the next chapter which we will prove together in class. The number $\sqrt{2}$ arises as the hypotenuse of a right triangle whose legs are each 1.

I. $\pi \notin \mathbb{Q}$. That is, π cannot be written as a fraction (using integers, of course), but I have no idea how to prove this fact. The number π can be defined as the ratio of the circumference of a circle to its diameter — this ratio is the same regardless of the size of the circle.

Note

The rational numbers are simply the fractions that can be created by using integers (with no zero on the bottom, of course). But every fraction can be converted to a decimal by doing long division. It's also a fact that eventually the decimal <u>must</u> begin repeating in a pattern that will continue forever and ever. For example,

$$\frac{2}{3} = 0.6666\ldots \qquad \text{(the "6" repeats)}$$

$$-\frac{1}{12} = -0.083333\ldots \qquad \text{(the "3" repeats)}$$

$$\frac{2}{7} = 0.285714285714\ldots \qquad \text{(the "285714" repeats)}$$

$$-\frac{9}{2} = -4.50000\ldots \qquad \text{(the "0" repeats)}$$

Therefore, not only can a rational number be defined as a ratio of two integers, but it can also be described as a repeating decimal:

$\mathbb{Q}$ = {all decimals with a *repeating block* of digits}

Final Example

In Example H above we said that $\sqrt{2}$ is not a rational number, and in the Note above we characterized the rational numbers as those numbers whose decimal expansion eventually repeats in a block pattern. Therefore, we can conclude the very important statement that the decimal expansion of $\sqrt{2}$ never begins to repeat in a block pattern that lasts forever. We can also conclude that π is another number that can never be written as a repeating decimal.

$\frac{43}{99}$ is rational because it can be written as the repeating decimal 0.434343 . . .

But $\sqrt{2}$ and π are <u>not</u> rational because they can <u>never</u> be written as a repeating decimal.

Equations

Recall the homework problem, "Does the equation $2x - 1 = 6$ have a solution in $\mathbb{Z}$?". There may be no solution in $\mathbb{Z}$, but there is a solution in $\mathbb{Q}$; it's $x = \frac{7}{2}$ (or 3.5). This is one of the reasons we have the set of rationals — we can solve equations that we couldn't otherwise solve.

Homework

15. Which of the following numbers are rational?

a. $\frac{5}{19}$ b. $-\frac{12}{5}$ c. 187 d. –14.75 e. 1.222

f. π g. $\sqrt{3}$ h. $\frac{\pi}{\pi}$ i. $\sqrt{100}$ j. $\sqrt{-1}$

k. 2.343434 . . . l. –0.123123123 . . .

m. 2.07932857 . . . [Note: there's no repeating block here]

n. 14.08978865334298

16. What's the importance of the stipulation that $b \neq 0$ in our first definition of $\mathbb{Q}$?

17. Prove that each of the following is rational:

a. $-\frac{2}{3}$ b. –801 c. 0.83 d. 1.125 e. $\frac{7}{16}$

18. Prove that 0.737373. . . $\in \mathbb{Q}$ by showing that this decimal can be written as a fraction (where the top and bottom are integers). [This is a hard problem — it's <u>not</u> cheating in this class to simply check out the full solution given at the end of the chapter.]

19. Prove that 0.5555. . . $\in \mathbb{Q}$ by writing the decimal as a fraction.

20. Explain why $\mathbb{Z} \subseteq \mathbb{Q}$. Is it clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$?

21. a. $\mathbb{N} \cup \mathbb{Q} =$ _____ b. $\mathbb{N} \cap \mathbb{Q} =$ _____

22. a. $\mathbb{Z} \cup \mathbb{Q} =$ _____ b. $\mathbb{Z} \cap \mathbb{Q} =$ _____

23. a. Is $\sqrt{3} \in \mathbb{Q}$? b. Is $\sqrt{49} \in \mathbb{Q}$?

24. Prove that $\frac{3}{7} \in \mathbb{Q}$ by using the decimal definition of a rational number.

25. Explain why the contention that
"π is rational, since it can be written as the ratio of integers, $\frac{22}{7}$"
is pure hokum.

26. T/F: $0.79799799979999\ldots \in \mathbb{Q}$

27. T/F: $0.28288288828888 \in \mathbb{Q}$

28. T/F: The sum of two rational numbers is rational.

29. T/F: The difference of two rational numbers is rational.

30. T/F: The product of two rational numbers is rational.

31. T/F: The quotient of two rational numbers is rational. (You may assume that the divisor is not zero.)

32. Let $q_1, q_2 \in \mathbb{Q}$, where $q_1 < q_2$. Prove that there exists a number $q \in \mathbb{Q}$ such that $q_1 < q < q_2$. Rephrased in English, given any two different rational numbers, show that there is always another rational number somewhere between them. [Note: This property did <u>not</u> hold in $\mathbb{Z}$.]

33. Does the equation $2x - 1 = 6$ have a solution in $\mathbb{Q}$?

34. Does the equation $x^2 - 2 = 0$ have a solution in $\mathbb{Q}$?

35. [EXTRA CHALLENGING] Convert $0.7383838\ldots$ to a fraction.

Review Problems

36. Give an element of $\mathbb{Z}$ which is not in $\mathbb{N}$.

37. Prove: If x is even, then x^2 is even.

38. a. The <u>definition</u> of the rationals is $\mathbb{Q} = \{ ____ \mid m, n \in ___, \text{ and } ____ \}$.

    b. Define $\mathbb{Q}$ as a set of decimals.

39. Use the definition (part a. above) to prove that $2.75 \in \mathbb{Q}$.

40. Use the definition (part a. above) to prove that $0.232323\ldots \in \mathbb{Q}$.

41. T/F: For any natural number n, it's always true that $\sqrt{n} \in \mathbb{Q}$.

42. T/F: The difference of two rational numbers is rational.

43. T/F: Let $q_1, q_2 \in \mathbb{Q}$, where $q_1 \neq q_2$. Then there exists a $q \in \mathbb{Q}$ such that q is between q_1 and q_2.

44. Construct an equation which has a solution in $\mathbb{Z}$ but not in $\mathbb{N}$.

45. Construct an equation which has a solution in $\mathbb{Q}$ but not in $\mathbb{Z}$.

46. Give an element of $\mathbb{Q}$ which is not in $\mathbb{Z}$.

47. a. Prove the statement: If x is odd, then x^2 is odd.
    b. State the converse of the statement. Is the converse true?
    c. State the contrapositive of the statement. Is the contrapositive true?

48. a. The <u>definition</u> of the rationals is $\mathbb{Q} = \{ \frac{m}{n} \mid m, n \in __, \text{ and } __ \neq 0 \}$.

    b. What set of numbers consists of all repeating decimals?

49. Use the definition (part a. above) to prove that $0.111 \in \mathbb{Q}$.

50. Use the definition (part a. above) to prove that $9.555\ldots \in \mathbb{Q}$.

51. T/F: There's at least one natural number such that $\sqrt{n} \in \mathbb{Q}$.

52. T/F: The product of two rational numbers is rational.

53. T/F: Let $q_1, q_2 \in \mathbb{Q}$. Then there exists a $q \in \mathbb{Q}$ such that q is between q_1 and q_2.

54. Construct an equation which has a solution in $\mathbb{N}$ but not in $\mathbb{Q}$.

55. Construct an equation which has a solution in $\mathbb{Z}$ but not in $\mathbb{Q}$.

56. True/False:

    a. $\pi \in \mathbb{N}$

    b. $10^6 \in \mathbb{N}$

    c. The product of two natural numbers is a natural number.

    d. The difference of two natural numbers is a natural number.

    e. The equation $x + 99 = 14$ has a solution in $\mathbb{N}$.

    f. $\mathbb{N} \subseteq \mathbb{Z}$

    g. The equation $x + 132 = 44$ has a solution in $\mathbb{Z}$.

    h. If $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$ and $p < q$, then there exists $r \in \mathbb{Z}$ such that $p < r < q$.

    i. The number 13/7 is called a rational number.

    j. $\mathbb{Q} = \{\frac{a}{b} \mid a,b \in \mathbb{Z} \}$.

    k. $7 \in \mathbb{Q}$.

l. $\sqrt{5} \in \mathbb{Q}$.

m. $-\sqrt{144} \in \mathbb{Q}$

n. $0.29343434\ldots \in \mathbb{Q}$.

o. $2.3874082365139\ldots \in \mathbb{Q}$.

p. If $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$ and $p < q$, then there exists $r \in \mathbb{Q}$ such that $p < r < q$.

q. The equation $13n + 17 = -100$ has a solution in $\mathbb{Q}$.

r. The equation $x^2 - 225 = 0$ has its solutions in $\mathbb{Q}$.

s. The equation $y^2 - 125 = 0$ has its solutions in $\mathbb{Q}$.

t. The sum of two odd numbers is odd.

u. $17.09385760225318 \in \mathbb{Q}$.

Solutions

1. a. Yes b. No **2**. ∞ **3**. a. What do you think? b. $\mathbb{N} \subseteq \mathbb{W}$

4. a. T b. F c. T d. F **5**. Yes **6**. No **7**. class exercises

8. a. It's not 51. b. Find primes whose product is 504.

9. $\mathbb{N} \subseteq \mathbb{Z}$ **10**. $\{\ldots -3, -2, -1, 0\}$

11. a. T b. T c. T d. F

12. For the example, take $a = -3$ and $b = 7$. Now choose $0 \in \subseteq$. Note that $-3 < 0 < 7$, as required by the conjecture. As for the counterexample, you're on your own.

13. Yes **14**. No **15**. a, b, c, d, e, h, i, k, l, n

16. Division by zero is undefined.

17. a. $-\frac{2}{3} = \frac{-2}{3}$, a ratio of integers; or, it's equal to –0.6666. . ., a repeating decimal.

b. $-801 = \frac{-801}{1}$, a ratio of integers; or, it's the same as –801.0000. . ., a repeating decimal.

c. $0.83 = \frac{83}{100}$, a ratio of integers; or, it's a repeating decimal 0.83000. . .

d. 1.125 is a repeating decimal (by attaching zeros); or, it can be written as $1\frac{125}{1000} = 1\frac{1}{8} = \frac{9}{8}$, which is a ratio of integers.

e. $\frac{7}{16}$ is a ratio of integers, so it's rational. Also, it can be written as 0.4376, which is a repeating decimal, another indication that it's rational.

18. Let

$$n = 0.737373\ldots$$

Multiply each side of the equation by 100:

$$100n = 73.737373\ldots$$

Subtract the first equation from the second equation:

$$\begin{aligned} 100n &= 73.737373\ldots \\ -\quad n &= 0.737373\ldots \\ \hline 99n &= 73 \end{aligned}$$

$n = \frac{73}{99}$ Since n stood for the original infinite decimal, and since n has been shown to equal 73/99, it follows that the fraction form of the decimal is $\frac{73}{99}$.

19. Similar to the previous problem, except multiply by 10.

20. Every integer $z \in \mathbb{Z}$ can be written as the fraction $\frac{z}{1} \in \mathbb{Q}$.

21. a. $\mathbb{Q}$ b. $\mathbb{N}$ **22**. a. $\mathbb{Q}$ b. $\mathbb{Z}$ **23**. a. No b. Yes

24. Convert the fraction to a repeating decimal.

25. If π were truly and exactly $\frac{22}{7}$, then π would be rational. The trouble is, π is only approximately equal to that fraction.

26. F **27**. T **28**. T **29**. T **30**. T **31**. T

32. What number lies halfway between the two given numbers?

33. Yes **34**. No

35. $\frac{731}{990}$

36. 0 and –3 are examples of numbers that are in $\mathbb{Z}$ but not in $\mathbb{N}$.

37. If x is even then $x = 2k$, in which case
$x^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is even (since it's twice something)
Therefore, x^2 is even.

38. a. $\frac{m}{n}$; $\mathbb{Z}$; $n \neq 0$ b. $\mathbb{Q}$ is the set of all repeating decimals.

39. $2.75 = 2\frac{3}{4} = \frac{11}{4}$, which is the ratio of two integers.

40. Let $n = 0.232323\ldots$ Multiplying both sides by 100 gives
$100n = 23.2323\ldots$ Subtracting the equations gives
$99n = 23$, from whence we deduce that $n = \frac{23}{99}$, which is rational.

41. False; for example, $5 \in \mathbb{N}$, but $\sqrt{5} \notin \mathbb{Q}$.

42. True; Basically, when you subtract two fractions, you get a fraction. Specifically, the difference of the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ad-bc}{bd}$.

43. True; The midpoint (average) works well, so define $q = \frac{q_1 + q_2}{2}$, and we have a rational number which is (halfway) between q_1 and q_2.

44. The solution of the equation $4x + 20 = 16$ is $x = -1$, which is in $\mathbb{Z}$ but not in $\mathbb{N}$.

45. Consider the equation $3x = 7$. Its solution is 7/3, which is rational but not an integer.

46. $\frac{3}{4} \in \mathbb{Q}$, but it's not in $\mathbb{Z}$.

47. a. If x is odd then $x = 2k + 1$, in which case

$x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd (because it's 1 more than twice something). Therefore, x^2 is odd.

b. Converse: If x^2 is odd, then x is odd. It's true.

c. Contrapositive: If x^2 is even, then x is even. It's true.

48. a. $\mathbb{Z}$; n b. $\mathbb{Q}$

49. $0.111 = \frac{111}{1000}$, which is the ratio of two integers.

50. Let $n = 9.555\ldots$ Then $10n = 95.555\ldots$ Subtracting gives $9n = 86$, and so $n = \frac{86}{9}$.

51. True; For example if $n = 81$, then $\sqrt{n} = \sqrt{81} = 9 \in \mathbb{Q}$.
Of course, the statement's not true for every natural number.

52. True; The product of the rational numbers $\frac{p}{q}$ and $\frac{r}{s}$ is $\frac{pr}{qs}$, which is rational.

53. False; this is a trick question. The statement does not specify that the two rational numbers must be different from each other. So if we take the two rationals to be the same, say $q_1 = 9$ and $q_2 = 9$, then it is clearly not the case that there's a rational numbers between them. [In fact, there's no number whatsoever that could be between 9 and itself.]

54. This is impossible, since $\mathbb{N} \subseteq \mathbb{Q}$, so how could something be in $\mathbb{N}$ but not in $\mathbb{Q}$?

55. Another impossibility, since $\mathbb{Z} \subseteq \mathbb{Q}$.

56. a. F b. T c. T d. F e. F f. T g. T h. F i. T
j. F k. T l. F m. T n. T o. F p. T q. T r. T
s. F t. F u. T

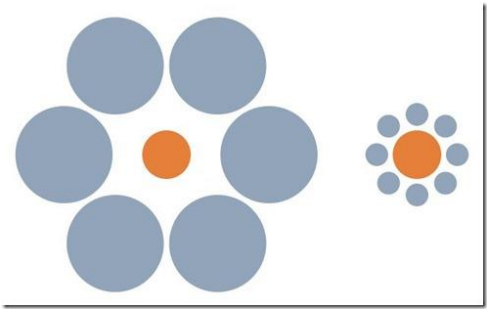
"Education, then, beyond all other devices of human origin, is the great equalizer of the conditions of men – the balance-wheel of the social machinery."

Horace Mann

RATIOS AND PROPORTIONS

There are two major ways to compare two things. If Joseph is 40 years old and Ellie is 10 years old, we can say either:

1) Joseph is 30 years *older* than Ellie, or
2) Joseph is 4 *times as old* as Ellie.



The first way is found by **subtacting**: $40 - 10 = \mathbf{30}$.

The second way is found by **dividing**: $\frac{40}{10} = \mathbf{4}$

It's this second way, using division, that is very powerful in explaining many of the numerical relationships we see in the world. When we compare two things by *division*, we're creating a ***ratio***.

❑ *PROPORTIONS*

Since a ratio is calculated by division, a ratio is basically a fraction. A ***proportion***, however, is a statement that <u>two ratios are equal</u>. An example of a proportion is

$$\frac{16}{12} = \frac{8}{6}$$ Each fraction is a ratio.

These ratios are equal because both fractions could be reduced to $\frac{4}{3}$, or if you prefer, they both equal 1.333....

❑ *SOLVING SIMPLE PROPORTIONS*

Example 1: Solve the proportion; that is, solve for x:

$$\frac{5}{13} = \frac{x}{1287}$$

We can use basic algebra to isolate the x by multiplying each side of the equation by 1287. It might be useful to write 1287 as a fraction with denominator 1:

$$\frac{5}{13}\left[\frac{1287}{1}\right] = \frac{x}{\cancel{1287}}\left[\frac{\cancel{1287}}{1}\right]$$

This simplifies to

$$\frac{6435}{13} = x\text{, or } \boldsymbol{x = 495}$$

Example 2: To solve the proportion

$$\frac{18}{n} = \frac{30}{17}$$

we notice that the variable n is on the bottom of the first ratio — we have to get it to the top so that we can solve for it. So first we'll multiply each side of the equation by n:

$$\frac{18}{n}[n] = \frac{30}{17}[n]$$

As before, we'll write each of the n's that we've put into the problem as a fraction with denominator 1. Thus, we can write

$$\frac{18}{\cancel{n}}\left[\frac{\cancel{n}}{1}\right] = \frac{30}{17}\left[\frac{n}{1}\right]$$

which simplifies to

$$18 = \frac{30n}{17}$$

Now multiply each side of the equation by 17 (or $\frac{17}{1}$ if you'd prefer):

$$18\left[17\right] = \frac{30n}{17}\left[17\right],$$

which (after flipping around the equal sign) leads to the equation

$$30n = 306$$

Now divide each side by 30:

$$\frac{30n}{30} = \frac{306}{30}$$

and we have our solution: ***n* = 10.2**

Homework

1. Which of the following statements are true proportions? [Try reducing fractions or converting fractions to decimals.]

   a. $\frac{60}{120} = \frac{1}{2}$ b. $\frac{37}{111} = \frac{10}{30}$ c. $\frac{4}{5} = \frac{5}{6}$

   d. $\frac{1}{7} = \frac{2}{13}$ e. $\frac{336}{84} = \frac{68}{17}$ f. $\frac{14}{6} = \frac{7}{2}$

2. Solve each proportion (leave answers rounded to 2 digits):

   a. $\frac{x}{7} = \frac{2}{9}$ b. $\frac{14}{a} = \frac{1}{20}$ c. $\frac{2}{7.2} = \frac{b}{5}$

   d. $\frac{100}{9.1} = \frac{2.3}{u}$ e. $\frac{c}{2.3} = \frac{1}{1.1}$ f. $\frac{14}{y} = \frac{14}{37}$

3. Solve each proportion (leave answers rounded to 2 digits):

a. $\frac{a}{13} = \frac{14}{156}$

b. $\frac{23}{b} = \frac{1}{14}$

c. $\frac{3}{4} = \frac{n}{128}$

d. $\frac{9}{2} = \frac{7.8}{z}$

e. $\frac{13}{u} = \frac{23}{10}$

f. $\frac{17}{15} = \frac{0.23}{k}$

g. $\frac{x}{5} = -7$

h. $\frac{17}{n} = -\frac{4}{5}$

i. $\frac{a}{9} = \frac{9}{a}$ [2 answers]

j. $\frac{x}{3} = \frac{1}{7}$

k. $\frac{2.1}{y} = \frac{5}{7}$

l. $\frac{8}{9} = \frac{w}{0.8}$

m. $\frac{2.1}{5} = \frac{9.3}{z}$

n. $\frac{5.6}{a} = \frac{2.33}{0.04}$

o. $\frac{x}{2} = \frac{18}{x}$ [2 answers]

❑ *SOLVING MORE PROPORTIONS*

I'm now going to show you a "shortcut" to solving proportions. Take the proportion:

$$\frac{a}{b} = \frac{c}{d}$$

Multiply both sides of the equation by the LCD, which is bd:

$$\frac{a}{b}\left[bd\right] = \frac{c}{d}\left[bd\right]$$

Now a little canceling,

$$\frac{a}{\not{b}}\left[\not{b}d\right] = \frac{c}{\not{d}}\left[b\not{d}\right]$$

which simplifies to

$$ad = bc$$

In short, we can take the proportion

$$\frac{a}{b} = \frac{c}{d},$$

"cross-multiply," and end up with the equivalent equation

$$ad = bc$$

<u>Example 3:</u> **Solve for x:** $\dfrac{3x-9}{5} = \dfrac{2x+7}{4}$

This proportion looks more complicated than the previous ones, but you know what? We can use our new "cross-canceling" shortcut to solve it. Let's begin by writing the original problem:

$\dfrac{3x-9}{5} = \dfrac{2x+7}{4}$ (the original equation)

$4(3x-9) = 5(2x+7)$ (cross-multiplying)

$12x - 36 = 10x + 35$ (distributing)

$2x - 36 = 35$ (subtract $10x$ from each side)

$2x = 71$ (add 36 to each side)

$\boxed{x = \dfrac{71}{2}}$ (divide each side by 2)

Homework

4. Solve each proportion (leave your answer in fractional form):

a. $\frac{x-1}{5} = \frac{2x+3}{4}$

b. $\frac{n}{3} = \frac{5-n}{10}$

c. $\frac{3a+9}{10} = \frac{2a}{3}$

d. $\frac{3}{m-2} = \frac{5}{m+12}$

e. $\frac{-2}{u-7} = \frac{-6}{9-u}$

f. $\frac{7}{-3y+7} = \frac{-1}{2-3y}$

g. $\frac{w+5}{w-3} = \frac{7}{8}$

h. $\frac{2a-1}{3} = \frac{2-a}{5}$

i. $\frac{2}{3} = \frac{4c-1}{4c+1}$

j. $\frac{-5}{2x+1} = \frac{3}{2-3x}$

k. $\frac{2y+1}{3y-4} = 8$

l. $\frac{x+3}{x-5} = \frac{x+1}{x-1}$

Solutions

1. a, b, and e are true

2. a. $x = 1.56$ b. $a = 280$ c. $b = 1.39$ d. $u = 0.21$

e. $c = 2.09$ f. $y = 37$

3. a. $a = 1.17$ b. $b = 322$ c. $n = 96$ d. $z = 1.73$

e. $u = 5.65$ f. $k = 0.2$ g. $x = -35$ h. $n = -21.25$

i. $a = \pm 9$ j. $x = 0.43$ k. $y = 2.94$ l. $w = 0.71$

m. $z = 22.14$ n. $a = 0.1$ o. $x = \pm 6$

4. a. $x = -\frac{19}{6}$ b. $n = \frac{15}{13}$ c. $a = \frac{27}{11}$

d. $m = 23$ e. $u = \frac{15}{2}$ f. $y = \frac{7}{8}$

g. $w = -61$ h. $a = \frac{11}{13}$ i. $c = \frac{5}{4}$

j. $x = \frac{13}{9}$ k. $y = \frac{3}{2}$ l. $x = -\frac{1}{3}$

"Only the curious will learn and only the resolute will overcome the obstacles to learning. The *quest quotient* has always excited me more than the intelligence quotient."

Edmund Wilson

THE REAL NUMBERS AND THEIR PROPERTIES

You have seen many different kinds of numbers in your algebra studies:

7 $\quad 0$ $\quad -9$ $\quad 2.835$ $\quad -\frac{15}{4}$

$\frac{1}{3}$ $\quad \frac{23}{75}$ $\quad \pi$ $\quad -\sqrt{2}$ $\quad 2.7383838\ldots$

$12.09358573008628\ldots$ $\quad -7.03003000300003\ldots$



As different as all these numbers may seem from each other, they actually share one critical common characteristic: They can all be written as **decimal numbers**.

❑ *THE REAL NUMBERS*

Here are all the numbers in the Introduction written as decimals:

$7 = 7.0$	A finite decimal
$0 = 0.0$	A finite decimal
$-9 = -9.0$	A finite decimal
2.835	A finite decimal

Actually, all the finite decimals listed here can be viewed as infinite, repeating decimals. How? By realizing that each finite decimal could have an unending supply of zeros attached to their back end. Thus, any finite decimal is really a repeating decimal.

$-\frac{15}{4} = -3.75$	A finite decimal
$\frac{1}{3} = 0.33333\ldots$	An infinite, repeating decimal
$\frac{23}{75} = 0.2787878\ldots$	An infinite, repeating decimal (the 78)
$\pi = 3.14159265\ldots$	An infinite, non-repeating decimal
$-\sqrt{2} = -1.41421356\ldots$	An infinite, non-repeating decimal
$12.09358573008628\ldots$	An infinite, non-repeating decimal
$-7.03003000300003\ldots$	An infinite, non-repeating decimal

Note that the first seven decimals listed above repeat a block of digits forever (the ***rational*** numbers). In contrast, the last four don't repeat (and are called ***irrational*** numbers). Nevertheless, they are all decimals.

F.Y.I.

π is the ratio of the circumference of any circle to its diameter.

$\sqrt{2}$ is the length of the hypotenuse of a right triangle where each leg has length 1.

But what about a number like $\sqrt{-9}$? This number must be a number whose square is –9. Now, what number do we know which, when squared, would come out –9?

Does 3 work? No, since $3^2 = 9$.

Does –3 work? No, since $(-3)^2 = 9$, as well.

We conclude that no decimal in the world can represent the number $\sqrt{-9}$, or for that matter, the square root of any negative number.

To distinguish between the numbers that are decimals and numbers like $\sqrt{-9}$, which can never be written as a decimal, the term **real number** was given to the decimals, and the term **imaginary number** was given to numbers like $\sqrt{-9}$.

Hundreds of years ago, mathematicians thought it was obvious which numbers were real and which were imaginary. But this demonstrates a rather arrogant attitude. After all, to a beginning algebra student, a

real number like $\sqrt{2}$ (which is an infinite, non-repeating decimal) may not seem very "real" at all. Moreover, imaginary numbers, like $\sqrt{-1}$, seem very real to people in fields such as electronics and quantum physics, who use them every day. The bottom line is that the terms *real* and *imaginary* are entirely arbitrary — one person's reality is another's imagination. But we're stuck with these terms, so we might as well learn them.

The REAL NUMBERS, $\mathbb{R}$, can be divided into two disjoint subsets:

The Rationals – the repeating decimals (which includes the finite decimals).

The Irrationals – the non-repeating decimals.

In summary, we call any number that can be written as a decimal a ***real number***. The set of real numbers is often denoted by writing $\mathbb{R}$. As far as the first courses in Algebra are concerned, the only numbers you will come across that won't be real numbers are the square roots of negative numbers, (as well as fourth roots, sixth roots, etc.).

Homework

1. Classify each number as **real** or **imaginary**:

a. 123	b. -42	c. 0
d. 2.3	e. $\sqrt{-8}$	f. $\sqrt{144}$
g. $-\sqrt{81}$	h. $\sqrt{10}$	i. -23.78
j. $-\pi$	k. $\sqrt{3}$	l. $\sqrt{-121}$
m. 0.239057	n. 2.787878. . .	o. 3.092748526
p. 3.1428669. . .	q. $\sqrt{8765}$	r. $\sqrt{-0.25}$
s. $-\sqrt{-25}$	t. $\sqrt{-(-71)}$	u. 10^6

If it looks like a decimal, swims like a decimal, and quacks like a decimal, IT'S A *REAL NUMBER*!



❑ THE CLOSURE PROPERTIES

Now that we know what a real number is, we can analyze some of the properties of these real numbers. We are likely familiar with all of these properties; it's the official names of these properties that might cause us some grief.



When a collie mates with a collie, it's quite certain that the offspring will be a collie. The same idea holds in the real numbers. For example, when two real numbers are added, the sum is a real number. Some examples:

$17 + 14.56 = 31.56$, which is a real number.

Let's add the real numbers π and $\sqrt{2}$. The sum is $\pi + \sqrt{2}$, which cannot be simplified, but I guarantee that the sum is a real number. Here's a little proof: Recall that a real number is any number which can be written as a decimal, so

$$\pi + \sqrt{2} = 3.1415926\ldots + 1.4142135\ldots = 4.5558062\ldots$$

which is certainly a decimal, and therefore a real number. Hence $\pi + \sqrt{2}$ is a real number.

Because of this property — that adding real numbers always results in a real number — we say that the set of real numbers is ***closed under addition***. The set of real numbers is also closed under subtraction and multiplication. And with one critical exception, it's also closed under division. Do you know what that exception is?

❑ THE COMMUTATIVE PROPERTIES

Just as a governor can commute your death sentence to a life sentence by "switching" things around, we can switch, or reverse, the factors of 7×3 and write it as 3×7, and we've known since we were little tykes that they're the same product, 21.

More generally, the factors of the product xy can be commuted, or switched, and be written yx, without changing the answer. We say that multiplication is a ***commutative*** operation. Anytime we want to switch the order of two factors, we can simply do it.

Addition is also a commutative operation; after all, it's pretty clear that $x + y = y + x$ for any real numbers x and y.

Now it gets interesting: Is subtraction a commutative operation? That is, does $x - y = y - x$ for all values of x and y? Of course not; does $10 - 2 = 2 - 10$?

You decide whether division is a commutative operation.

In summary, the ***commutative properties*** state the following: For any real numbers x, y, and z:

$$x + y = y + x$$
$$xy = yx$$

❑ THE ASSOCIATIVE PROPERTIES

Consider the sum $5 + 3 + 2$. We've learned via the *Order of Operations* chapter that we can certainly work the problem left to right, which means we're going to add the 5 and 3 first; let's use parentheses around the $5 + 3$:

$(5 + 3) + 2 = 8 + 2 = \mathbf{10}$ Here we "associate" the 5 and 3.

Now let's see if we get the same answer if we add the 3 and 2 first:

$5 + (3 + 2) = 5 + 5 = \mathbf{10}$ Here we "associate" the 3 and 2.

We get the same answer. It appears that $(5 + 3) + 2 = 5 + (3 + 2)$. That is, we can "shift" the parentheses left-to-right or right-to-left.

The same property holds for multiplication, too. For example,

$(7 \times 5) \times 2 = 35 \times 2 = \mathbf{70}$

and, $7 \times (5 \times 2) = 7 \times 10 = \mathbf{70}$

In summary, the ***associative properties*** state the following: For any real numbers x, y, and z:

$$(x + y) + z = x + (y + z)$$
$$(xy)z = x(yz)$$

❑ THE DISTRIBUTIVE PROPERTY

This property is the cornerstone of algebra, and is utilized constantly throughout all branches of mathematics. It is the only property in this chapter which involves two different operations, multiplication and addition.

For any real numbers a, b, and c:

$$a(b + c) = ab + ac$$

❑ THE ADDITIVE IDENTITY

Would you agree that adding **0** to a real number results in that same number? In other words, $x + 0 = x$ for all real numbers x. Of course you would agree; you knew that when you were a little kid. Leave it to

the math geeks to come up with a fancy term for something so simple. We call the number zero the ***additive identity*** because when zero is **added** to a number, the sum is **identical** to that number. There is only one additive identity in the set of real numbers. In other words, zero is the only real number that acts like zero.

❑ THE MULTIPLICATIVE IDENTITY

More silliness — we know that any number times **1** is itself: $\boldsymbol{x \cdot 1 = x}$ for any real number x. This special number, 1, is given the name ***multiplicative identity*** (accent on the "plic"), because when a number is **multiplied** by 1, the product is **identical** to that number. And it should be clear that 1 is the only number that possesses this property.

❑ THE ADDITIVE INVERSE

Remember the term "opposite"? [The opposite of 7 is –7, the opposite of –99 is 99, and the opposite of 0 is 0.] The official term for "opposite" is ***additive inverse***, so now we can talk fancy and say things like, "The additive inverse of –23 is 23." Every real number has an additive inverse; in general, the additive inverse of n is $-n$. Also notice that when a number and its additive inverse are <u>added</u>, the <u>sum</u> is always 0, the additive identity. That is, for every real number n,

$$\boldsymbol{n + (-n) = 0}$$

❑ THE MULTIPLICATIVE INVERSE

Here's some more jargon: The phrase *multiplicative inverse* simply means "reciprocal." Thus, the multiplicative inverse of $\frac{3}{5}$ is $\frac{5}{3}$. In general, the multiplicative inverse of x is $\frac{1}{x}$, and the multiplicative

inverse of $\frac{a}{b}$ is $\frac{b}{a}$. Does every real number have a multiplicative inverse? Absolutely NOT. Consider zero; if zero had a multiplicative inverse, it would be $\frac{1}{0}$, which is undefined. Therefore, every real number except zero has a multiplicative inverse. Notice that when a number and its multiplicative inverse are multiplied, the product is always 1, the multiplicative identity. In other words, for any real number x (other than 0),

$$x\left(\frac{1}{x}\right) = 1$$

Homework

2. a. The set of real numbers, $\mathbb{R}$, is **closed** under division, with one exception. What is that exception?

   b. Explain what it means to say that the set of real numbers is closed under multiplication.

3. Which operations (addition, subtraction, multiplication, division) are **commutative** operations in $\mathbb{R}$? Give examples which demonstrate which operations are commutative and which are not.

4. Prove that subtraction and division are NOT **associative** operations in $\mathbb{R}$.

5. What's the official term for the number **0** in $\mathbb{R}$?

6. What's the official term for the number **1** in $\mathbb{R}$?

7. Find the **additive inverse** of each real number:

   a. 17 b. -3 c. 0 d. $-\pi$

e. $-\frac{7}{4}$ f. x g. $-n$

8. The sum of any real number and its additive inverse is ___.

9. Find the **multiplicative inverse** of each real number:

a. 22 b. –9 c. 0

d. $\frac{1}{\pi}$ e. $-\frac{\sqrt{3}}{\sqrt{2}}$ f. N

10. The product of any real number and its multiplicative inverse is ___.

11. a. T/F: Every real number has an opposite.

b. T/F: Every real number has a reciprocal.

MATCHING:

12. __	$a(b + c) = ab + ac$	a. commutative
13. __	$a + b = b + a$	b. associative
14. __	xy is a real number if x and y are.	c. distributive
15. __	$-a$	d. closure
16. __	1	e. additive identity
17. __	$\frac{1}{a}$	f. additive inverse of a
18. __	$x(yz) = (xy)z$	g. multiplicative identity
19. __	0	h. multiplicative inverse of a
20. __	$uw = wu$	
21. __	$(a + b) + c = a + (b + c)$	
22. __	$a - b$ is a real number if a and b are real numbers.	

23. ___ $x + (y + z) = x + (z + y)$

24. ___ $a^2b^3 = b^3a^2$

25. ___ If w is a real number, so is w^6.

26. ___ $\sqrt{2} - \sqrt{3} + 10\pi$ is a real number.

27. ___ $(ab)(cd) = a(bc)d$

Solutions

1.

a.	Real	b.	Real	c.	Real	d.	Real
e.	Imaginary	f.	Real	g.	Real	h.	Real
i.	Real	j.	Real	k.	Real	l.	Imaginary
m.	Real	n.	Real	o.	Real	p.	Real
q.	Real	r.	Imaginary	s.	Imaginary	t.	Real
u.	Real						

2. a. Dividing any number by 0 does <u>not</u> produce a real number.

b. It means that if x and y are any two real numbers, their product xy is also a real number. For example, since π and $\sqrt{7}$ are real numbers, so is their product $\pi\sqrt{7}$.

3. Addition and multiplication are commutative operations. For example,

$$1 + \sqrt{2} = \sqrt{2} + 1 \quad \text{and} \quad 100 \times \pi = \pi \times 100$$

Subtraction and division are NOT commutative operations. For instance,

$$34 - 10 \neq 10 - 34 \quad \text{and} \quad \frac{10}{2} \neq \frac{2}{10}$$ (Confirm these statements!)

4. If subtraction were associative, the following would have to be true:

$$10 - (7 - 4) = (10 - 7) - 4$$
$$\Rightarrow \quad 10 - 3 = 3 - 4$$
$$\Rightarrow \quad 7 = -1, \text{ which, of course, it isn't.}$$

If division were associative, the following would hold:

$$20 \div (10 \div 2) = (20 \div 10) \div 2$$

The problem is, the left side equals 4, while the right side equals 1.

5. Additive identity

6. Multiplicative identity

7. a. -17 b. 3 c. 0 d. π e. $\frac{7}{4}$ f. $-x$ g. n

8. 0

9. a. $\frac{1}{22}$ b. $-\frac{1}{9}$ c. Does not exist d. π e. $-\frac{\sqrt{2}}{\sqrt{3}}$ f. $\frac{1}{N}$

10. 1

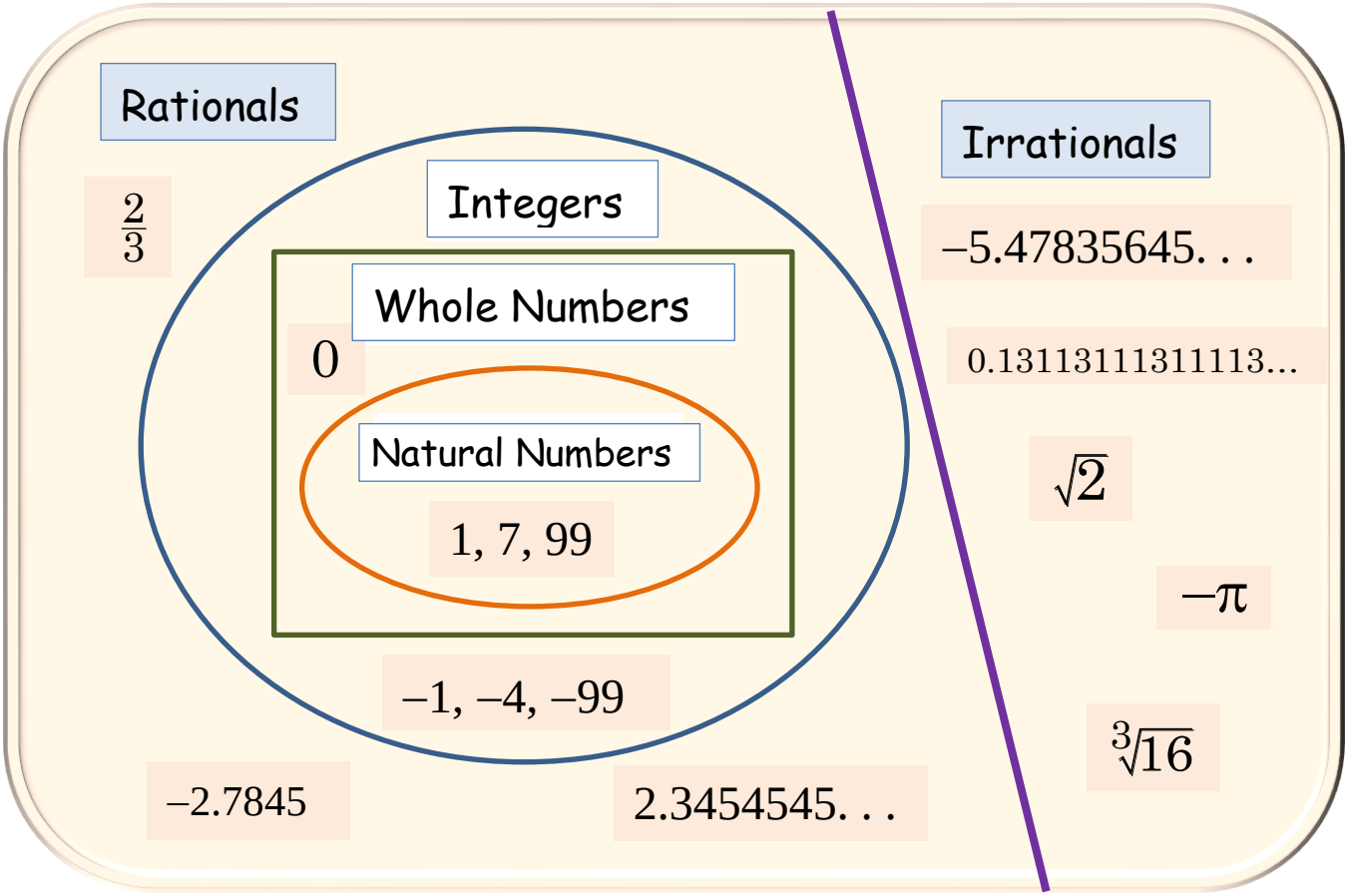
11. a. True b. False

12. c. **13**. a. **14**. d. **15**. f. **16**. g. **17**. h.

18. b. **19**. e. **20**. a. **21**. b. **22**. d. **23**. a.

24. a. **25**. d. **26**. d. **27**. b.

The Set of Real Numbers



– Every natural number is a whole number.

– Every whole number is an integer.

Therefore, every natural number is an integer.

– The only whole number that is not a natural number is 0.

– Every integer is rational.

Therefore, every natural number is rational,

and every whole number is rational.

– The rationals can be described as all <u>repeating</u> decimals.

– The irrationals can be described as all non-repeating decimals.

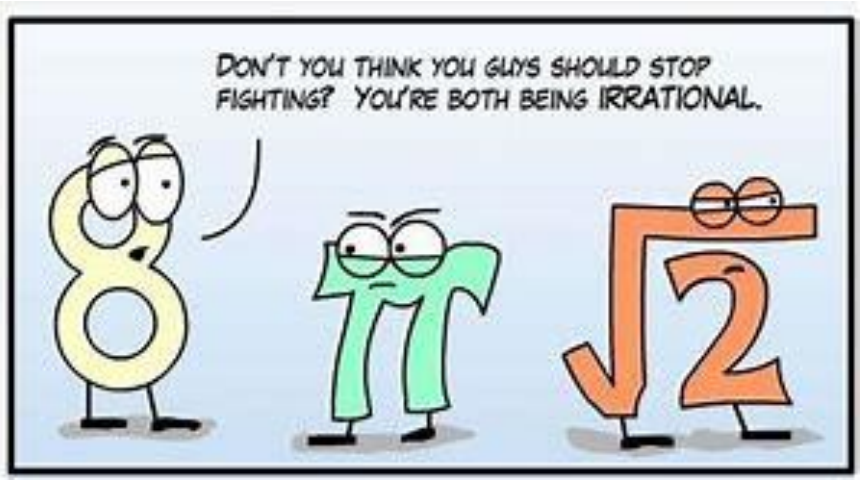
– The rationals and irrationals are *disjoint*; they have nothing in common:

Rationals ∩ Irrationals = ∅ [the null, or empty, set]

– The rationals and irrationals make up the real numbers.

Rationals ∪ Irrationals = Real Numbers

See Sets for more info on ∩, ∪, and ∅.

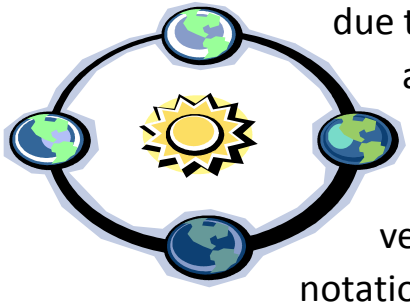


"It is harder
to crack a prejudice
than an atom."

– A. Einstein

SCIENTIFIC NOTATION

You might read that the distance from the Earth to the Sun is approximately **9.3×10^7 miles**. [The distance varies somewhat throughout the year due to the Earth's elliptical (oval) orbit around the sun.] This way of writing a number is called ***scientific notation***, and is used to write very large and very small numbers in a compact notation that's easier to understand than long strings of digits.



❑ *THE DISTANCE FROM THE EARTH TO THE SUN*

To convert 9.3×10^7 (distance from the Earth to the Sun) into a regular number, we need only our knowledge of exponents:

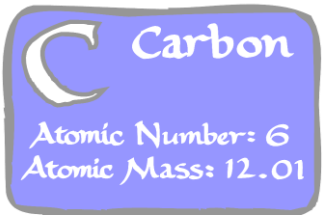
$$\begin{aligned} & \mathbf{9.3 \times 10^7} \\ = \quad & 9.3 \times (10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10) \\ = \quad & 9.3 \times 10{,}000{,}000 \\ = \quad & \mathbf{93{,}000{,}000} \end{aligned}$$

Thus, 9.3×10^7 miles is simply 93 million miles. Why not just write "93 million miles," or 93,000,000 miles? Because with other numbers it might become quite cumbersome.

If we were to expand, for instance, the number 6.02×10^{23}, we'd get "602" followed by 21 zeros:

$$6.02 \times 10^{23} = \mathbf{602{,}000{,}000{,}000{,}000{,}000{,}000{,}000}$$

Not only would it be horrendous to write it this way, it would overflow a calculator (or convert it right back to scientific notation!), and most people wouldn't even know the name for a number that big — thus I hope you see the need for scientific notation.



6.02×10^{23} is the number of **atoms** in 12 grams of Carbon-12.

❑ *Multiplying and Dividing by Powers of 10*

Before we begin the actual conversions, let's review how we multiply and divide by powers of 10.

The number $\mathbf{7.06 \times 10^{6}}$ **miles** is calculated by moving the decimal point 6 places to the right, resulting in **7,060,000 miles.**

To multiply by a negative power of 10, move the decimal point to the left. For example, $\mathbf{6.09 \times 10^{-5}}$ grams = **0.0000609 grams.**

Can you see how we know whether to move the decimal point to the right or to the left?

❑ *The Definition of Scientific Notation*

A number is in ***scientific notation*** if it's written in the form

$$M \times 10^{C}$$

where M is a real number such that $1 \le M < 10$, and C is an integer (whole numbers and their negatives).

Also notice that the M is being multiplied by the power of **10**. In the Introduction, we stated that the distance from the

$$M \times 10^{C}$$

M is any number between 1 and 10 (including the 1, but excluding the 10), and C must come from the set of integers:

$\{ \ldots -3, -2, -1, 0, 1, 2, 3, \ldots \}$

Earth to the Sun was 9.3×10^7 miles, a number written in *scientific notation*.

The following <u>are</u> in scientific notation, and the reason is that in each case:

1) the operation is multiplication,
2) the front number is between 1 and 10 (possibly 1, but not 10),
3) the base is 10, and
4) the exponent is an integer.

4.56×10^7 9.7023×10^{-9} 8×10^0 1×10^{34}

The following, however, are <u>not</u> in scientific notation:

23.9×10^{17}	The front number is too big.
0.72×10^{-67}	The front number is too small.
10×10^{44}	The front number is too big.
7.023×2^{34}	The base (the 2) is not 10, as it must be.
$3.4 \times 10^{1/2}$	The exponent is not an integer.
$1.13 + 10^{19}$	It's an addition problem, not multiplication.

Homework

1. Is the given number in scientific notation?

a. 1.0007×10^{77} b. 9.999×10^{-5} c. 13×10^8

d. 0.77×10^{23} e. $2.34 + 10^7$ f. $4.5 \times 10^{1.3}$

g. 3.44×2^9 h. 10.1×10^{-9} i. 9.99×10^1

j. 2.66×10^0 k. $4.34^2 \times 10^{-23}$ l. $2.39 \times 10^{-1/3}$

❑ *CONVERTING WITH SCIENTIFIC NOTATION*

EXAMPLE 1: **Convert each scientific notation number into a regular number (sometimes called standard notation):**

A. $\mathbf{2.73 \times 10^4}$

Move the decimal point 4 places to the right ⇒ **27,300**

B. $\mathbf{8.9 \times 10^1}$

Move the point 1 place to the right ⇒ **89**

C. $\mathbf{1.33 \times 10^0}$

Move the point 0 places to the right ⇒ **1.33**

[Or, since $10^0 = 1$, the answer must be 1.33.]

D. $\mathbf{4.1 \times 10^{-3}}$

Move the point 3 places to the left ⇒ **0.0041**

E. $\mathbf{7.827 \times 10^{-8}}$

Move the point 8 places to the left ⇒ **0.00000007827**

Homework

2. Convert each scientific notation number into a regular number:

a. 8.73×10^2 b. 2.09×10^5 c. 8.0×10^1 d. 2.36×10^{10}

e. 5.99×10^{-2} f. 6.01×10^{-4} g. 1.0×10^{-6} h. 8.35×10^0

EXAMPLE 2: **Convert each regular number into scientific notation:**

A. **540.92**

We need the decimal part of the scientific notation number to be between 1 and 10, so we move the decimal point 2 places to the left, giving 5.4092, which is equivalent to dividing by 100. We compensate for this dividing by multiplying the new decimal by 100, or 10^2. We end up with

$$\mathbf{5.4092 \times 10^2}$$

B. **0.000056**

To get the decimal in the right spot for scientific notation, we need to move it 5 places to the right, which is essentially multiplying the number by 100,000; this produces 5.6. We balance this maneuver by "dividing" by 100,000, which is equivalent by multiplying by 10^{-5}. Thus, the given decimal is equal to

$$\mathbf{5.6 \times 10^{-5}}$$

Homework

3. Convert each regular number into scientific notation:

a. 9,200,000	b. 23,400,000,000	c. 34.56
d. 0.0000123	e. 76,996.7	f. 0.0000000077

EXAMPLE 3: **Convert each number into scientific notation:**

Note that the following two problems may look vaguely like scientific notation, but they're not.

A. $\mathbf{238.2 \times 10^{7}}$

First, be sure you understand why the number is not already in scientific notation. To convert the decimal to a number between 1 and 10 (but not 10), shift the decimal point 2 places to the left (giving 2.382), which is equivalent to dividing by 100. Now balance this by multiplying the power of 10 by 100, which gives $10^{7} \times 10^{2} = 10^{9}$. Final result:

$$\mathbf{2.382 \times 10^{9}}$$

B. $\mathbf{0.0076 \times 10^{-5}}$

The decimal is not between 1 and 10, so we shift the decimal point 3 places to the right to make it between 1 and 10, so now it's 7.6. That shift is the same as multiplying by 1000, so we compensate by dividing 10^{-5} by 1000, which is accomplished by subtracting 3 from the exponent, giving 10^{-8}. And we're done:

$$\mathbf{7.6 \times 10^{-8}}$$

Homework

4. Convert each number into scientific notation:

a. 23.9×10^{5} b. 127.05×10^{14} c. 0.19×10^{-8}

d. 0.015×10^{-10} e. 832.44×10^{-7} f. 0.004×10^{-5}

❑ *Multiplying and Dividing with Scientific Notation*



Important: Notice that in all of the following calculations, you are not allowed to use the EE button on your calculator. Your calculator is to be used only for simple multiplication and division of decimals.

EXAMPLE 4: **Calculate the product: $(9.2 \times 10^4)(1.7 \times 10^7)$**

Solution: Notice that the problem contains nothing but multiplications; we can therefore rearrange the four factors in any manner we please (associative and commutative properties).

$$
\begin{aligned}
& (9.2 \times 10^4)(1.7 \times 10^7) & \\
= \quad & 9.2 \times 1.7 \times 10^4 \times 10^7 & \text{(rearrangement)} \\
= \quad & (9.2 \times 1.7) \times (10^4 \times 10^7) & \text{(regroup)} \\
= \quad & 15.64 \times 10^{11} & \text{(add the exponents)} \\
= \quad & \mathbf{1.564 \times 10^{12}} & \text{(convert to scientific notation)}
\end{aligned}
$$

EXAMPLE 5: **Find the quotient: $\dfrac{3.91 \times 10^{15}}{7.2 \times 10^{23}}$**

Solution: This one's a little tricky. Remember how we multiply fractions: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. Reading this rule the other way, we can take something like $\frac{xw}{yz}$ and split it into $\frac{x}{y} \times \frac{w}{z}$. Now for the problem at hand:

$$\frac{3.91 \times 10^{15}}{7.2 \times 10^{23}} = \frac{3.91}{7.2} \times \frac{10^{15}}{10^{23}} = 0.543 \times 10^{-8} = \mathbf{5.43 \times 10^{-9}}$$

Homework

5. Calculate each product:

   a. $(1.1 \times 10^{6})\,(2.42 \times 10^{4})$

   b. $(2.0 \times 10^{5})\,(3.7 \times 10^{-8})$

   c. $(4.1 \times 10^{6})\,(2.1 \times 10^{-4})$

   d. $(1.9 \times 10^{-23})\,(2.3 \times 10^{-7})$

6. Calculate each quotient:

   a. $\dfrac{4.42\times10^{15}}{1.7\times10^{13}}$

   b. $\dfrac{8.0\times10^{9}}{2.0\times10^{20}}$

   c. $\dfrac{9.2\times10^{-9}}{4.6\times10^{4}}$

   d. $\dfrac{4.095\times10^{15}}{1.5\times10^{-8}}$

❑ *Adding and Subtracting with Scientific Notation*

❑ *Applications*

EXAMPLE 6: **Light (and all electromagnetic radiation) travels in a vacuum at a speed of 3.0×10^{8} m/s (meters /second). Find the distance (in meters) that light will travel in 7.32×10^{7} seconds.**



Solution: Since distance is the product of rate (speed) and time, we start with the formula $d = rt$, and then substitute the given numbers in the correct places:

$$d = rt$$

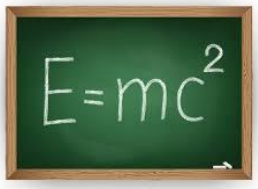
$$\Rightarrow \quad d = (3.0 \times 10^8)(7.32 \times 10^7)$$

$$\Rightarrow \quad d = (3.0 \times 7.32)(10^8 \times 10^7) \quad \text{(rearranging the factors)}$$

$$\Rightarrow \quad d = 12.96 \times 10^{15}$$

$$\Rightarrow \quad \boxed{d = 1.296 \times 10^{16} \text{ m}}$$

EXAMPLE 7: **Einstein's formula, $E = mc^2$, describes the equivalence of mass and energy. How much energy (in joules) is created when 1.45×10^7 kilograms of mass are converted to energy?**



Solution: Start with the mass/energy formula, and plug in the numbers:

$$E = mc^2$$

$$\Rightarrow \quad E = (1.45 \times 10^7)(3.0 \times 10^8)^2$$

$$\Rightarrow \quad E = (1.45 \times 10^7)(9.0 \times 10^{16}) \quad [\text{from the rule: } (ab)^2 = a^2 b^2]$$

$$\Rightarrow \quad E = (1.45 \times 9.0)(10^7 \times 10^{16}) \quad \text{(rearranging the factors)}$$

$$\Rightarrow \quad \boxed{E = 1.305 \times 10^{23} \text{J}}$$

EXAMPLE 8: **Carbon 12 (C-12) is the isotope of carbon containing 6 neutrons in its nucleus. It is known that 12 g of C-12 contains 6.02×10^{23} atoms. How many atoms are there in 24 g of C-12?**



Solution: Since 12 grams contains 6.02×10^{23} atoms, and since 24 is twice 12, it follows that 24 grams must contain twice as many atoms:

$$\text{Number of atoms} = 2\left(6.02\times10^{23}\right)$$

$$= 12.04\times10^{23}$$

$$= \boxed{1.204\times10^{24}\text{ atoms}}$$

EXAMPLE 9: **A spaceship travels at a rate of 23,500 mi/hr. How long (in hours) would it take to travel 150 million miles? Use *scientific notation* to perform all the calculations.**



Solution: Since the problem refers to speed (rate) and distance, and asks for time, the formula $d = rt$ seems relevant. Solving this formula for t gives

$$t = \frac{d}{r}$$

Plugging in the given values gives

$$t = \frac{150\,\text{million}}{23{,}500}$$

$$= \frac{150{,}000{,}000}{23{,}500}$$

$$= \frac{1.5\times10^{8}}{2.35\times10^{4}}$$

$$= 0.638\times10^{4}$$

$$= 6.38\times10^{3}$$

It will take **6.38×10^{3} hours** to travel 150 million miles.

Homework

7. How far (in meters, m) can light travel in the given amount of time?

   a. 4.05×10^{9} seconds b. 7.3×10^{-5} seconds

   c. 1.09×10^{11} seconds d. 1 hour

8. For each given mass (in kilograms), use scientific notation to calculate the amount of energy created (in joules, J) when the mass is converted to energy:

   a. 2.73×10^{14} kg b. 5.09×10^{-12} kg c. 500 kg

9. How many atoms of C-12 are there in the given mass?

   a. 36 g of C-12 b. 120 g of C-12 c. 6 g of C-12

10. Find the travel time given the distance and the speed:

    a. d = 800 million miles r = 18,500 mi/hr

    b. d = 1.7 billion miles r = 29,200 mi/hr

    c. d = 8 trillion miles r = 95,000 mi/hr

❑ LINKS

Math is Fun

Solutions

1. a. Yes b. Yes c. No d. No e. No f. No
g. No h. No i. Yes j. Yes k. No l. No

2. a. 873 b. 209,000 c. 80 d. 23,600,000,000
e. 0.0599 f. 0.000601 g. 0.000001 h. 8.35

3. a. 9.2×10^{6} b. 2.34×10^{10} c. 3.456×10^{1}
d. 1.23×10^{-5} e. 7.69967×10^{4} f. 7.7×10^{-9}

4. a. 2.39×10^{6} b. 1.2705×10^{16} c. 1.9×10^{-9}
d. 1.5×10^{-12} e. 8.3244×10^{-5} f. 4.0×10^{-8}

5. a. 2.662×10^{10} b. 7.4×10^{-3}
c. 8.61×10^{2} d. 4.37×10^{-30}

6. a. 2.6×10^{2} b. 4×10^{-11}
c. 2×10^{-13} d. 2.73×10^{23}

7. a. 1.215×10^{18} m b. 2.19×10^{4} m
c. 3.27×10^{19} m d. 1.08×10^{12} m

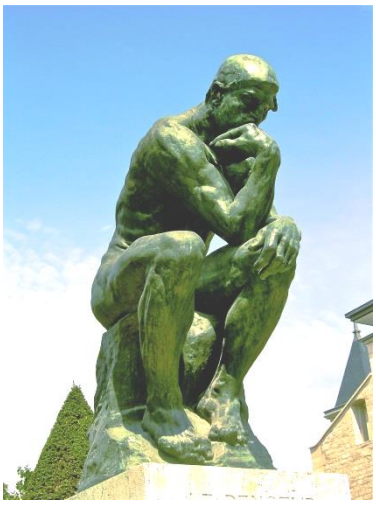
8. a. 2.457×10^{31} J b. 4.581×10^{5} J c. 4.5×10^{19} J

9. a. 1.806×10^{24} atoms b. 6.02×10^{24} atoms c. 3.01×10^{23} atoms

10. a. 4.32×10^{4} hrs b. 5.82×10^{4} hrs c. 8.42×10^{7} hrs

"What sculpture is
to a block of marble,
education is
to the human soul."

Joseph Addison



ADDING AND SUBTRACTING SIGNED NUMBERS

❑ *LET'S PLAY JEOPARDY!*



Each answer in Jeopardy is worth a specific number of dollars. Your score goes up or down depending on whether you get the question right or wrong. It is also possible for scores to go below zero — into the negative numbers.

In our first example, you should use your intuition, if possible. But until you're comfortable with that method, you might try picturing a number line: A correct response (that is, adding) moves you to the right on the number line, while an incorrect response (that is, subtracting) moves you to the left on the number line.

EXAMPLE 1:

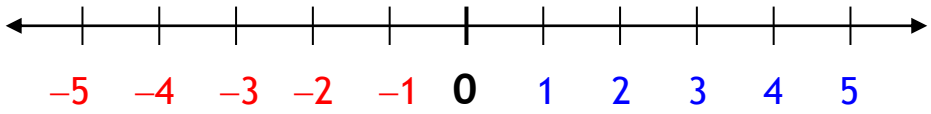
$ Current Amount $	$ Value of Question $	Right or Wrong	Calculation	$ New Amount $
20	10	right	20 + 10	30
15	5	wrong	15 – 5	10
12	15	wrong	12 – 15	–3
–10	5	right	–10 + 5	–5
–20	30	right	–20 + 30	10
–8	5	wrong	–8 – 5	–13
25	25	wrong	25 – 25	0
–200	200	right	–200 + 200	0
0	400	right	0 + 400	400
0	500	wrong	0 – 500	–500

Homework

1. Fill in the blanks:

$ Current Amount $	$ Value of Question $	Right or Wrong	Calculation	$ New Amount $
25	10	right		
12	5	wrong		
10	15	wrong		
–15	5	right		
–25	30	right		
–8	7	wrong		
29	29	wrong		
–300	300	right		
0	800	right		
0	300	wrong		

❑ *THE NUMBER LINE APPROACH TO ADDING AND SUBTRACTING SIGNED NUMBERS*



EXAMPLE 2:

A. $3 + 2 = 5$

Start at 3 on the number line; move 2 units to the right, and you end up at 5.

B. $-4 + 5 = 1$

Start at –4; move 5 units to the right, and you end up at 1.

C. $-5 + 3 = -2$

Start at –5; move 3 units to the right, and you end up at –2.

D. $5 - 3 = 2$

Start at 5; move 3 units to the left, and you end up at 2.

E. $3 - 7 = -4$

Start at 3; move 7 units to the left, and you end up at –4.

F. $-3 - 2 = -5$

Start at –3; move 2 units to the left, and you end up at –5.

All of these results could have been obtained by thinking about Jeopardy. You may pick any method you like to add and subtract positive numbers, including anything you learned in a previous course; just be sure you can do it really well. Now for some problems where we add and subtract negative numbers.

EXAMPLE 3:

A. $5 + (-2) = 3$

Since we know that adding a positive number moves you to the right on the number line, it's reasonable to suppose that adding a negative number moves you to the left on the number line.

Start at 5 on the number line; move 2 units to the left, and you end up at 3.

B. $3 + (-6) = -3$

Start at 3; move 6 units to the left, and you end up at –3.

C. $-1 + (-4) = -5$

Start at –1; move 4 units to the left, and you wind up at –5.

Now for the strangest example: $3 - (-2)$. Here's the logic: Since we've learned that subtracting a positive number moves you to the left on the number line, we figure that subtracting a *negative* number moves you to the right on the number line.

D. $3 - (-2) = 5$

Start at 3; move 2 units to the right, and you end up at 5.

E. $-7 - (-5) = -2$

Start at –7; move 5 units to the right, and you end up at –2.

Let's look at the four conclusions we've reached regarding the adding and subtracting of positive and negative numbers. Assuming b is a positive number:

(i)	$a + b$	Move b units to the right.
(ii)	$a - b$	Move b units to the left.
(iii)	$a + (-b)$	Move b units to the left.
(iv)	$a - (-b)$	Move b units to the right.

In other words, if you see either $a + b$ or $a - (-b)$, start at a and move b units to the right. [Subtracting a negative is the same as adding.]

If you see $a - b$ or $a + (-b)$, start at a and move b units to the left. [Adding a negative is the same as subtracting.]

Homework

2. Perform each addition or subtraction problem:

a. $17 + 3$	b. $16 - 5$	c. $7 - 8$	d. $12 - 12$
e. $23 - 30$	f. $-2 + 10$	g. $-3 + 2$	h. $-12 - 3$
i. $8 - 100$	j. $-30 + 40$	k. $-20 + 5$	l. $-80 - 20$
m. $0 + 32$	n. $0 - 32$	o. $88 + 0$	p. $34 - 0$
q. $-10 - 15$	r. $-23 + 80$	s. $-7 + 2$	t. $-18 - 18$
u. $9 - 12$	v. $-5 + 9$	w. $-10 - 3$	x. $-10 + 2$
y. $-30 - 4$	z. $3 - 100$		

3. Evaluate (simplify) each expression:

a. $17 + 3$	b. $-8 + 3$	c. $-8 + (-1)$	d. $7 + (-3)$
e. $(-3) + (-10)$	f. $5 + (-13)$	g. $-9 + 0$	h. $-7 + 12$
i. $-20 + 10$	j. $-30 + (-2)$	k. $0 + (-3)$	l. $-10 + 10$
m. $-12 + 7$	n. $-3 + (-20)$	o. $9 + (-9)$	p. $1 + (-12)$

4. Evaluate (simplify) each expression:

a. $7 - 5$	b. $9 - 10$	c. $6 - 6$	d. $12 - 25$
e. $-5 - 4$	f. $-10 - 10$	g. $13 - 20$	h. $-1 - 99$
i. $12 - (-3)$	j. $-1 - (-4)$	k. $-14 - (-3)$	l. $-2 - (-2)$
m. $-3 - 17$	n. $20 - (-21)$	o. $-5 - (-13)$	p. $-9 - (-7)$

NOTE: You may have already learned that addition and multiplication are **commutative** operations, since $a + b = b + a$ and $ab = ba$ for any values of a and b. Is subtraction commutative? That is, does $a - b = b - a$ for all choices of a and b? Of course not. After all, $7 - 2 = 5$, while $2 - 7 = -5$. Thus, "commuting" the 7 and the 2 around the subtraction sign results in different answers. Thus, subtraction is not a commutative operation (nor is division).

EXAMPLE 4:

A. $7 + (-3) + 8$

$= 4 + 8$ (start at 7; move 3 to the left)

$= \mathbf{12}$ (now move 8 to the right)

B. $-13 - 3 + (-5)$

$= -16 + (-5)$ (start at –13; move 3 to the left)

$= \mathbf{-21}$ (move 5 to the left)

C. $-8 - (-1) - (-3)$

$= -7 - (-3)$ (start at –8; move 1 to the right

$= \mathbf{-4}$ (move 3 to the right)

D. $13 + (-7) - (-1)$

$= 6 - (-1)$ (start at 13; move 7 to the left)

$= \mathbf{7}$ (move 1 to the right)

E. $-12 + (-3) - 9 - (-2)$

Start at –12; move 3 to the left:

$= -15 - 9 - (-2)$

Now you're at –15; move 9 to the left:

$= -24 - (-2)$

Now you're at –24; move 2 to the right:

= **–22** and we're finally done!

Homework

5. Evaluate (simplify) each expression:

a. –7 – (–3) – (–12)

b. 8 – (–2) – (–9)

c. 5 + (–11) + 1 + (–8)

d. –3 – (–2) – (–11) – 7 – 3

e. –4 – (–7) + 12

f. 11 + 6 + (–6) + 6

g. –6 + 9 – 12 + 11

h. 5 – (–6) + (–4) – 7 – (–3)

i. –2 + 2 + 9 – (–6) + 8

j. –6 – 6 – 1

k. 6 – 4 – 7 – (–10)

l. 2 – (–12) + (–7)

m. 6 + 2 – 8 + 4 + 7

n. –9 – (–5) + 10 + (–11)

o. –9 + 9 – 5 – 12

p. –2 – 11 – 3

Solutions

1.

$ Current Amount $	$ Value of Question $	Right or Wrong	Calculation	$ New Amount $
25	10	right	**25 + 10**	**35**
12	5	wrong	**12 – 5**	**7**
10	15	wrong	**10 – 15**	**–5**
–15	5	right	**–15 + 5**	**–10**

$ Current Amount $	$ Value of Question $	Right or Wrong	Calculation	$ New Amount $
–25	30	right	**–25 + 30**	**5**
–8	7	wrong	**–8 – 7**	**–15**
29	29	wrong	**29 – 29**	**0**
–300	300	right	**–300 + 300**	**0**
0	800	right	**0 + 800**	**800**
0	300	wrong	**0 – 300**	**–300**

2. a. 20 b. 11 c. –1 d. 0 e. –7 f. 8 g. –1 h. –15 i. –92
j. 10 k. –15 l. –100 m. 32 n. –32 o. 88 p. 34 q. –25 r. 57
s. –5 t. –36 u. –3 v. 4 w. –13 x. –8 y. –34 z. –97

3. a. 20 b. –5 c. –9 d. 4 e. –13 f. –8 g. –9 h. 5 i. –10
j. –32 k. –3 l. 0 m. –5 n. –23 o. 0 p. –11

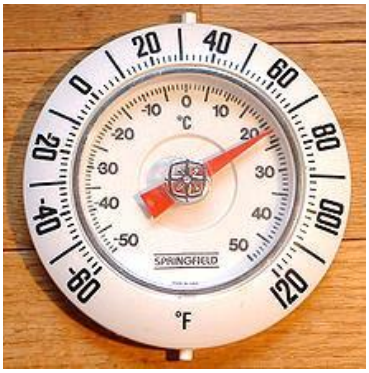
4. a. 2 b. –1 c. 0 d. –13 e. –9 f. –20 g. –7 h. –100
i. 15 j. 3 k. –11 l. 0 m. –20 n. 41 o. 8 p. –2

5. a. 8 b. 19 c. –13 d. 0 e. 15 f. 17 g. 2 h. 3
i. 23 j. – 13 k. 5 l. 7 m. 11 n. –5 o. –17 p. –16

"The important thing is to not stop questioning."

– Albert Einstein

INTRO TO SIGNED NUMBERS



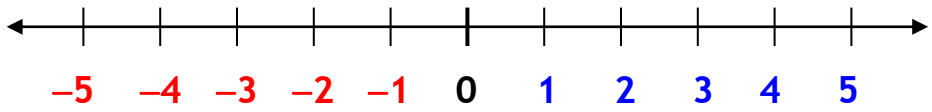
If you look at an equation like $x + 2 = 10$, you easily see that x must be 8, since $8 + 2 = 10$. But what about an equation like

$$x + 9 = 7\ ?$$

Is there a number x that when you add 9 to it, you get a sum of 7? You might already know that there is such a number, but it requires us to define some new numbers that will have to be smaller than zero. We call these the ***negative numbers***.

❑ *POSITIVE AND NEGATIVE NUMBERS*

Let's begin with a picture of the positive and negative numbers (and zero) called the ***number line***.



The ***positive*** numbers are to the right of zero, while the ***negative*** numbers are to the left of zero. The positive number 4 could be written +4, but it's not necessary; we know that the plus sign is understood. By the way, 0 is neither positive nor negative — it's considered neutral, and is called the ***origin*** of the number line.

The arrows at each end of the number line indicate that it keeps going forever in both directions. We might even say that the line goes toward

positive infinity (+∞) to the right and toward negative infinity (–∞) to the left.

If a number x is further to the right on the number line than a number y, we say "x is ***greater*** than y." So, for instance, 5 is greater than 1; 2 is greater than –5; and –3 is greater than –12. Notice that 0 is greater than any negative number and less than any positive number. Is it clear that if x is *greater* than y, then y is ***less*** than x?

There are other numbers on the number line besides the kind you see labeled on the line. For example, the fraction $\frac{3}{2}$ is halfway between the 1 and the 2. And if you remember your circles, you should know that the number π would lie a little to the right of 3, and that –π would lie a little to the left of –3. If you've seen square roots before, the number $\sqrt{2}$ (which has infinitely many non-repeating digits in it) is about 1.414, so it's also on the number line.

The ***opposite*** of a number is the "mirror image" of the number on the number line. It's on the opposite side of the origin but the same distance from the origin. For example, the opposite of 5 is –5, and the opposite of –12 is 12. We will agree that the opposite of 0 is 0, since 0 and 0 are mirror images of each other. (Do you buy that?) Notice that <u>every</u> number has an opposite.

> The opposite of x is $-x$.
>
> The opposite of $-x$ is x.

❑ *Practical Uses for Negative Numbers*

In your savings account, a deposit could be represented by a positive number and a withdrawal by a negative number.

A temperature of 14° *below* zero on the Fahrenheit scale might be written –14°F.

The nation's lowest elevation is 282 feet *below* sea level, which could be written –282 ft.



Death Valley

If the electric charge of a *proton* is **+1**, then the charge of an *electron* would be written **–1**, and the charge of a *neutron* (which has no electric charge at all) is **0**.

You could even extend the concept to football — a quarterback sack could result in a "gain" of –9 yards.

Homework

1. When a number has no sign in front of it, it's assumed to be ____.
2. How many numbers are there between 2 and 3?
3. True/False?

   a. 9 is greater than 4
   b. 3 is greater than 14
   c. 7 is greater than 0
   d. –5 is greater than 0
   e. 2 is greater than –1
   f. 3 is greater than –100
   g. –9 is greater than –2
   h. –5 is greater than –29
   i. If x is greater than y, then x is positive.
   j. If x is positive and y is negative, then x is greater than y.

4. a. $\frac{2}{9}$ lies between what two consecutive whole numbers?
   b. $\frac{17}{3}$ lies between what two consecutive whole numbers?
   c. 2π lies between what two consecutive whole numbers?
5. Find the *opposite* of each number:

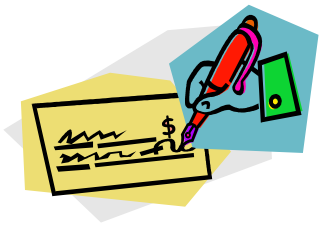
   a. 99 b. –17 c. $\sqrt{2}$ d. $-\pi$ e. 0

6. True/False?

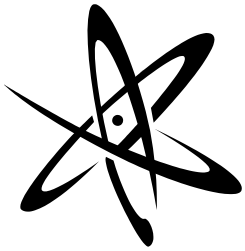
   a. The opposite of a negative number must be positive.
   b. The number 0 does not have an opposite.
   c. Every number has an opposite.

7. What's the opposite of the opposite of 99?

8. Write a number which best describes the situation:



a. a savings account deposit of $800
b. a withdrawal in the amount of $17.50
c. a credit card service charge of $10
d. savings account interest of $141
e. 8° above zero on the Fahrenheit scale
f. 17° below zero on the Celsius scale
g. 8 miles above sea level
h. 6000 meters below sea level
i. the electric charge of 7 protons
j. the electric charge of 12 electrons
k. the electric charge of 137 neutrons
l. the fullback gains 12 yards
m. a 10-yard penalty



9. a. Yesterday's checking account balance was $20. Today Jodie wrote a check for $6 and made a $13 deposit. What is the current balance?

b. Yesterday's checking account balance was $25. Today Mike wrote a check for $60 and made a $15 deposit. What is the current balance?

c. The morning temperature was –25°F, and rose 20° during the day. What was the temperature at the end of the day?



d. The morning temperature was –15°F, and rose 17° during the day. What was the temperature at the end of the day?

e. Carrie started at an elevation of 420 ft, and descended 600 ft into the canyon. What is Carrie's current elevation?

f. Anthony started at an elevation of –320 ft, and ascended 600 ft. What is Anthony's current elevation?

g. Margaret began at an elevation of –200 meters, and ascended 125 meters. What is Margaret's current elevation?

h. Jarod started at 300 ft below sea level, and then descended 150 ft. What is Jarod's current elevation?

i. One proton (+) and one electron (–) cancel each other out. What is the net charge of 8 electrons and 4 more electrons?

j. What is the net charge of 10 electrons and 7 protons?

k. What is the net charge of 20 electrons and 22 protons?

l. What is the net charge of 189 protons and 189 electrons?

m. A 42-yard passing gain was offset by a 15-yard penalty. Find the net gain.

n. A 3-yard running gain was offset by a 10-yard penalty. Find the net gain.

o. To add insult to injury, a quarterback sack which resulted in a 12-yard loss was also accompanied by both a 5-yard penalty and a 15-yard penalty. Find the net gain.



Review Problems

10. a. T/F –5 is greater than –2.
    b. T/F Every number has an opposite.
    c. What is the opposite of each number? 7 $-\sqrt{2}$ 0

11. What two consecutive whole numbers does each of the following numbers lie between?

    a. $\frac{4}{9}$ b. $\frac{40}{3}$ c. 2.05 d. 5π e. $\frac{122}{7}$

12. Write a number which best describes

    a. 35° below zero on the Celsius scale.
    b. 2 miles below sea level.
    c. a checking account service charge of $12.

13. Joe started at an altitude of 200 ft, and then descended 600 ft. What is Joe's current elevation?

14. The morning temperature was –30°C and then rose 26°. What's the current temperature?

15. Some people take out a mortgage (home loan) which involves "***negative*** *amortization*." Research this loan option.

Solutions

1. positive **2**. infinitely many

3. a. T b. F c. T d. F e. T f. T
g. F h. T i. F (it might be true, but not necessarily)
j. T

4. a. 0 and 1 b. $\frac{17}{3} = 5\frac{2}{3}$ is between 5 and 6 c. between 6 and 7

5. a. –99 b. 17 c. $-\sqrt{2}$ d. π e. 0

6. a. T b. F, the opposite of 0 is 0 c. T

7. 99

8. a. 800 b. –17.50 c. –10 d. 141 e. 8
f. –17 g. 8 h. –6000 i. 7 j. –12
k. 0 l. 12 m. –10

9. a. $27 b. –$20 c. –5°F d. 2°F e. –180 ft
f. 280 ft g. –75 m h. –450 ft i. –12 j. –3
k. 2 l. 0 m. 27 yd n. –7 yd o. –32 yd

10. a. False b. True c. $-7;\ \sqrt{2};\ 0$

11. a. 0 & 1 b. 13 & 14 c. 2 & 3
d. 15 & 16 e. 17 & 18

12. a. –35°C b. –2 miles c. –$12

13. –400 ft

14. –4°C

"Intelligence
plus
character
– that is the goal
of true education."

Martin Luther King, Jr.



MORE DIVISION, SIGNED NUMBERS, AND EQUATIONS

❑ *DIVISION AND THOSE PESKY ZEROS*

One of the most important facts in all of mathematics is that the denominator (bottom) of a fraction can NEVER be zero.

Sometimes this is phrased

"Never divide by zero."



This is the result of dividing by zero.

What's the big deal? Why can't we do it?

Recall from the discussion of dividing signed numbers that

$$\frac{56}{7} = 8 \quad \text{because } 8 \times 7 = 56.$$

We don't have to blindly accept the fact that you should never divide by 0. Let's put zeros in fractions and see what happens.

Zero on the Top

How shall we interpret the division problem

$$\frac{0}{7} = ???$$

What number times 7 yields an answer of 0? Well, 0 works; that is,

$$\frac{0}{7} = 0 \quad \text{because } 0 \cdot 7 = 0.$$

Moreover, no other number besides 0 will work (confirm this yourself).

Zero on the Bottom

Now let's put a zero on the bottom (but not the top) and see what happens:

$$\frac{9}{0} = ???$$

Your first guess might be 0; let's check it out:

$\frac{9}{0} = 0$ would be true only if $0 \cdot 0 = 9$, which it isn't.

How about we try an answer of 9?

$\frac{9}{0} = 9$ which is checked by seeing if $9 \cdot 0 = 9$. Sorry.

Could the answer be π? Nope; $\pi \cdot 0 = 0$, not 9.

In fact, <u>any</u> number we conjure up as a potential answer will have to multiply with 0 to make a product of 9. But this is <u>impossible</u>, since any number times 0 is always 0, never 9. In short, <u>*no number* in the whole world</u> will work as the answer to this division problem.

Zero on the Top AND the Bottom

Now for an even stranger problem with division and zeros:

$$\frac{0}{0} = ???$$

We can try 0: $\frac{0}{0} = 0$ which seems reasonable, since $0 \cdot 0 = 0$.

Let's try an answer of 5: $\frac{0}{0} = 5$, which — amazingly! — works out too, because $5 \cdot 0 = 0$.

Could π possibly work? $\frac{0}{0} = \pi$ — still true, since $\pi \cdot 0$ is surely 0.

That's three answers for this division problem: 0, 5, and π.

Is there any end to this madness? Apparently not, since <u>any</u> number we think up will multiply with 0 to make a product of 0. In

short, *every number* in the whole world will work in this division problem.

SUMMARY:

1) Zero on the top of a fraction is perfectly okay, as long as the bottom is NOT zero. The answer to this kind of division problem is always zero. For example, $\frac{0}{7} = 0$.

2) There is no answer to the division problem $\frac{9}{0}$. Clearly, we can never work a problem like this.

3) There are infinitely many answers to the division problem $\frac{0}{0}$. This may be a student's dream come true, but in mathematics we don't want a division problem with trillions of answers.

Each of the two kinds of division problems with a zero in the denominator leads to a major conundrum, so we summarize cases 2) and 3) by stating that

DIVISION BY ZERO IS UNDEFINED!

Thus,

$$\frac{0}{7} = 0$$

$$\frac{9}{0} \text{ is undefined}$$

$$\frac{0}{0} \text{ is undefined}$$

"Black holes are where God divided by zero."

Steven Wright

Note: The two division problems with zero in the denominator may both be undefined, but they're undefined for totally different reasons. Your teacher may require you to understand this.

Homework

1. Evaluate each expression, and explain your conclusion:

   a. $\frac{0}{15}$ b. $\frac{32}{0}$ c. $\frac{0}{0}$

2. Evaluate each expression:

   a. $\frac{0}{17}$ b. $\frac{0}{-9}$ c. $\frac{6-6}{17+3}$ d. $\frac{3^2-8-1}{100}$

   e. $\frac{98}{0}$ f. $\frac{-44}{0}$ g. $\frac{7+8}{2^3-8}$ h. $\frac{7^2-40}{-23+23}$

   i. $\frac{0}{0}$ j. $\frac{-9+9}{10-10}$ k. $\frac{5^2-25}{0^2+0^3}$ l. $\frac{4\cdot5-2\cdot10}{3^3-9}$

3. $\frac{0}{\pi} = 0$ because

   a. 0 is the only number that when multiplied by π gives 0.
   b. no number times π equals 0.
   c. every number times π equals 0.

4. $\frac{0}{0}$ is undefined because

   a. no number times 0 equals 0.
   b. every number times 0 equals 0.
   c. any number divided by itself is 1.

5. $\frac{7}{0}$ is undefined because

   a. 0 is the only number multiplied by 0 to get 7.
   b. no number times 0 equals 7.
   c. every number times 0 equals 7.

6. a. The numerator of a fraction is 0. What can you conclude?
   b. The denominator of a fraction is 0. What can you conclude?

❑ *POWERS OF SIGNED NUMBERS*

Positive Base

This is identical to what we've learned before. For example, squaring a positive number results in a positive number:

$(12)^2 = 12 \times 12 = \mathbf{144}$

The same holds for cubing and higher powers:

$(5)^3 = 5 \cdot 5 \cdot 5 = \mathbf{125}$

$(10)^4 = 10 \times 10 \times 10 \times 10 = \mathbf{10{,}000}$

Negative Base

Now it gets interesting. Let's square a negative number:

$(-7)^2 = (-7)(-7) = \mathbf{49}$ (notice the parentheses around the –7)

The square of a negative number is positive. Now for the cube of a negative number:

$(-4)^3 = (-4)(-4)(-4) = 16(-4) = \mathbf{-64}$

This example shows that *the cube of a negative number is negative*.

A Tricky One What is the value of -5^2 ?

Ask yourself: What's being squared? Well, according to the Order of Operations, an exponent has a much higher priority than a lowly

minus sign. Therefore, the square applies only to the 5; the minus sign is then attached to the result of squaring the 5. Thus, believe it or not:

$$-5^2 = \mathbf{-25}$$

SUMMARY: $\mathbf{(-5)^2 = 25}$, because the –5 is being squared.

$\mathbf{-5^2 = -25}$, because only the 5 is being squared.

Moral: The parentheses make all the difference!

Homework

7. For each number below, find two numbers whose square is that number:

   a. 1 b. 9 c. 25 d. 196 e. 225 f. 529

8. Simplify each power:

   a. 13^2 b. 1^2 c. 0^2 d. 6^2

   e. $(-3)^2$ f. $(-12)^2$ g. $(-9)^2$ h. $(-1)^2$

   i. $\left(\frac{2}{5}\right)^2$ j. $\left(-\frac{1}{3}\right)^2$ k. $(1.2)^2$ l. $(-0.5)^2$

9. Simplify each power:

   a. 3^3 b. 10^3 c. $(-5)^3$ d. $(-1)^3$

   e. 0^3 f. $(-12)^3$ g. $\left(-\frac{1}{2}\right)^3$ h. $[-(-4)]^3$

   i. $(2-5)^3$ j. $(0.2)^3$ k. $(-0.4)^3$ l. $(\pi-\pi)^3$

10. Simplify each expression:

a. $3^2 - 4^2$	b. $0^2 - 7^2$	c. $0 - 9^2$	d. $1 - 10^2$
e. -8^2	f. -10^2	g. $(-10)^2$	h. -1^2
i. $(-1)^2$	j. -11^2	k. -12^2	l. -100^2

❑ *SQUARE ROOTS OF SIGNED NUMBERS*

Review

We saw square roots of positive numbers and zero in the chapter *Pythagorean Theorem – Finding a Leg*. Here's a brief review:

$\sqrt{49} = 7$, since $7^2 = 49$

$\sqrt{144} = 12$, because $12^2 = 144$

$-\sqrt{64} = -8$, since the minus sign signifies the *opposite* of $\sqrt{64}$

$\sqrt{1} = 1$, due to the fact that $1^2 = 1$

$\sqrt{0} = 0$, because $0^2 = 0$

$\sqrt{21} \approx 4.583$, since $4.583^2 = 21.003889$, which is about 21

$\approx$ is approximately equal to

$-\sqrt{2} \approx -1.414$

The Real Purpose of This Section

Now we tackle the square root of a negative number. Let's consider

$$\sqrt{-9}$$

Could it be 3? NO, since $3^2 = +9$, not -9. How about -3? Still NO, because $(-3)^2$ also equals $+9$, not -9. What's happening here? Why

can't we find a number whose square is –9? Because, as we saw in the preceding section, whether a number is positive or negative, its square is positive! There does not exist any number whose square is –9, and therefore there's no square root of –9.

Thus, (in this course) one can never take the square root of a negative number. It simply doesn't exist (yet). What does your calculator say about this situation?

Homework

11. Simplify each square-root expression:

a. $\sqrt{64}$	b. $\sqrt{-64}$	c. $\sqrt{16+9}$	d. $\sqrt{0}$
e. $\sqrt{400}$	f. $\sqrt{-25}$	g. $\sqrt{-1}$	h. $-\sqrt{100}$
i. $\sqrt{1}$	j. $-\sqrt{-2}$	k. $\sqrt{-(-169)}$	l. $-\sqrt{-5}$
m. $-\sqrt{81}$	n. $-\sqrt{-36}$	o. $\sqrt{-(-121)}$	p. $-\sqrt{-0}$
q. $-\sqrt{-(-144)}$	r. $-\sqrt{49}$	s. $-\sqrt{-49}$	t. $\sqrt{225}$

12. Explain the difference between $\sqrt{-3}$ and $-\sqrt{3}$.

❑ *SOLVING EQUATIONS*

An equation like $2x = -14$ can now be solved, even with the negative number in it. Dividing each side by 2 produces the equation $x = \frac{-14}{2}$, which implies that $x = -7$, by the rule that a negative divided by a positive is negative.



EXAMPLE 1: **Solve each equation:**

A. $-3n = 18 \Rightarrow \frac{-3n}{-3} = \frac{18}{-3} \Rightarrow \boldsymbol{n = -6}$

B. $-8y = -34 \Rightarrow \frac{-8y}{-8} = \frac{-34}{-8} \Rightarrow \boldsymbol{y = \frac{17}{4}}$

C. $\frac{x}{2} = -17 \Rightarrow \left[\frac{x}{2}\right]2 = \left[-17\right]2$

$\Rightarrow \left[\frac{x}{2}\right]2 = -34 \Rightarrow \boldsymbol{x = -34}$

D. $\frac{a}{-10} = -20 \Rightarrow a = (-20)(-10) \Rightarrow \boldsymbol{a = 200}$

EXAMPLE 2: **Solve each equation:**

A. $2a + 8 = -14$

$2a + 8 - 8 = -14 - 8$ (subtract 8 from each side)

$2a = -22$ (simplify each side)

$\frac{2a}{2} = \frac{-22}{2}$ (divide each side by 2)

$\boldsymbol{a = -11}$ (simplify each side)

B. $-3b - 7 = -10$

$-3b - 7 + 7 = -10 + 7$ (add 7 to each side)

$-3b = -3$ (simplify each side)

$\frac{-3b}{-3} = \frac{-3}{-3}$ (divide each side by –3)

$\boldsymbol{b = 1}$ (simplify each side)

C. $-x + 5 = -2$

$-x + 5 - 5 = -2 - 5$ (subtract 5 from each side)

$-x = -7$ (simplify each side)

At this point, it's handy to write $-x$ as $-1x$.

$-1x = -7$

$\frac{-x}{-1} = \frac{-7}{-1}$ (divide each side by –1)

$\boldsymbol{x = 7}$ (simplify each side)

D. $-6 - x = -4$

$-6 + 6 - x = -4 + 6$ (add 6 to each side)

$-x = 2$ (simplify each side)

$\frac{-x}{-1} = \frac{2}{-1}$ (divide each side by –1)

$\boldsymbol{x = -2}$ (simplify each side)

E. $3y + 7 = 5$

$3y + 7 - 7 = 5 - 7$ (subtract 7 from each side)

$3y = -2$ (simplify)

$\frac{3y}{3} = \frac{-2}{3}$ (divide each side by 3)

$\boldsymbol{y = -\frac{2}{3}}$ (simplify)

Homework

13. Solve each equation:

a. $5x = -15$ b. $-3y = 24$ c. $-2n = -20$

d. $-3z = 14$ e. $8n = -6$ f. $-12Q = -40$

g. $\frac{x}{3} = -23$ h. $\frac{w}{-5} = 40$ i. $\frac{a}{-5} = -6$

j. $-5x = -15$ k. $-3y = -24$ l. $2n = -20$

m. $-3z = 15$ n. $8n = -2$ o. $-14Q = -40$

p. $\frac{x}{3} = -12$ q. $\frac{w}{-5} = 20$ r. $\frac{a}{-6} = -6$

14. Solve each equation:

a. $3x + 7 = 5$ b. $8n - 3 = -9$ c. $4y + 8 = -13$

d. $-5t - 1 = -8$ e. $-3d + 8 = 4$ f. $-4p - 1 = -8$

g. $-x - 8 = 7$ h. $23 - y = -1$ i. $-g - 4 = -5$

j. $12 = 2x + 20$ k. $8 - 3w = 8$ l. $-6 + 2x = -20$

m. $-23 = 7 - 3r$ n. $12 + 12u = 12$ o. $-20 = 23 - 7z$

p. $2x + 7 = 5$ q. $9n - 3 = -9$ r. $3y + 8 = -13$

Review Problems

15. a. Find two numbers whose square is 1,000,000.
b. There's only one number whose square is 0. What is it?
c. There's no number (in this book) whose square is –25. Explain.

16. a. $\frac{0}{0}$ b. $\frac{0}{23}$ c. $\frac{-5}{0}$ d. $(13)^2$

e. $(-12)^2$ f. $(6)^3$ g. -15^2 h. $(-7)^3$

i. $\sqrt{196}$ j. $\sqrt{-81}$ k. $-\sqrt{-144}$

l. Solve for x: $\frac{x}{-5} = -6$ m. Solve for n: $\frac{n}{-6} = 10$

n. Solve for a: $-3a - 9 = 5$ o. Solve for w: $8 - w = 10$

17. a. $-20^2 =$ b. $(-6)^2 =$ c. $-6^2 =$

d. $-0^2 =$ e. $-4^2 =$ f. $-15^2 =$

18. Which statement is true?

a. $\frac{8}{0}$ is undefined AND $\frac{0}{3} = 0$ AND $\frac{0}{0} = 0$.

b. $\frac{8}{0} = 0$ AND $\frac{0}{3} = 0$ AND $\frac{0}{0} = 0$.

c. $\frac{8}{0}$ is undefined AND $\frac{0}{3}$ is undefined AND $\frac{0}{0} = 0$.

d. $\frac{8}{0}$ is undefined AND $\frac{0}{3} = 0$ AND $\frac{0}{0}$ is undefined.

e. $\frac{8}{0} = 0$ AND $\frac{0}{3} = 0$ AND $\frac{0}{0} = 0$.

Solutions

1. a. $\frac{0}{15} = 0$ since $0 \times 15 = 0$, and 0 is the only number that accomplishes this.

b. $\frac{32}{0}$ is undefined because any number times 0 is 0, <u>never</u> 32; thus NO number works.

c. $\frac{0}{0}$ is undefined because any number times 0 is 0; thus EVERY number works.

2. a. 0 b. 0 c. 0 d. 0 e. Undefined f. Undefined
g. Undefined h. Undefined i. Undefined j. Undefined
k. Undefined l. 0

3. a. **4**. b. **5**. b.

6. a. You can't conclude anything — it depends on what's on the bottom. If the bottom is a non-zero number (like 7), then $\frac{0}{7} = 0$. If the bottom is zero, then $\frac{0}{0}$ is undefined.

b. This time we can conclude that the fraction is undefined, since division by 0 is undefined, no matter what's on the top of the fraction.

7. a. 1 & –1 b. 3 & –3 c. 5 & –5 d. 14 & –14
e. 15 & –15 f. 23 & –23

8. a. 169 b. 1 c. 0 d. 36 e. 9 f. 144
g. 81 h. 1 i. $\frac{4}{25}$ j. $\frac{1}{9}$ k. 1.44 l. 0.25

9. a. 27 b. 1000 c. –125 d. –1 e. 0 f. –1728
g. $-\frac{1}{8}$ h. 64 i. –27 j. 0.008 k. –0.064 l. 0

10. a. –7 b. –49 c. –81 d. –99 e. –64 f. –100
g. 100 h. –1 i. 1 j. –121 k. –144 l. –10,000

11. a. 8 b. Does not exist c. 5 d. 0
e. 20 f. Does not exist g. Does not exist h. –10
i. 1

j. Does not exist — the square root of –2 does not exist, so the problem is over right there. The minus sign in front has no bearing on the problem.

k. 13 — This time, the minuses cancel, leaving $\sqrt{169}$.

l. Does not exist m. –9 n. Does not exist

o. 11 p. 0 q. –12

r. –7 s. Does not exist t. 15

12. $\sqrt{-3}$ is the square root of –3, which does not exist in Algebra 1. But $-\sqrt{3}$ does exist. It's simply the <u>opposite</u> of $\sqrt{3}$, and is therefore approximately –1.732.

13. a. –3 b. –8 c. 10 d. $-\frac{14}{3}$ e. $-\frac{3}{4}$ f. $\frac{10}{3}$

g. –69 h. –200 i. 30 j. 3 k. 8 l. –10

m. –5 n. $-\frac{1}{4}$ o. $\frac{20}{7}$ p. –36 q. –100 r. 36

14. a. $-\frac{2}{3}$ b. $-\frac{3}{4}$ c. $-\frac{21}{4}$ d. $\frac{7}{5}$ e. $\frac{4}{3}$ f. $\frac{7}{4}$

g. –15 h. 24 i. 1 j. –4 k. 0 l. –7

m. 10 n. 0 o. $\frac{43}{7}$ p. –1 q. $-\frac{2}{3}$ r. –7

15. a. 1000 & –1000 b. 0 c. The square of any number in this class is 0 or positive, never negative; this is because a positive times a positive is positive, and a negative times a negative is also positive.

16. a. Undefined b. 0 c. Undefined d. 169

e. 144 f. 216 g. –225 h. –343

i. 14 j. Does not exist k. Does not exist

l. $x = 30$ m. $n = -60$ n. $a = -\frac{14}{3}$ o. $w = -2$

17. a. –400 b. 36 c. –36 d. 0 e. –16 f. –225

18. d.

"We cannot hold a torch to light another's path without brightening our own."

– Ben Sweetland

MORE SIGNED NUMBERS AND EQUATIONS

❑ *MORE ON OPPOSITES*

Sometimes we come across an expression like the following:

$$-(-4)$$

This can be read "the opposite of negative 4," which we know is positive 4; that is, $-(-4) = 4$.

$$-(-x) = x$$

Advertisers take advantage of this idea of the "double negative" when they claim that "Nobody Doesn't Like Sara Lee."

This "double negative" idea can be used to solve a problem like this:

$$-(-7) + (-5) = 7 + (-5) = \mathbf{2}$$

What happens if we add a number and its opposite together? If we add 7 and its opposite, we get

$$7 + (-7) = \mathbf{0}$$ (start at 7; move 7 units to the left)

If we add –13 and its opposite, we get the same sum:

$$-13 + 13 = \mathbf{0}$$ (start at –13; move 13 to the right)

It seems that

> The sum of a number and its opposite is zero.

We can write this as a formula. For any number n, whether it's positive, negative, or zero, we have:

$$n + (-n) = 0$$

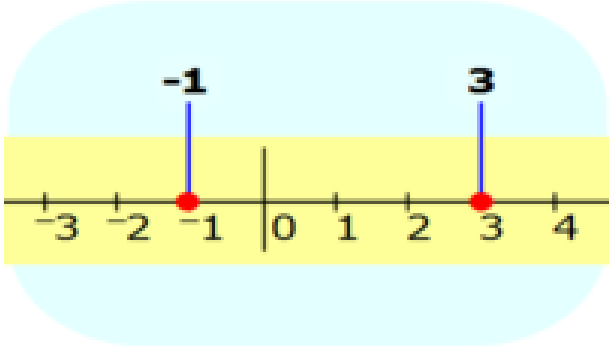
In fact, we can say that two numbers are opposites of each other precisely when their sum is 0. (This might explain to you why the opposite of 0 is 0.)

Homework

1. Simplify: $-(-9)$
2. Simplify: $-(-\pi)$
3. Prove that 23 and -23 are opposites of each other.
4. Prove that $-\pi$ and π are opposites of each other.
5. Prove that 0 and 0 are opposites of each other.
6. What is the opposite of -9? Explain.
7. What is the opposite of 123? Explain.
8. What is the opposite of 0? Explain.

❑ *Another Approach to Adding and Subtracting*

We recall the method of using the number line to add and subtract positive and negative numbers: by starting at the first number, and then moving left or right, as appropriate, to arrive at the answer.



Another method, totally equivalent to the number-line method, is as follows:

> If two adjacent signs are *opposites*, replace them with a single *minus* sign.
>
> If two adjacent signs are the *same*, replace them with a single *plus* sign.

Then we proceed from there using the Jeopardy method, the number line, or common sense.

EXAMPLE 1:

A. $10 + (-3) = 10 - 3 = \mathbf{7}$

Change the two adjacent signs to a minus sign.

B. $-9 + (-5) = -9 - 5 = \mathbf{-14}$

Change the two adjacent signs to a minus sign.

C. $8 - (-2) = 8 + 2 = \mathbf{10}$

Change the two adjacent signs to a plus sign.

D. $-7 - (-3) = -7 + 3 = \mathbf{-4}$

Change the two adjacent signs to a plus sign.

E. $-10 - (-14) = -10 + 14 = 4$

Change the two adjacent signs to a plus sign.

EXAMPLE 2: **Simplify: $-(-7) + 8 + (-10) - 3 - (-2)$**

Here's the original expression:

$-(-7) + 8 + (-10) - 3 - (-2)$

Change the double negative in front to a positive 7:

$7 + 8 + (-10) - 3 - (-2)$

Change the adding of –10 to simple subtraction:

$7 + 8 - 10 - 3 - (-2)$

Change the subtracting of –2 to adding:

$7 + 8 - 10 - 3 + 2$

Now that all double signs have been removed, just proceed from left to right:

$15 - 10 - 3 + 2$

$5 - 3 + 2$

$2 + 2$

$\boxed{4}$

Homework

9. Evaluate (simplify) each expression:

a. $6 + (-6) + (-1)$

b. $-(-4) + 8 + (-8) + (-7)$

c. $-3 + (-4) + 5$

d. $-5 + (-5) + (-3)$

e. $12 + (-3) + (-12)$

f. $(-1) + (-2) + (-3)$

g. $-(-2) + (-2) + (-3) + 3$

h. $8 + 7 + 8 + (-1)$

10. Evaluate (simplify) each expression:

a. $-(-10) - 2$

b. $2 - 10$

c. $-3 - 8$

d. $-4 - 2$

e. $5 - (-1)$	f. $10 - (-3)$	g. $-3 - (-3)$	h. $-1 - (-7)$
i. $18 - 22$	j. $-7 - 5$	k. $-23 - (-4)$	l. $8 - (-2)$
m. $-(-7) - 20$	n. $-99 - 1$	o. $-1 - (-2)$	p. $12 - (-4)$
q. $13 - 21$	r. $-4 - 9$	s. $-(-12) - (-9)$	t. $-5 - (-1)$

11. Evaluate (simplify) each expression:

a. $7 + (-3) - 8$	b. $-3 + 0 - (-1)$	c. $-8 - 2 - 1$
d. $-2 - (-3) + 1$	e. $7 - (-3) - 8$	f. $9 - 2 - 3 - 4$
g. $-2 + 5 + 9 - 1$	h. $-1 - (-1) - (-1)$	i. $8 + (-2) - 8 - (-1)$
j. $-(-3) + 7$	k. $-(-1) - 3$	l. $-(-5) - (-4)$
m. $-(-11) - (-8)$	n. $-(-7) - (+5)$	o. $-(-3) + (-3)$

12. True/False?

a. The opposite of a negative number must be positive.

b. The number 0 does not have an opposite.

c. Every number has an opposite.

❑ *MORE EQUATIONS*

Now we can solve a wider variety of equations. Let's start with the equation: $x + 9 = 7$, which has no positive solutions. We remove the 9 just as we would expect — subtract it from each side of the equation:

	$x + 9 = 7$	(the original equation)
$\Rightarrow$	$x + 9 \mathbf{- 9} = 7 \mathbf{- 9}$	(subtract 9 from each side)
$\Rightarrow$	$\boldsymbol{x = -2}$	($9 - 9 = 0$, and $7 - 9 = -2$)

To check this solution, we see that $-2 + 9 = 7$ ✓

EXAMPLE 1: **Solve each equation:**

A. $n - 3 = -10$

Add 3 to each side of the equation:

$n - 3 \mathbf{+ 3} = -10 \mathbf{+ 3} \Rightarrow \boldsymbol{n = -7}$

B. $y - 7 = -2$

Add 7 to each side of the equation:

$y - 7 \mathbf{+ 7} = -2 \mathbf{+ 7} \Rightarrow \boldsymbol{y = 5}$

C. $z + 18 = 15$

Subtract 18 from each side of the equation:

$z + 18 \mathbf{- 18} = 15 \mathbf{- 18} \Rightarrow \boldsymbol{z = -3}$

D. $a + 10 = -5$

Subtract 10 from each side of the equation:

$a + 10 \mathbf{- 10} = -5 \mathbf{- 10} \Rightarrow \boldsymbol{a = -15}$

E. $5x - 30 = -20$

Add 30 to each side of the equation:

$5x - 30 \mathbf{+ 30} = -20 \mathbf{+ 30}$

$\Rightarrow 5x = 10$

Last, divide each side of the equation by 5:

$\Rightarrow \boldsymbol{x = 2}$

Homework

13. Solve each equation:

a. $x + 5 = 9$ b. $y - 13 = 12$ c. $n + 7 = 7$

d. $t - 8 = -4$ e. $x - 5 = -3$ f. $g - 9 = -13$

g. $a + 8 = 7$ h. $b + 9 = -2$ i. $c + 7 = -10$

j. $d - 9 = -10$ k. $q - 19 = -10$ l. $t - 9 = 9$

m. $x - 17 = -2$ n. $x + 17 = -3$ o. $z + 7 = -4$

p. $w - 11 = -11$ q. $n - 2.3 = -0.7$ r. $m + 0.3 = -2$

14. Solve each equation:

a. $3a - 8 = -3$ b. $4t - 3 = -1$ c. $8w - 9 = -6$

d. $2x - 4 = -2$ e. $5n - 12 = -9$ f. $6u - 10 = -5$

g. $9w + 8 = 13$ h. $6a - 7 = -4$ i. $12g - 13 = -9$

j. $7a - 9 = 14$ k. $13w - 12 = 1$ l. $12P - 12 = 14$

Review Problems

15. Evaluate (simplify) each expression:

a. $-7 - (-3) - (-12)$ b. $-(-8) - 10 + 3 + (-10)$

c. $8 - (-2) - (-9)$ d. $-5 + (-8) + 4 - 11$

e. $5 + (-11) + 1 + (-8)$ f. $-3 - (-2) - (-11) - 7 - 3$

g. $-(-1) - 7 - 11 + (-5)$ h. $-4 - (-7) + 12$

i. $-(-2) - (-2) + (-7) - (-7) - 4$ j. $11 + 6 + (-6) + 6$

k. $-6 + 9 - 12 + 11$ l. $5 - (-6) + (-4) - 7 - (-3)$

m. $-2 + 2 + 9 - (-6) + 8$ n. $-6 - 6 - 1$

o. $6 - 4 - 7 - (-10)$ p. $2 - (-12) + (-7)$

q. $-(-9) + 8 - (-5) + 9$

r. $6 + 2 - 8 + 4 + 7$

s. $-9 - (-5) + 10 + (-11)$

t. $-12 + 9 - 2 - (-5) - (-1)$

u. $-9 + 9 - 5 - 12$

v. $-2 - 11 - 3$

16. Solve each equation:

a. $x - 5 = 9$

b. $y + 13 = 12$

c. $n - 7 = 7$

d. $t + 8 = -4$

e. $x + 5 = -3$

f. $g - 9 = 13$

g. $a + 8 = -7$

h. $b + 9 = 2$

i. $c - 7 = -10$

j. $d + 9 = -10$

k. $q + 19 = -10$

l. $t - 9 = -9$

m. $x - 17 = 2$

n. $x - 17 = -3$

o. $z - 7 = -24$

17. Solve each equation:

a. $3x - 20 = -9$

b. $6n - 13 = 20$

c. $5q - 20 = -15$

d. $6a - 23 = -22$

e. $10w + 13 = 100$

f. $12p - 44 = -8$

g. $4T - 90 = -12$

h. $18a - 10 = -8$

i. $7x - 99 = -99$

j. $5g - 25 = -10$

Solutions

1. 9

2. π

3. $23 + (-23) = 0$ ✓

4. $-\pi + \pi = 0$ ✓ **5**. $0 + 0 = 0$ ✓ **6**. 9; because $-9 + 9 = 0$

7. -123; because $123 + (-123) = 0$

8. 0; because $0 + 0 = 0$

9. a. -1 b. -3 c. -2 d. -13
e. -3 f. -6 g. 0 h. 22

10. a. 8 b. -8 c. -11 d. -6 e. 6 f. 13
g. 0 h. 6 i. -4 j. -12 k. -19 l. 10
m. -13 n. -100 o. 1 p. 16 q. -8 r. -13
s. 21 t. -4

11. a. -4 b. -2 c. -11 d. 2 e. 2
f. 0 g. 11 h. 1 i. -1 j. 10
k. -2 l. 9 m. 19 n. 2 o. 0

12. a. T b. F, the opposite of 0 is 0 c. T

13. a. $x = 4$ b. $y = 25$ c. $n = 0$ d. $t = 4$ e. $x = 2$
f. $g = -4$ g. $a = -1$ h. $b = -11$ i. $c = -17$ j. $d = -1$
k. $q = 9$ l. $t = 18$ m. $x = 15$ n. $x = -20$ o. $z = -11$
p. $w = 0$ q. $n = 1.6$ r. $m = -2.3$

14. a. $a = \frac{5}{3}$ b. $t = \frac{1}{2}$ c. $w = \frac{3}{8}$ d. $x = 1$
e. $n = \frac{3}{5}$ f. $u = \frac{5}{6}$ g. $w = \frac{5}{9}$ h. $a = \frac{1}{2}$
i. $g = \frac{1}{3}$ j. $a = \frac{23}{7}$ k. $w = 1$ l. $P = \frac{13}{6}$

15. a. 8 b. -9 c. 19 d. -20 e. -13 f. 0 g. -22 h. 15
i. 0 j. 17 k. 2 l. 3 m. 23 n. -13 o. 5 p. 7
q. 31 r. 11 s. -5 t. 1 u. -17 v. -16

16. a. $x = 14$ b. $y = -1$ c. $n = 14$ d. $t = -12$ e. $x = -8$ f. $g = 22$

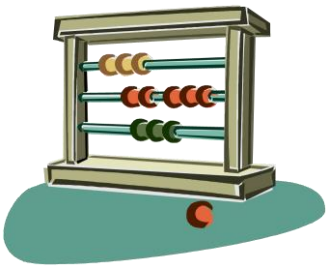
g. $a = -15$ h. $b = -7$ i. $c = -3$ j. $d = -19$ k. $q = -29$ l. $t = 0$
m. $x = 19$ n. $x = 14$ o. $z = -17$

17. a. $x = \frac{11}{3}$ b. $n = \frac{11}{2}$ c. $q = 1$ d. $a = \frac{1}{6}$ e. $w = \frac{87}{10}$ f. $p = 3$
g. $T = \frac{39}{2}$ h. $a = \frac{1}{9}$ i. $x = 0$ j. $g = 3$

"Education is not merely a means for earning a living or an instrument for the acquisition of wealth. It is an initiation into life of spirit, a training of the human soul in the pursuit of truth and the practice of virtue."

– Vijaya Lakshmi Pandit

MULTIPLYING AND DIVIDING SIGNED NUMBERS



You may have learned how to solve an equation like $7n = 35$: You divide each side of the equation by 7,

$$\frac{7n}{7} = \frac{35}{7},$$

and conclude that $n = 5$. What if we come across an equation like $2x = -14$?

Our instinct should be to divide each side of the equation by 2 in order to isolate the x. But now we're faced with dividing a negative number (the -14) by a positive number (the 2). Not to worry — we'll figure it out.

❑ *MULTIPLYING SIGNED NUMBERS*

Recall that the result of multiplying two numbers is called the **product** of the two numbers. So, for instance, the product of 3 and 5 is 15. Some ways to represent the product of 3 and 5 are

$3 \cdot 5$ 3×5 $3(5)$ $(3)5$ $(3)(5)$ All these products equal 15

Instead of using parentheses to indicate multiplication, square brackets are sometimes used. So $[3][5] = 15$ also.

Recall the ***commutative property of multiplication***, where the order in which you

With variables, it's even easier: The product of 7 and x can be written $\mathbf{7x}$, and the product of a and b can be written $\boldsymbol{ab}$.

multiply a pair of numbers makes no difference in the product: $ab = ba$ for any numbers a and b.

Also recall the ***associative property of multiplication***, where parentheses can be "shifted" without changing the answer:

$(ab)c = a(bc)$ for any numbers a, b, and c

Positive Times Positive

What should 6×4 be? Luckily, the answer is exactly what you would expect: **24**. In other words, what we learned as kids still holds: **A positive number times a positive number is positive**.

Positive Times Negative

This one's not so obvious. Consider the multiplication problem $(7)(-1)$. We'll determine the answer by sneaking up on the problem while it's not looking:

$(7)(2) = 14$	since a positive times a positive is positive
$(7)(1) = 7$	same rule – or – anything times 1 is itself
$(7)(0) = 0$	anything times 0 is 0
$(7)(-1) =$???	what should the product be?

What comes next in the sequence of products: 14, 7, 0, ? Since this sequence of numbers is <u>decreasing</u> by 7 at each step, the next number in the sequence must be –7, and we see that $(7)(-1) = \mathbf{-7}$. It appears that **a positive number times a negative number is negative**.

Negative Times Positive

To calculate $(-3)(4)$, first reverse the order of the factors (commutative property): $(-3)(4) = (4)(-3)$, which is now a product of a positive with a negative. By the previous rule, we know the



answer is **–12**. Thus, **a negative number times a positive number is negative**.

Negative Times Negative

Now for the most interesting situation, the product of two negative numbers — for example (–5)(–1). We'll get a running start and see what emerges.

(–5)(2)	= –10	since a negative times a positive is negative
(–5)(1)	= –5	same rule – or – anything times 1 is itself
(–5)(0)	= 0	anything times 0 is 0
(–5)(–1)	= ???	what should the product be?

What comes next in the sequence of answers –10, –5, 0 ? Since this sequence of numbers is increasing by 5 at each step, it follows that the next number is 5, giving us the result:

(–5)(–1) = 5

We've reached the inescapable conclusion that **a negative number times a negative number is positive**!

See if you can deduce the two rules in the following box from the four rules stated above.

Multiplying Signed Numbers:

If the signs are the same, the product is positive.

If the signs are different, the product is negative.

Homework

1. Find the product:

a. (17)(3)	b. (–4)(7)	c. 3(–10)	d. (–3)(–4)
e. –7(–2)	f. 2(–7)	g. –1(8)	h. –1(–9)
i. (1)(–134)	j. (–765)(0)	k. -3×-4	l. $7 \cdot -10$
m. (–18)(–2)	n. 7(–3)	o. –8(7)	p. $3 \cdot -9$
q. (–2)(99)	r. (–1)(–7)	s. (7)(–10)	t. (–4)(–5)

2. Find the product:

a. (0.2)(0.3)	b. (–0.1)(0.1)	c. (2.1)(–3)	d. (–0.3)(–0.4)
e. (–0.3)(5)	f. 2(–1.77)	g. (–0.1)(–0.2)	h. 17(–0.2)

3. Find the product in reduced form:

a. $\frac{2}{3} \times \frac{1}{5}$	b. $-\frac{2}{3} \cdot \frac{3}{4}$	c. $\frac{4}{5}\left(-\frac{5}{4}\right)$	d. $(-2)\left(-\frac{7}{2}\right)$
e. $\frac{4}{5} \times -\frac{10}{3}$	f. $\left[-\frac{1}{2}\right]\left[\frac{2}{99}\right]$	g. $\left(-\frac{2}{3}\right)\left(-\frac{3}{2}\right)$	h. $10\left(-\frac{3}{40}\right)$

4. Find the product:

Example: (–2)(3)(–4)
= (–6)(–4) (multiply the first two factors)
= 24

a. (–2)(–3)(4)	b. $7 \times -6 \times 3$	c. (–1)(–2)(–3)(–4)
d. $4 \cdot 4 \cdot 4(-1)(3)$	e. 5(–1)(–1)(–2)	f. 4(–3)(2)(–8)
g. (–2)(–1)(–3)	h. (–7)(6)(2)(–1)	i. (–1)(–1)(–1)(8)
j. (–1)(–3)(–5)(–7)	k. –3(–4)(5)2	l. (–2)(–2)(–1)(–3)(–5)0

❑ *DIVIDING SIGNED NUMBERS*

The secret to the division rules for signed numbers is the fact that division is checked by multiplication. For example,

$$\frac{56}{7} = 8, \text{ precisely because } 8 \times 7 = 56.$$

Positive Divided by Positive

$\frac{6}{2} = 3$, since $(3)(2) = 6$.

Therefore, **a positive number divided by a positive number is positive**.

Negative Divided by Positive

$\frac{-6}{2} = -3$, because $(-3)(2) = -6$.

Thus, **a negative number divided by a positive number is negative.**

Positive Divided by Negative

$\frac{6}{-2} = -3$, which is checked by seeing that $(-3)(-2) = 6$.

Conclusion: **a positive number divided by a negative number is negative.**

Negative Divided by Negative

$\frac{-6}{-2} = 3$, which is confirmed by the fact that $(3)(-2) = -6$.

We see that **a negative number divided by a negative number is positive.**

NOTES:

1. Do these four rules for dividing signed numbers remind you of anything? The rules for <u>dividing</u> signed numbers are the <u>same</u> as the rules for <u>multiplying</u> signed numbers. (Now, that's nice.)

2. Note that $\frac{-10}{2} = \frac{10}{-2} = -\frac{10}{2}$, because each of these three division problems has a

quotient of –5. In general, the following three fractions are equal:

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

Homework

5. Using the logic above, which explains how signed numbers are divided, explain why each division problem is correct:

a. $\frac{22}{11} = 2$

b. $\frac{-21}{7} = -3$

c. $\frac{88}{-22} = -4$

d. $\frac{-100}{-20} = 5$

Here's the bottom line for dividing signed numbers:

Dividing Signed Numbers:

If the signs are the same, the quotient is positive.

If the signs are different, the quotient is negative.

EXAMPLE 1: **Find each quotient:**

A. $\frac{45}{-9} = -\mathbf{5}$ (positive divided by negative is negative)

B. $\frac{-250}{-25} = \mathbf{10}$ (negative divided by negative is positive)

C. $\frac{-21}{15} = -\frac{21}{15} = -\frac{\mathbf{7}}{\mathbf{5}}$ (negative divided by positive is negative)

D. $\frac{-8}{-12} = \frac{8}{12} = \mathbf{\frac{2}{3}}$ (negative divided by negative is positive)

E. $\frac{18}{-7} = -\mathbf{\frac{18}{7}}$ (positive divided by negative is negative)

F. $\frac{-4}{-9} = \mathbf{\frac{4}{9}}$ (negative divided by negative is positive)

Homework

6. Find the quotient (leave answers as simplified fractions):

a. $\frac{30}{5}$ b. $\frac{60}{-10}$ c. $\frac{-100}{20}$ d. $\frac{-14}{-2}$

e. $\frac{3}{6}$ f. $\frac{-18}{4}$ g. $\frac{24}{-5}$ h. $\frac{-34}{-24}$

i. $\frac{-1}{9}$ j. $\frac{-345}{-345}$ k. $\frac{10}{-10}$ l. $\frac{-120}{-70}$

7. Evaluate each expression:

a. $\frac{5-1}{1-2}$ b. $\frac{10(-2)}{-4}$ c. $\frac{3^2-9}{7}$

d. $\frac{18-2}{2-18}$ e. $\frac{1+2+3}{-2-4}$ f. $\frac{3^2-1^3}{6-5}$

8. Evaluate each expression:

Example: $4 - 5[-3 + 2(3 - 6)]$

$= 4 - 5[-3 + 2(-3)]$ (start with innermost parentheses)

$= 4 - 5[-3 - 6]$ (multiply before adding)

$= 4 - 5[-9]$ (now for the brackets)

$= 4 + 45$ (multiply before subtracting)

$= \mathbf{49}$ (and we're done)

a. $2[5 - 9(-2)]$ b. $-3[1 + 2(-5)]$

c. $4 + 3[2 - 3(4)]$

d. $4 - 5[1 + 5(1 - 4)]$

e. $10 - 8[2(3) - 4(2)]$

f. $7 - 2[8 - 2(3 - 4)]$

g. $4 - 7[-5 - (2 - 9)]$

h. $10 - 2[10 - 2(3 - 7)]$

Review Problems

9. A negative divided by a negative is positive. Explain why.

10. a. $(-5)(-2)(-3) =$ b. $\frac{45}{-9} =$ c. $\frac{-80}{25} =$

d. $\frac{-12}{-48} =$ e. $\frac{2-8-10}{12-1-5} =$

11. a. $(-5)(2)(-4) =$ b. $\frac{-45}{-9} =$ c. $\frac{80}{-15} =$

d. $\frac{-36}{-48} =$ e. $\frac{2-8+10}{12+1-5} =$

12. $4 - 3[2(-3) - 5(8 - 6)] =$

The entire chapter can be summarized like this:

For both <u>multiplying and dividing</u>,

Same signs $\Rightarrow$ Positive answer

Different signs $\Rightarrow$ Negative answer

Solutions

1. a. 51 b. −28 c. −30 d. 12 e. 14 f. −14
g. −8 h. 9 i. −134 j. 0 k. 12 l. −70
m. 36 n. −21 o. −56 p. −27 q. −198 r. 7
s. −70 t. 20

2. a. 0.06 b. −0.01 c. −6.3 d. 0.12 e. −1.5
f. −3.54
g. 0.02 h. −3.4

3. a. $\frac{2}{15}$ b. $-\frac{1}{2}$ c. −1 d. 7 e. $-\frac{8}{3}$ f. $-\frac{1}{99}$
g. 1 h. $-\frac{3}{4}$

4. a. 24 b. −126 c. 24 d. −192 e. −10 f. 192
g. −6 h. 84 i. −8 j. 105 k. 120 l. 0

5. a. (2)(11) = 22 b. (−3)(7) = −21
c. (−4)(−22) = 88 d. (5)(−20) = −100

6. a. 6 b. −6 c. −5 d. 7 e. $\frac{1}{2}$
f. $-\frac{9}{2}$ g. $-\frac{24}{5}$ h. $\frac{17}{12}$ i. $-\frac{1}{9}$ j. 1
k. −1 l. $\frac{12}{7}$

7. a. −4 b. 5 c. 0 d. −1 e. −1 f. 8

8. a. 46 b. 27 c. −26 d. 74
e. 26 f. −13 g. −10 h. −26

9. $\frac{\text{neg}}{\text{neg}}$ = pos because when the division is checked by multiplication, we

confirm that pos × neg = neg ✓

10. a. −30 b. −5 c. $-\frac{16}{5}$ d. $\frac{1}{4}$ e. $-\frac{8}{3}$

11. a. 40 b. 5 c. $-\frac{16}{3}$ d. $\frac{3}{4}$ e. $\frac{1}{2}$

12. 52

"Let us think of education as the means of developing our greatest abilities, because in each of us there is a private hope and dream which, fulfilled, can be translated into benefit for everyone and greater strength of the nation."

– John F. Kennedy



SLOPE = $\Delta y / \Delta x$

Finding the slope, $m = \frac{\text{rise}}{\text{run}}$, of a line by plotting two points and counting the squares to determine the rise and the run works fine only when it's convenient to plot the points and you're in the mood to count squares. Indeed, consider the line connecting the two points $(\pi, 2000)$ and $(3\pi, -5000)$. Certainly these points determine a line, and of course that line has some slope, but plotting these points is not feasible. We need a simpler way to calculate slope.



❑ *A NEW VIEW OF SLOPE*

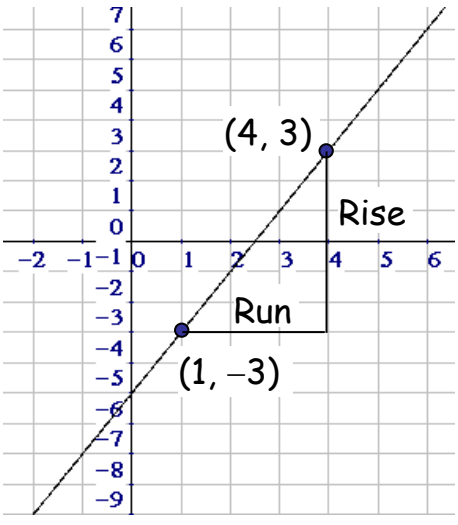
Recall Example 1 from the chapter *Slope = Rise/Run*,

$$y = 2x - 5$$

We plotted the points (4, 3) and (1, –3) and then counted squares (as we moved from left to right) to get a rise of 6 and a run of 3, giving us a slope of

$$m = \frac{\text{rise}}{\text{run}} = \frac{6}{3} = \mathbf{2}$$

How can we get the numbers 6 and 3 without referring to the points on the graph? Notice that if we subtract the



y-coordinate of one point from the y-coordinate of the other point, we get

$$rise = 3 - (-3) = 3 + 3 = \mathbf{6}$$

Similarly, if we subtract one x-coordinate from the other, we get

$$run = 4 - 1 = \mathbf{3}$$

Now, dividing the rise by the run gets us our slope of **2**. We can now think of our $m = \frac{\text{rise}}{\text{run}}$ formula as

$$m = \frac{\text{change in } y}{\text{change in } x}$$

The only issue we need to worry about is that we are consistent in the direction in which we do our subtractions. For example, using the same two points, (1, –3) and (4, 3), we can subtract in the reverse order from above, as long as both subtractions are reversed.

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{-3-3}{1-4} = \frac{-6}{-3} = \mathbf{2}$$

the same value of slope we calculated before.

EXAMPLE 1: **Find the slope of the line connecting the points (–7, –13) and (12, –10). Calculate the slope again by subtracting in the reverse direction.**

Solution: Subtracting in one direction computes the slope as

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{-13-(-10)}{-7-12} = \frac{-13+10}{-7-12} = \frac{-3}{-19} = \mathbf{\frac{3}{19}}$$

Reversing the direction in which we subtract the coordinates:

$$\text{Slope} = \Delta y / \Delta x$$

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{-10-(-13)}{12-(-7)} = \frac{-10+13}{12+7} = \mathbf{\frac{3}{19}}$$

Either way, we get the same slope; thus, the order in which you subtract is entirely up to you, as long as each subtraction (top and bottom) is done in the same direction.

New Notation

We're just about ready to calculate the slope of a line using the points mentioned in the Intro to this chapter: (π, 2000) and (3π, –5000). But first we introduce some new notation.

The natural world is filled with changes. In slope, we've seen changes in x and y in the notions of rise and run. In chemistry, there are changes in the volume and pressure of a gas. In nursing, there are changes in temperature and blood pressure, and in economics there are changes in supply and demand. This concept occurs so often that there's a special notation for a "change" in something. We use the Greek capital letter delta, Δ, to represent a change in something. A change in volume might be denoted by ΔV and a change in time by Δt. And so now we can redefine ***slope*** as

$$m = \frac{\Delta y}{\Delta x}$$

Slope is the *ratio* of the change in *y* to the change in *x*.

which is, of course, just fancy notation for what we already know.

EXAMPLE 2: **Find the slope of the line connecting the points (π, 2000) and (3π, –5000).**

Solution: A simple ratio calculation will give us the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{2000-(-5000)}{\pi-3\pi} = \frac{7000}{-2\pi} = \frac{\cancel{2}\cdot 3500}{-\cancel{2}\pi} = -\mathbf{\frac{3500}{\pi}}$$

Slope = $\Delta y / \Delta x$

In the last step of this calculation, we used the fact that a positive number divided by a negative number is negative. Also, we could obtain an approximate answer by dividing 3500 by 3.14, then attaching the negative sign, to get a slope of about **–1,114.65**.

Notice that there's no need to plot points and count squares on a grid. We've turned the geometric concept of slope into an arithmetic problem. Try reversing the order of the above subtractions to ensure you get the same slope.



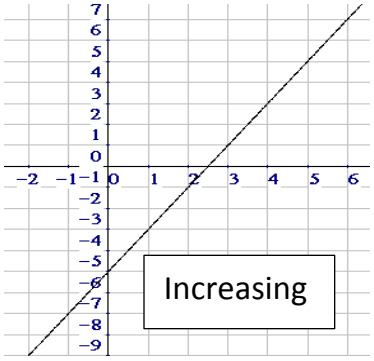
Why use delta, Δ, to represent a *change* in something? Because "delta" begins with a *d*, and *d* is the first letter of the word "difference," and difference means "subtract," and subtract is what you do when you want to calculate the change in something. (And it's the logo for Delta Airlines!)

Homework

1. Use the formula $m = \frac{\Delta y}{\Delta x}$ to find the slope of the line connecting the given pair of points:

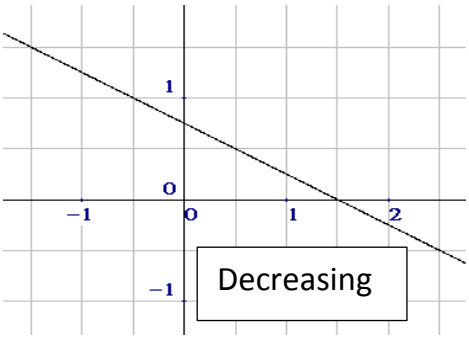
   a. (2, 3) and (4, 7)
   b. (–3, 0) and (0, 6)
   c. (1, –3) and (–2, 5)
   d. (2, 2) and (7, 7)
   e. (–3, –3) and (0, 0)
   f. (–1, –2) and (3, –5)
   g. (1, 1) and (–2, 3)
   h. (1, 4) and (0, 0)
   i. (–3, –2) and (1, –3)
   j. (–1, 3) and (1, –3)
   k. (–4, 5) and (0, 0)
   l. (–1, –1) and (4, –2)

❑ THE SLOPES OF INCREASING AND DECREASING LINES



Looking back at Example 1 from the chapter *Slope = Rise/Run*, let's make a quick sketch of the line. We can call this an "increasing" line because *as we move from left to right*, the line is rising, or going up, since the y-values are getting bigger. Now notice that the slope of this line, as calculated before, was 2, a <u>positive</u> number.

Referring to Example 2 from that chapter, we find that its graph, unlike the previous one, is falling *as we move from left to right* — that is, we have a "decreasing" line. And this is because the y-values are getting smaller. Next, we note that the slope was calculated to be the <u>negative</u> number $-\frac{1}{2}$.



This connection between the "increasing/decreasing" of a line and the sign of its slope is always true. Our conclusion is:

> An increasing line has a positive slope, while a decreasing line has a negative slope.

Slope = $\Delta y / \Delta x$

Homework

2. First find the slope of the line connecting the given pair of points. Then use that slope to determine whether the graph of the line is *increasing* or *decreasing*.

   a. (–10, 7) and (–12, –8)
   b. (12, –10) and (8, –5)
   c. (12, 3) and (–3, 10)
   d. (1, 3) and (10, 5)
   e. (–8, 10) and (12, 8)
   f. (–9, 1) and (–10, 11)
   g. (–2, –1) and (1, 5)
   h. (6, –1) and (–12, –1)
   i. (4, 6) and (9, –5)
   j. (3, –3) and (12, 6)

Solutions

1. a. 1 b. 2 c. –2 d. 3 e. –3 f. –1

g. $-\frac{1}{2}$ h. $\frac{2}{3}$ i. 3 j. $\frac{3}{2}$ k. $-\frac{2}{5}$ l. $\frac{3}{4}$

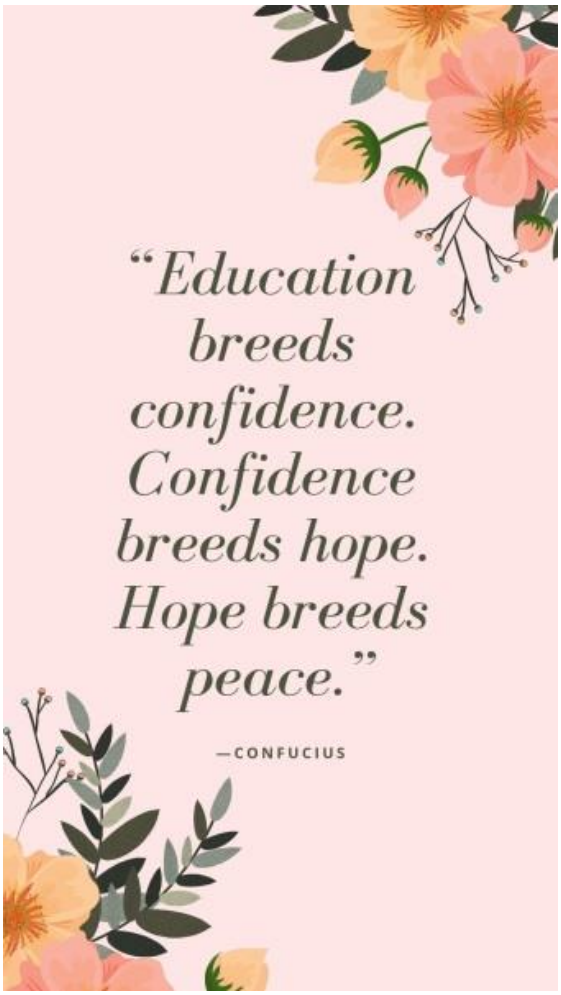
2. If the slope is positive, the line is increasing; if the slope is negative, the line is decreasing. But what about part h. of this problem?

a. $\frac{15}{2}$; inc b. $-\frac{5}{4}$; dec c. $-\frac{7}{15}$; dec

d. $\frac{2}{9}$; inc e. $-\frac{1}{10}$; dec f. –10; dec

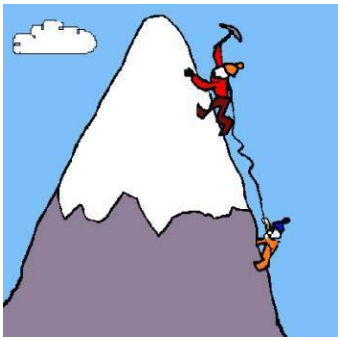
g. 2; inc h. 0; ??? i. $-\frac{11}{5}$; dec

j. 1; inc



Slope = $\Delta y / \Delta x$

SLOPE = RISE/RUN



A line has many attributes, or characteristics. Two of the most important are its *intercepts* and its *slope*. The intercepts tell us where the line crosses the *x*-axis and the *y*-axis; they are excellent reference points. On the other hand, the ***slope*** of a line tells us how *steep* the line is — it's like the angle that a line makes, and is a concept used in economics, chemistry, calculus, statistics, construction, and mountain climbing.

❑ *SLOPE*



A trucker is keenly aware of the *grade*, or angle, of the road on which a truck travels — it determines the speed limit and the proper gear that the truck needs to be in. A roofer is concerned with a roof's *pitch*, or steepness. A construction worker must ensure that a wheelchair ramp has the correct *angle* with the street or sidewalk. All of these ideas are examples of the concept of "*steepness*."



We'll use the term ***slope*** to represent steepness, and give it the letter *m* (I don't know why — maybe *m* for mountain?).

In other countries, the slope is called the **gradient.**

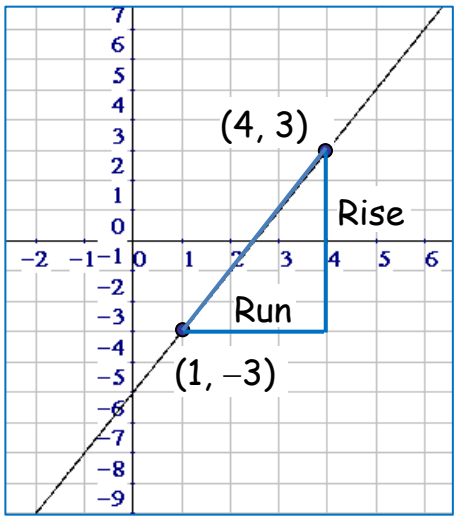
Our definition of slope in this course and all future math courses (and chemistry, economics, and nursing courses) is as follows:

$$m = \frac{\text{rise}}{\text{run}}$$

As we'll see shortly, a **rise** is a vertical (up/down) change, while a **run** is a horizontal (left/right) change. Slope is defined as the **ratio** of the rise to the run; we can also say that slope is the **quotient** of the rise and the run. In other words, to find slope, divide the rise by the run.

EXAMPLE 1: **Graph the line $y = 2x - 5$ and determine its slope.**

Solution: Let's calculate a couple of points by choosing some random x-values. If we let $x = 1$, then $y = -3$, so we get the point $(1, -3)$ on the line. And if we let $x = 4$, then $y = 3$, giving us the point $(4, 3)$. We could calculate more points for our line, but let's cut to the chase and graph the line given the two points just computed.



Notice that we've constructed a right triangle using the line segment between the two given points as the hypotenuse. The rise and run are then just the lengths of the legs of the triangle.

Slope = Rise/Run

Counting squares from left to right along the bottom of the triangle, we see that the run is **3**. Counting squares up the side of the triangle yields a rise of **6**. Using the slope formula, we can calculate the slope of the line:

$$m = \frac{\text{rise}}{\text{run}} = \frac{6}{3} = \boxed{2}$$

Note: The concept of slope is dimensionless; that is, slope has no units. Here's why: Suppose that the units in the triangle are in feet. Then the slope is

$$m = \frac{\text{rise}}{\text{run}} = \frac{6\,\text{ft}}{3\,\text{ft}} = \frac{6\,\cancel{\text{ft}}}{3\,\cancel{\text{ft}}} = \mathbf{2} \text{ (since the feet cancel out)}$$

EXAMPLE 2: **Find the slope of the line $2x + 4y = 3$.**

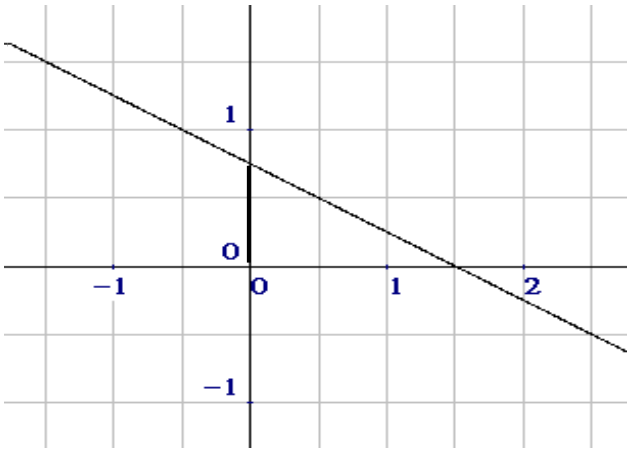
Solution: To graph this line, let's calculate the two intercepts (since they're generally the easiest points to calculate). Set $x = 0$ to get

$$2(\mathbf{0}) + 4y = 3 \;\Rightarrow\; 4y = 3 \;\Rightarrow\; y = \frac{3}{4}$$

Thus, the y-intercept is $(0, \frac{3}{4})$. If we set $y = 0$, we can solve for x:

$2x + 4(\mathbf{0}) = 3 \;\Rightarrow\; 2x = 3 \;\Rightarrow\; x = \frac{3}{2}$, which implies that the x-intercept is $(\frac{3}{2}, 0)$.

Plotting these two intercepts gives us our line:



As we move from left to right along the line between the two points, from the y-intercept to the x-intercept, we notice that the rise is actually a drop — this means that the rise is negative. Since the height of the triangle is $\frac{3}{4}$, we conclude that the "rise" is $-\frac{3}{4}$. Since the run is from left to right, the run is positive $\frac{3}{2}$. Now we're ready for the calculation:

$$m = \frac{\text{rise}}{\text{run}} = \frac{-\frac{3}{4}}{\frac{3}{2}} = -\frac{3}{4} \div \frac{3}{2} = -\frac{3}{4} \cdot \frac{2}{3} = \boxed{-\frac{1}{2}}$$

Note: Instead of moving from left to right, from the y-intercept to the x-intercept, we could also have moved from right to left, from the x-intercept to the y-intercept. In this case, the rise is positive because we're moving up, but the run is negative because we're moving to the left. This will still give us the same answer, since now the calculation looks like:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\frac{3}{4}}{-\frac{3}{2}} = \frac{3}{4} \div -\frac{3}{2} = \frac{3}{4} \cdot -\frac{2}{3} = \mathbf{-\frac{1}{2}}$$

Homework

1. For each pair of points, plot them on a grid, find the rise and the run, and then use the formula for slope to calculate the *slope* of the line connecting the two points:

   a. (2, 3), (4, 7) b. (–3, 0), (0, 6) c. (1, –3), (–2, 5)

   d. (2, 2), (7, 7) e. (–3, –3), (0, 0) f. (–1, –2), (3, –5)

2. Find the *slope* of the given line by graphing the line and using the rise and run. You may, of course, use any two points on the line to calculate the rise and the run:

   a. $y = x + 3$ b. $y = 2x - 1$ c. $y = -2x + 3$

   d. $y = 3x + 1$ e. $y = -3x - 2$ f. $y = -x + 2$

   g. $x + 2y = 4$ h. $2x - 3y = 1$ i. $3x - y = 3$

   j. $-3x + 2y = 6$ k. $2x + 5y = 10$ l. $3x - 4y = -8$

Solutions

1. a. 2 b. 2 c. $-\frac{8}{3}$ d. 1 e. 1 f. $-\frac{3}{4}$

2. a. 1 b. 2 c. –2 d. 3 e. –3 f. –1

g. $-\frac{1}{2}$ h. $\frac{2}{3}$ i. 3 j. $\frac{3}{2}$ k. $-\frac{2}{5}$ l. $\frac{3}{4}$

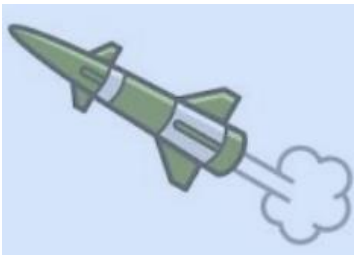
"Human history becomes more and more a race between education and catastrophe."

– H.G. Wells (1866-1946)



Slope = Rise/Run

3×3 Systems of Linear Equations and More Parabolas



There are many problems in math, science, and business that require more than two variables, for example, trying to find the path of a missile. To solve this problem, we will solve a system of *three equations* with *three variables*. [Note that a solution of such a system will consist of a set of three numbers that will satisfy all three of the equations.]

❑ *EXAMPLE*

Solve the system: $4x - 2y + 3z = -14$ [Equ 1]

$6x + y - z = 13$ [Equ 2]

$-x + 3y - 4z = 24$ [Equ 3]

Start with Equ 1 and Equ 2 to eliminate the x:

$4x - 2y + 3z = -14$ (times 3) $\Rightarrow$ $12x - 6y + 9z = -42$

$6x + y - z = 13$ (times −2) $\Rightarrow$ $-12x - 2y + 2z = -26$

Add: $-8y + 11z = -68$ [Equ 4]

Now use Equ 1 and Equ 3 to eliminate the x:

$4x - 2y + 3z = -14$ (leave it) $\Rightarrow$ $4x - 2y + 3z = -14$

$-x + 3y - 4z = 24$ (times 4) $\Rightarrow$ $-4x + 12y - 16z = 96$

Add: $10y - 13z = 82$ [Equ 5]

Solve the 2×2 system consisting of Equ 4 and Equ 5:

[Equ 4] $-8y + 11z = -68$ (times 5) $\Rightarrow$ $-40y + 55z = -340$

[Equ 5] $10y - 13z = 82$ (times 4) $\Rightarrow$ $40y - 52z = 328$

Add: $3z = -12$

$\Rightarrow$ $z = -4$

We now calculate y using Equ 4 and our newly found value of z:

$-8y + 11z = -68$

$\Rightarrow$ $-8y + 11(\underline{\mathbf{-4}}) = -68$

$\Rightarrow$ $-8y - 44 = -68$

$\Rightarrow$ $-8y = -24$

$\Rightarrow$ $y = 3$

And lastly, we solve for x using Equ 1 and our values of y and z:

$4x - 2y + 3z = -14$

$\Rightarrow$ $4x - 2(\underline{\mathbf{3}}) + 3(\underline{\mathbf{-4}}) = -14$

$\Rightarrow$ $4x - 6 - 12 = -14$

$\Rightarrow$ $4x - 18 = -14$

$\Rightarrow$ $4x = 4$

$\Rightarrow$ $x = 1$

We conclude that the solution of the system of 3 equations in 3 variables is

$$x = 1,\ y = 3,\ z = -4$$

We could CHECK our solution by plugging the values of x, y, and z into ALL THREE of the *original* equations.

Homework

Solve each system of equations:

1. $\begin{aligned} 2x - y + 3z &= 9 \\ x + 4y - z &= 6 \\ -3x + 2y + 2z &= 7 \end{aligned}$

2. $\begin{aligned} 3a - 4b + c &= 9 \\ -2a - 4b + 3c &= 19 \\ 5a + 3b - c &= -8 \end{aligned}$

3. $\begin{aligned} 2p + 3q - r &= 2 \\ -p - 3r + 2r &= 5 \\ 7p - 4p - 3r &= 23 \end{aligned}$

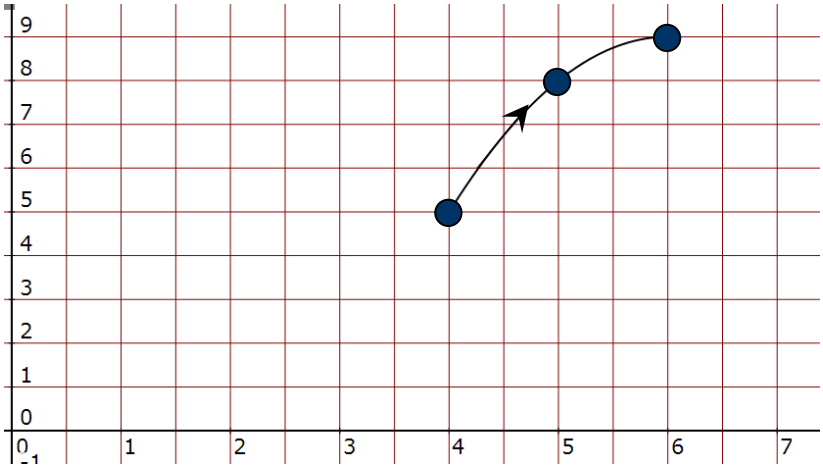
4. $\begin{aligned} 7x - 2y + 3z &= -12 \\ -5x + 7y - 4z &= 3 \\ 6x + 3y - 2z &= 6 \end{aligned}$

❑ *APPLICATION*



A missile is fired from an unknown location on the *x*-axis of the radar grid. Its first three positions are reported as (4, 5), (5, 8), and (6, 9), at which time the radar jams. Assume that the missile follows a parabolic trajectory (path) traveling east. Calculate both the *x*-value (on the *x*-axis) from where the missile was fired and the *x*-value (on the *x*-axis) where the missile will hit the ground.

Solution: Here's a graphic of the missile's trajectory, as determined by the three reported positions:



Where was it launched, and where will it land?

Our goal is to "extrapolate" (extend) this curve to a full parabola formula, so we can deduce where the missile came from (pretty important) and where it's going to land (REALLY important).

Beginning with $y = ax^2 + bx + c$, we get the following three equations, obtained by plugging each of the points on the parabola into the generic parabola equation:

$$y = ax^2 + bx + c$$

(4, 5): $\mathbf{5} = a(\mathbf{4}^2) + b(\mathbf{4}) + c \Rightarrow 16a + 4b + c = 5$

(5, 8): $\mathbf{8} = a(\mathbf{5}^2) + b(\mathbf{5}) + c \Rightarrow 25a + 5b + c = 8$

(6, 9): $\mathbf{9} = a(\mathbf{6}^2) + b(\mathbf{6}) + c \Rightarrow 36a + 6b + c = 9$

We now have a system of three equations in three unknowns, which can be solved using the techniques of this chapter. When you (yes, you!) solve the system you will determine that $a = -1$, $b = 12$, and $c = -27$, from which we deduce that the parabola which describes the trajectory is

$$y = -x^2 + 12x - 27$$

But we're still not done. (By the way, we will not be blown to bits by the missile while we wade through all these calculations. A computer can solve the problem in a billionth of a second — but remember: someone has to program the computer.) To find the firing point and the landing point on the x-axis, we now calculate the two x-intercepts of the parabola, which we do by setting y to 0:

$0 = -x^2 + 12x - 27$

$\Rightarrow \quad x^2 - 12x + 27 = 0$

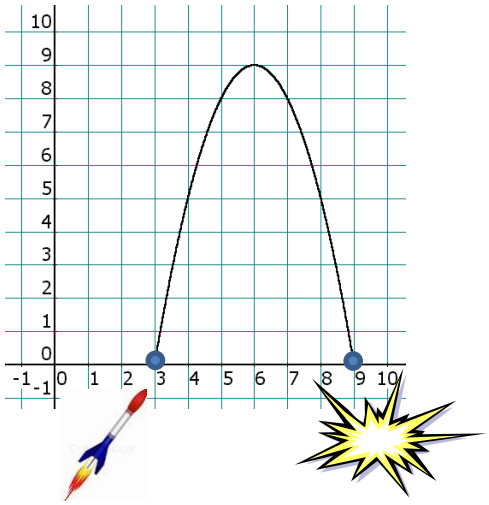
$\Rightarrow \quad (x - 3)(x - 9) = 0$

$\Rightarrow \quad x - 3 = 0$ or $x - 9 = 0$

$\Rightarrow \quad x = 3$ or $x = 9 \Rightarrow$ The x-intercepts are **(3, 0)** and **(9, 0)**.

We're now ready to answer the question at hand:

> The missile was fired from $\boldsymbol{x = 3}$, and will land at $\boldsymbol{x = 9}$.



Homework

5. Solve the missile trajectory problem if the missile was spotted at the points (4, 14), (5, 18), (7, 20) before the radar failed.

6. Solve the missile trajectory problem if the missile was spotted at the points (7, 10), (8, 12) and (9, 12).

7. Solve the missile trajectory problem if the missile was spotted at the points (1, 0), (3, 8) and (5, 8).

8. Solve the missile trajectory problem if the missile was spotted at the points (3, 3), (4, 4) and (5, 3).

Solutions

1. $x = 1, y = 2, z = 3$

2. $a = 0, b = -1, c = 5$

3. $p = 4, q = -1, r = 3$

4. $x = 1/3, y = -10/3, z = -7$

5. Fired from $x = 2$; will land at $x = 11$

6. Fired from $x = 5$; will land at $x = 12$

7. Fired from $x = 1$; will land at $x = 7$

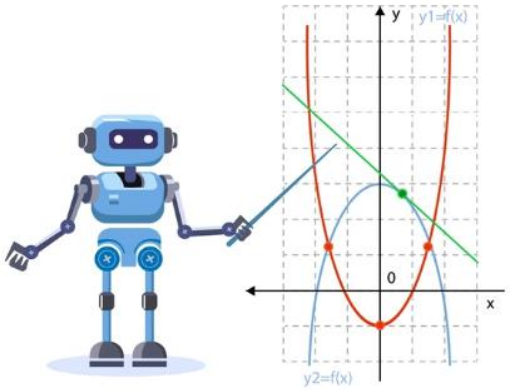
8. Fired from $x = 2$; will land at $x = 6$

"An education isn't how much you have committed to memory, or even how much you know. It's being able to differentiate between what you know and what you don't."

Anatole France

SYSTEMS OF LINEAR EQUATIONS – ELIMINATION

To solve the applications in future chapters, we need to be able to solve a system of two equations in two variables. A method that works very well in many cases is called the ***Elimination Method***. We multiply one or both equations by appropriate numbers (whatever that means), **add** the resulting equations to eliminate a variable, and then solve for the variable that survived.



❑ *EXAMPLES*

EXAMPLE 1: **Solve the system:** $\begin{aligned} 5x - 2y &= 20 \\ 3x + 7y &= -29 \end{aligned}$

Solution: In the Elimination Method we may eliminate either variable. But there's a certain "orderliness" that comes in handy in future math courses if we always eliminate the first variable, so in this case we will eliminate the x. As mentioned above, we multiply one or both equations by some numbers, add the resulting equations to kill off one of the variables, and then solve for the variable that still lives. How do we find these numbers? Rather than some mystifying explanation, just watch — you'll catch on.

$$5x - 2y = 20 \xrightarrow{\text{times } 3} 15x - 6y = 60$$
$$3x + 7y = -29 \xrightarrow{\text{times } -5} -15x - 35y = 145$$

Add the equations: $0x - 41y = 205$

The x's are gone!

Divide by –41: $\frac{-41y}{-41} = \frac{205}{-41} \Rightarrow \underline{y = -5}$

Now that we have the value of y, we can substitute its value of –5 into either of the two original equations to find the value of x. Using the first equation:

$$\begin{aligned} & 5x - 2(\mathbf{-5}) = 20 \\ \Rightarrow\ & 5x + 10 = 20 \\ \Rightarrow\ & 5x = 10 \\ \Rightarrow\ & \underline{x = 2} \end{aligned}$$

Therefore, the final solution to the system of equations is

$$\boxed{x = 2 \ \& \ y = -5}$$

EXAMPLE 2: **Solve the system:** $\mathbf{12a + 7b = 9}$, $\mathbf{-18a - 5b = 3}$

Solution: Notice how the numbers we choose to multiply each equation by will accomplish our goal: They make the a's disappear when the equations are added.

$$12a + 7b = 9 \xrightarrow{\text{times } 3} 36a + 21b = 27$$
$$-18a - 5b = 3 \xrightarrow{\text{times } 2} -36a - 10b = 6$$

Add the equations: $0a + 11b = 33$

Divide by 11: $\underline{b = 3}$

Placing this value of b into the first equation gives us

$$12a + 7(\mathbf{3}) = 9 \;\Rightarrow\; 12a + 21 = 9 \;\Rightarrow\; 12a = -12 \;\Rightarrow\; \underline{a = -1}$$

We now have our complete solution to the system of equations:

$$\boxed{a = -1 \;\&\; b = 3}$$

Let's use this example to learn how to check our solution. The main theme is this: The values of a and b must work in <u>both</u> of the <u>original</u> equations in order to constitute a valid solution.

1st equation:

$12a + 7b = 9$

$12(\mathbf{-1}) + 7(\mathbf{3}) \stackrel{?}{=} 9$

$-12 + 21 \stackrel{?}{=} 9$

$9 = 9$ ✓

2nd equation:

$-18a - 5b = 3$

$-18(\mathbf{-1}) - 5(\mathbf{3}) \stackrel{?}{=} 3$

$18 - 15 \stackrel{?}{=} 3$

$3 = 3$ ✓

Our conclusion is that the values $a = -1$ and $b = 3$ work perfectly. The solution can also be written as the ordered pair $(-1, 3)$.

Homework

1. Solve each system using the Elimination Method, and be sure you practice checking your solution (your pair of numbers) in <u>both</u> of the <u>original</u> equations:

a. $\begin{aligned} 2x + y &= 5 \\ -2x + 7y &= 19 \end{aligned}$

b. $\begin{aligned} 5a - 3b &= 5 \\ 10a + 4b &= -40 \end{aligned}$

c. $\begin{aligned} -2u - 3v &= -16 \\ -7u + 8v &= -56 \end{aligned}$

d. $\begin{aligned} 7x + 12y &= -24 \\ 6x - 7y &= 14 \end{aligned}$

e. $\begin{aligned} 3m - 2n &= 34 \\ -6m + n &= -62 \end{aligned}$

f. $\begin{aligned} -3s - 3t &= -24 \\ 10s + 8t &= 64 \end{aligned}$

g. $\begin{aligned} 2c - 3d &= 13 \\ 5c + 6d &= -8 \end{aligned}$

h. $\begin{aligned} -5w - 4x &= -20 \\ 20w + 3x &= 15 \end{aligned}$

i. $\begin{aligned} -5x - 4n &= -8 \\ 11x + 6n &= -2 \end{aligned}$

j. $\begin{aligned} 2w - 4a &= 6 \\ -3w + 9a &= -12 \end{aligned}$

k. $\begin{aligned} 2n - 3y &= -2 \\ 8n - 11y &= -2 \end{aligned}$

l. $\begin{aligned} 4c + 9y &= 4 \\ -5c - 11y &= 12 \end{aligned}$

m. $\begin{aligned} 5g - 2h &= -6 \\ 4g + 2h &= 3 \end{aligned}$

n. $\begin{aligned} -4w + 3h &= -1 \\ -3w + 4h &= 5 \end{aligned}$

o. $\begin{aligned} -3w + 4m &= 6 \\ -3w - m &= 1 \end{aligned}$

p. $\begin{aligned} 3a + 3q &= 1 \\ -5a + 5q &= 6 \end{aligned}$

Solutions

1. a. $x = 1, y = 3$

Complete Check:

$2x + y = 5$ | $-2x + 7y = 19$

$2(\mathbf{1}) + \mathbf{3} = 5$ | $-2(\mathbf{1}) + 7(\mathbf{3}) = 19$

$2 + 3 = 5$ | $-2 + 21 = 19$

$5 = 5$ ✓ | $19 = 19$ ✓

b. $a = -2, b = -5$

c. $u = 8, v = 0$

d. $x = 0, y = -2$

e. $m = 10, n = -2$

f. $s = 0, t = 8$

g. $c = 2, d = -3$

h. $w = 0, x = 5$

i. $x = -4, n = 7$

j. $w = 1, a = -1$

k. $n = 8, y = 6$

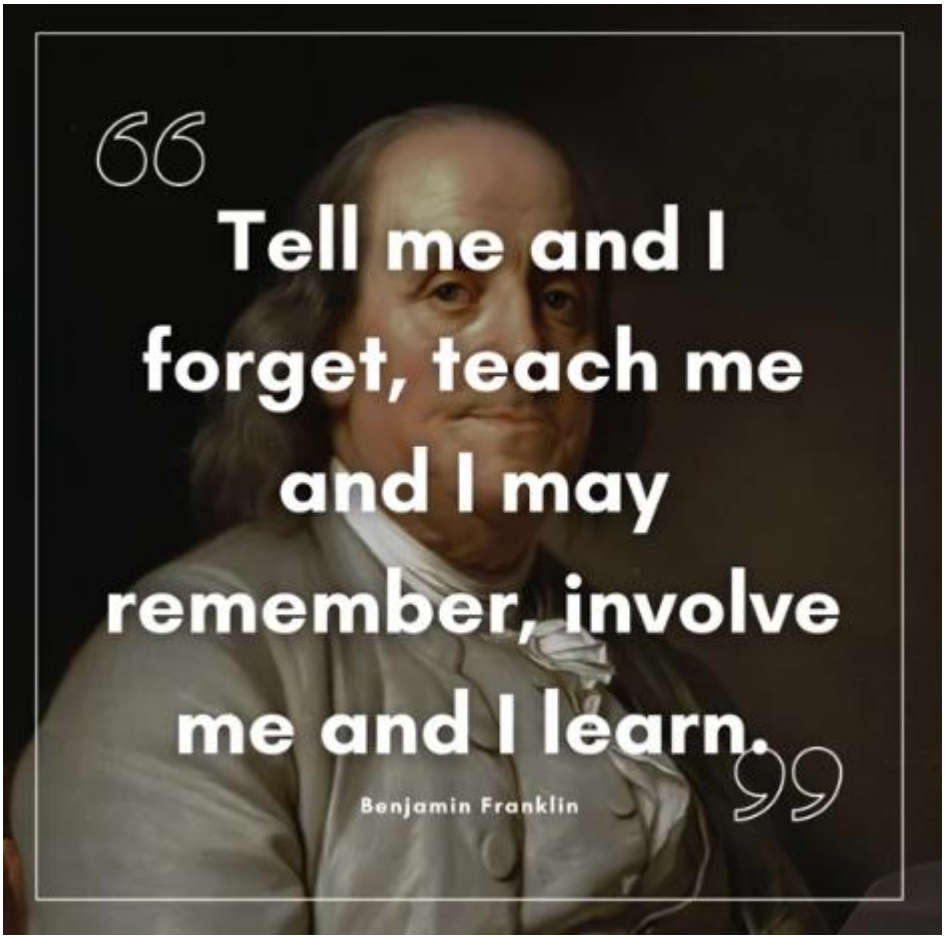
l. $c = -152, y = 68$

m. $g = -\frac{1}{3}, h = \frac{13}{6}$

n. $w = \frac{19}{7}, h = \frac{23}{7}$

o. $w = -\frac{2}{3}, m = 1$

p. $a = -\frac{13}{30}, q = \frac{23}{30}$


Tell me and I forget, teach me and I may remember, involve me and I learn.
Benjamin Franklin

SYSTEMS OF LINEAR EQUATIONS – GRAPHING

Consider the system of two equations in two variables:

$$\begin{aligned} x + y &= 7 \\ -2x + y &= 1 \end{aligned}$$

Assuming you know how to graph straight lines, we're going to learn a method to solve a system of two linear equations in two variables. Each equation in the system above is a line; assuming the lines intersect (and assuming they're not the same line), there will be one point of intersection. Since that point of intersection lies on both lines (is that clear?), the coordinates (the *x* and *y*) of that point must satisfy both of the equations.

❑ *TWO LINES*

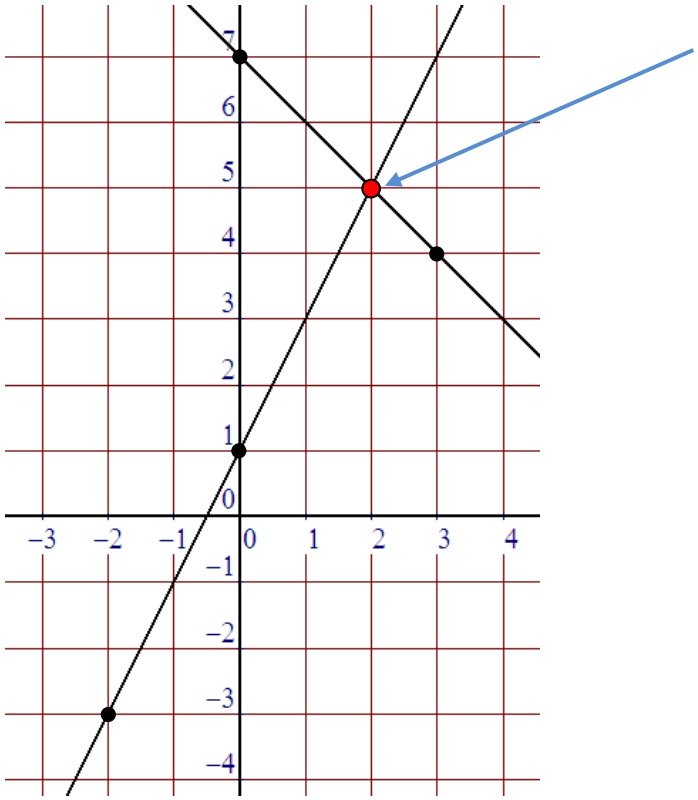
EXAMPLE 1: **Solve the system of equations** $\begin{aligned} x + y &= 7 \\ -2x + y &= 1 \end{aligned}$ **by graphing.**

Solution: Each equation is a line, so let's graph each of them and then turn our attention to their point of intersection.

Line #1: Solve for y to get $y = -x + 7$. If we let $x = 3$, then $y = 4$, and so **(3, 4)** is on the line. If we choose $x = 0$, then $y = 7$, in

which case **(0, 7)** is on the line. We'll use these two points for Line #1.

Line #2: Solve for y to get $y = 2x + 1$. If $x = 0$, then $y = 1$, and if $x = -2$, then $y = -3$. So we'll use the points **(0, 1)** and **(–2, –3)** for Line #2.



The point of intersection of the two lines appears to be the point (2, 5). This means that the solution of the given system of equations is

$x = 2, y = 5$

Important Note: Since we're reading points on a graph, it's very easy to misread them (imagine if the *x*- and *y*-coordinates were fractions or square roots). Thus, the graphing method yields only an *approximation*, but it's a great method when algebraic methods – for instance, Elimination and Substitution – fail to work.

Homework

Solve each system graphically:

1. $\begin{aligned} 2x + 3y &= 7 \\ -x - y &= 4 \end{aligned}$

2. $\begin{aligned} x - y &= 7 \\ x + 2y &= 0 \end{aligned}$

3. $\begin{aligned} x &= 4 \\ y &= 3x - 8 \end{aligned}$

4. $\begin{aligned} y &= -2 \\ 3x - 2y &= 0 \end{aligned}$

Solutions

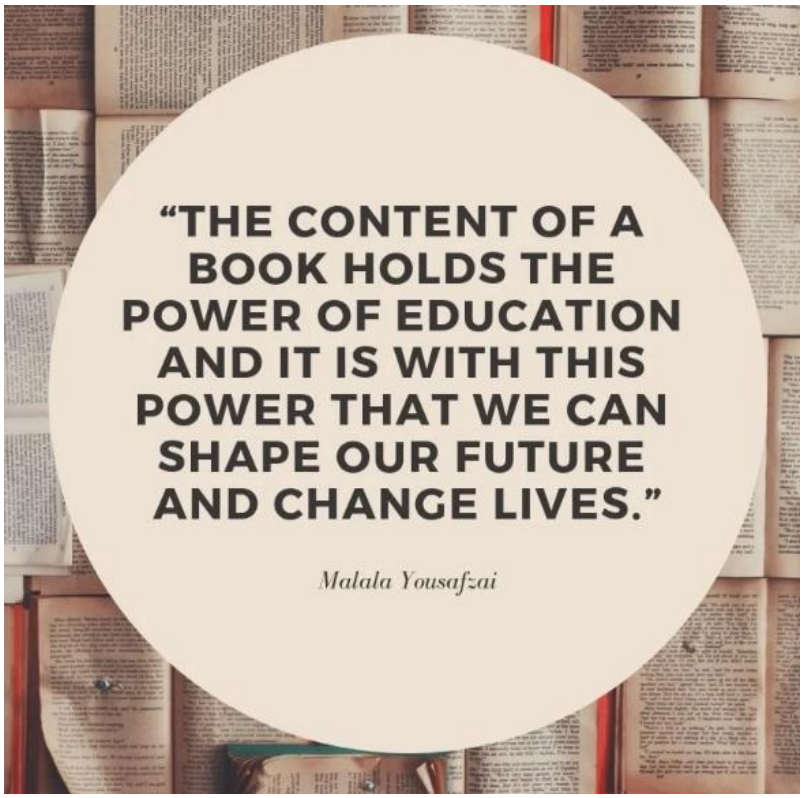
Remember, these are approximations from a graph.

1. $x = -9;\ y = 5$

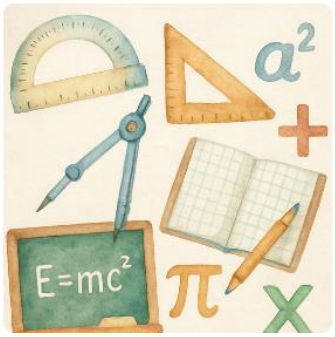
2. $x = 4.7;\ y = -2.3$

3. $x = 4;\ y = 4$

4. $x = -1.3;\ y = -2$


"THE CONTENT OF A BOOK HOLDS THE POWER OF EDUCATION AND IT IS WITH THIS POWER THAT WE CAN SHAPE OUR FUTURE AND CHANGE LIVES."
Malala Yousafzai

SYSTEMS OF EQUATIONS – AN INTRODUCTION



Here's an example of a ***system of two equations in two variables***:

$$x + y = 10$$
$$2x - 3y = 5$$

The *two equations* are easy to see, as are the *two variables*. The term *system* refers to the fact that these two equations are tied together — our final solution must be a pair of numbers, one for x and one for y, satisfying both equations.

❑ *WHAT DOES IT MEAN TO BE A SOLUTION OF A SYSTEM OF EQUATIONS?*

In the system above, the values $x = 8$ and $y = 2$ will satisfy the first equation [$8 + 2 = 10$ ✓], but will not satisfy the second equation [$2(8) - 3(2) = 16 - 6 = 10$, not 5]. Therefore, $x = 8$ and $y = 2$ is not a solution of the *system* of equations.

Sticking with the same system of equations above, the values $x = -8$ and $y = -7$ will satisfy the second equation [$2(-8) - 3(-7) = -16 + 21 = 5$ ✓] but will not satisfy the first equation [$-8 + (-7) = -15$, not 10]. Therefore, $x = -8$ and $y = -7$ is another pair of numbers which do not constitute a solution of the *system* of equations, either.

But $x = 7$ and $y = 3$ is a solution of the system of equations. Here's why:

$7 + 3 = 10$ ✓ *and* $2(7) - 3(3) = 14 - 9 = 5$ ✓

❑ *WHERE CAN SUCH A SYSTEM OF EQUATIONS ARISE?*

How does such a system of equations come about? How about a problem like the following:

Find two numbers whose sum is 17 and whose difference is 1.

If you wanted to solve this problem using Algebra, you would proceed something something like this: First give names to the two unknown numbers; maybe call them g and h.

Then the *sum* part of the problem yields the equation

$$g + h = 17$$

The *difference* part of the problem would imply that

$$g - h = 1 \qquad \text{[We will assume that } g > h\text{.]}$$

We now have a **system** of two equations in two variables: $\begin{aligned} g + h &= 17 \\ g - h &= 1 \end{aligned}$

Among the variety of methods available to solve such a system, pay attention to the method required by your instructor.

Homework

1. Consider the system of equations: $\begin{aligned} a + b &= 9 \\ a - b &= 7 \end{aligned}$

   a. Show that $a = 5$ and $b = 4$ <u>is</u> a solution of the first equation, but is <u>not</u> a solution of the system.

   b. Show that $a = 20$ and $b = 13$ <u>is</u> a solution of the second equation, but is <u>not</u> a solution of the system.

   c. Show that $a = 8$ and $b = 1$ <u>is</u> a solution of the system.

2. Try to solve the system $\begin{aligned} u+w &= 12 \\ u-w &= 0 \end{aligned}$ by guessing.

   Now try the system $\begin{aligned} 3x-17y &= 200 \\ -5x-12y &= 29 \end{aligned}$ by guessing (just kidding!).

3. Give an example of a system of three equations in three variables.

4. Translate the following word problem to a system of two equations in two variables: *The sum of two numbers is 25 and their product is 150.*

5. Translate the following word problem to a system of two equations in two variables: *The difference of two numbers is 25 and their quotient is 6.*

Solutions

1. a. $5 + 4 = 9$ ✓ But $5 - 4 = 1 \neq 7$.

b. $20 - 13 = 7$ ✓ But $20 + 13 = 33 \neq 9$.

c. $8 + 1 = 9$ ✓ And $8 - 1 = 7$ ✓

2. Both variables = 6, since 6 + 6 = 12, and 6 – 6 = 0.

3. Something like

$$\begin{aligned} 3x-2y+z &= 9 \\ 5x+y+10z &= 44 \\ -4x-5y-z &= -2 \end{aligned}$$

4. $x + y = 25$
$xy = 150$ [Of course, you can use any two variables you'd like.]

5. $a - b = 25$
$\frac{a}{b} = 6$ [We're assuming that $a > b$ and that $b \neq 0$.]



"To educate a man in mind and not in morals is to educate a menace to society."

Theodore Roosevelt
(1858–1919)

SYSTEMS OF LINEAR EQUATIONS – SUBSTITUTION

One of the methods that's really useful in solving systems of equations is the Substitution Method, where a variable is solved for in one equation, and then its value is substituted into another equation.

We'll start with the simplest substitution problem, which turns out to be very useful in calculus.

EXAMPLE 1: **Solve the system for *x*:** $y = 3x - 7$
$y = -9x + 8$

Solution: Since the two expression are both equal to y, they must be equal to each other:

$$3x - 7 = -9x + 8$$

$$\Rightarrow \quad 12x = 15$$

$$\Rightarrow \quad x = \frac{15}{12}$$

$$\Rightarrow \quad \boxed{x = \frac{5}{3}}$$

By the way, we've just calculated the *x*-coordinate of the *point of interection* of the lines y = 3x –7 and y = –9x + 8.

❑ *FRACTIONS NOT NEEDED*

EXAMPLE 2: **Solve the system:** $\begin{aligned} 3x + y &= 12 \\ -5x + 2y &= 13 \end{aligned}$

Solution: The command "Solve the system" means we must find the values of both x and y that make both equations true.

The first step is to choose an equation and a variable within that equation to solve for. For this system, the best variable to solve for is the y in the first equation. Why y? Because its coefficient is 1, which means we avoid fractions when solving for y.

Solving the first equation for y gives us

$y = 12 - 3x$ (by subtracting $3x$ from each side)

Now we substitute the expression $12 - 3x$ into the second equation where the y is.

$-5x + 2y = 13$ (the second equation)

$\Rightarrow \quad -5x + 2(\mathbf{12 - 3x}) = 13$ (since $y = 12 - 3x$)

$\Rightarrow \quad -5x + 24 - 6x = 13$ (distribute)

$\Rightarrow \quad -11x + 24 = 13$ (combine the x's)

$\Rightarrow \quad -11x = -11$ (subtract 24 from each side)

$\Rightarrow \quad x = 1$ (divide each side by -11)

Since $y = 12 - 3x$, and we just found out that $x = 1$, we can solve for y:

$y = 12 - 3(\mathbf{1})$
$y = 12 - 3$
$y = 9$

$x = 1$ & $y = 9$

Notes: First, if you'd like to check our solution, substitute the values of x and y into both of the original equations. Second, what if none of the variables has a coefficient of 1? Well, you'll just have to

pick a variable and then deal with whatever fractions may arise. In the real world, however, it's probably best to solve the system using the Elimination Method.

Homework

1. Solve each system using the Substitution Method, and be sure you practice checking your solution (your pair of numbers) in both of the original equations:

a. $\begin{aligned} 2x + y &= 5 \\ -2x + 7y &= 19 \end{aligned}$ b. $\begin{aligned} 3m - 2n &= 34 \\ -6m + n &= -62 \end{aligned}$ c. $\begin{aligned} -3w + 4m &= 6 \\ -3w - m &= 1 \end{aligned}$

❑ *FRACTIONS NEEDED*

EXAMPLE 2: **Solve the system:** $\begin{aligned} \mathbf{5x + 3y} &= \mathbf{7} \\ \mathbf{4x - 2y} &= \mathbf{1} \end{aligned}$

Solution: There's no way to avoid fractions (can you see why?), so even though we could solve for either variable in either equation, for no particular reason we'll solve for x in the first equation:

$5x + 3y = 7$ (the first equation)

$\Rightarrow \quad 5x = -3y + 7$ (subtract $3y$ from each side)

$\Rightarrow \quad x = \dfrac{-3y + 7}{5}$ (*) (divided each side by 5)

Now substitute this value of x into the second equation:

$$4\left(\frac{-3y + 7}{5}\right) - 2y = 1$$

$$\Rightarrow \quad \frac{-12y+38}{5} - 2y = 1$$

Multiply both sides of the equation (that is, all three terms) by 5:

$$\Rightarrow \quad \left(\frac{-12y+28}{5}\right)[5] - 2y[5] = 1[5]$$

$$\Rightarrow \quad \left(\frac{-12y+28}{\cancel{5}}\right)[\cancel{5}] - 2y[5] = 1[5]$$

$$\Rightarrow \quad -12y + 28 - 10y = 5$$

$$\Rightarrow \quad -22y = -23$$

$$\Rightarrow \quad y = \frac{23}{22}$$

Sorry, but we're not done — we still need to find x, so we substitute the value of y just obtained into the equation for x (the one with the * next to it):

$$x = \frac{-3y+7}{5}$$

And since $y = \frac{17}{22}$, it follows that (hold on tight!)

$$x = \frac{-3\left(\frac{23}{22}\right)+7}{5} = \frac{\frac{-69}{22}+7}{5} = \frac{\frac{-69}{22}+\frac{7(\mathbf{22})}{\mathbf{22}}}{5} = \frac{\frac{-69}{22}+\frac{154}{22}}{5} = \frac{\frac{85}{22}}{5}$$

$$= \frac{85}{22}\times\frac{1}{5} = \frac{5\cdot 17}{22}\times\frac{1}{5} = \frac{\cancel{5}\cdot 17}{22}\times\frac{1}{\cancel{5}} = \frac{17}{22}$$

And so our final solution is

$$\boxed{x = \frac{17}{22} \text{ and } y = \frac{23}{22}}$$

Homework

2. Solve each system using the Substitution Method:

a. $\begin{aligned} 2x+3y &= 7 \\ 4x-2y &= 11 \end{aligned}$ b. $\begin{aligned} 5a-2b &= 5 \\ 4a-3b &= 9 \end{aligned}$ c. $\begin{aligned} 2w+3z &= 7 \\ w-3z &= 13 \end{aligned}$

Solutions

1. a. $x = 1, y = 3$ b. $m = 10, n = -2$ c. $w = -\frac{2}{3}, m = 1$

2. a. $x = \frac{47}{16}, \ y = \frac{3}{8}$ b. $a = -\frac{3}{7}, \ b = \frac{-25}{7}$ c. $w = \frac{20}{3}, \ z = \frac{-19}{9}$

"There are no shortcuts to any place worth going."

– Beverly Sills, Opera Soprano

UNIT CONVERSIONS

❑ THE BASIC FACTS OF LENGTH IN THE U.S. SYSTEM: INCHES (IN), FEET (FT), YARDS (YD), AND MILES (MI)

12 in = 1 ft
3 ft = 1 yd
5,280 ft = 1 mi

Note that the smallest of the four units is inch, while the largest is mile. The word **inch** comes from the Latin word *unicia*, which means "twelfth part."

This system, along with units like ounces, tons, quarts, and gallons is sometimes called the U.S. system, since we're the only industrialized country in the world that still uses it.

❑ FOUR BASIC FACTS FROM ARITHMETIC

#1: Any quantity divided by 1 is itself. For example,

$$\frac{89}{1} = 89 \qquad \frac{x^2}{1} = x^2 \qquad \frac{8\,\text{yd}}{1} = 8\,\text{yd}$$

#2: Any quantity (except 0) divided by itself is 1. For instance,

$$\frac{23}{23} = 1 \qquad \frac{\pi}{\pi} = 1 \qquad \frac{12\,\text{inches}}{1\,\text{foot}} = 1$$

#3: Any quantity multiplied by 1 is itself.

$$\frac{23}{30}\times 1 = \frac{23}{30} \qquad N\cdot 1 = N \qquad 15\times \underbrace{\boxed{\frac{3\,\text{ft}}{1\,\text{yd}}}}_{=1} = 15$$

#4: We recall the procedure of "cross-canceling" when we multiply fractions:

$$\frac{8}{7}\times\frac{3}{8} = \frac{\cancel{8}}{7}\times\frac{3}{\cancel{8}} = \frac{3}{7}$$

This cross-canceling even works for units of measurement:

$$\frac{7\,\text{ft}}{1}\times\frac{12\,\text{in}}{1\,\text{ft}} = \frac{7\,\cancel{\text{ft}}}{1}\times\frac{12\,\text{in}}{1\,\cancel{\text{ft}}} = \frac{84\,\text{in}}{1} = 84\,\text{in}$$

This example is a preview of the task coming up, and shows you how to convert 7 ft into an equivalent number of inches.

Terminology: We call fractions like $\frac{12\text{ inches}}{1\text{ foot}}$ and $\frac{3\text{ ft}}{1\text{ yd}}$ **conversion factors**; the top and the bottom of the fraction are equal to each other, and therefore each fraction has the value 1. Can you explain why $\frac{1\text{ mi}}{5{,}280\text{ ft}}$ is a conversion factor? Notice that by flipping over each of these three conversion factors, we get three more conversion factors, for a total of six. It's these six conversion factors that are the basis of our method for converting between units of length in the U.S. system:

$$\frac{12\text{ inches}}{1\text{ foot}} \quad \frac{3\text{ ft}}{1\text{ yd}} \quad \frac{1\text{ mi}}{5{,}280\text{ ft}} \quad \frac{1\text{ foot}}{12\text{ inches}} \quad \frac{1\text{ yd}}{3\text{ ft}} \quad \frac{5{,}280\text{ ft}}{1\text{ mi}}$$

Each of these fractions is a *conversion factor*; the numerator and denominator are equal, and thus each fraction is equal to 1.

❑ *CONVERSION EXAMPLES*

There are many ways to convert from one unit of measurement to another. The method we'll use here is a little tricky at first, but it works beautifully in chemistry and other courses. In fact, the less you understand the units of measurement, the more valuable this method becomes.

EXAMPLE 1: **Convert 30 inches to feet.**

Solution:

Step 1: Put the given problem over 1: $\frac{30\text{ in}}{1}$

Step 2: To get rid of the inch unit, multiply by the conversion factor that will cancel that unit, $\frac{1\text{ ft}}{12\text{ in}}$:

$$\frac{30\text{ in}}{1}\times\frac{1\text{ ft}}{12\text{ in}}$$

Step 3: Now we cross-cancel the inches, do the arithmetic, leaving just the unit feet, our goal:

$$\frac{30\ \cancel{\text{in}}}{1}\times\frac{1\text{ ft}}{12\ \cancel{\text{in}}} = \frac{30\text{ ft}}{12} = \mathbf{2.5\ ft}$$

EXAMPLE 2: **Convert 3.2 miles to yards.**

Solution: We have no basic fact that directly connects miles and yards, but we can use feet as an intermediary, or in-between, step.

Step 1: Put the given problem over 1:

$$\frac{3.2\text{ mi}}{1}$$

Step 2: To rid ourselves of miles, multiply by the conversion factor $\frac{5{,}280\text{ ft}}{1\text{ mi}}$:

$$\frac{3.2\text{ mi}}{1} \times \frac{5{,}280\text{ ft}}{1\text{ mi}}$$

Step 3: Though the miles will cross-cancel, at this point we'd be at feet, not at our goal of yards. So, you know what? We tack on the additional conversion factor $\frac{1\text{ yd}}{3\text{ ft}}$; this will eliminate the feet and leave us with yards:

$$\frac{3.2\text{ mi}}{1} \times \frac{5{,}280\text{ ft}}{1\text{ mi}} \times \frac{1\text{ yd}}{3\text{ ft}}$$

Step 4: Now we cancel the miles, cancel the feet, do the arithmetic, and we're left with nothing but yards:

$$\frac{3.2\ \cancel{\text{mi}}}{1} \times \frac{5{,}280\ \cancel{\text{ft}}}{1\ \cancel{\text{mi}}} \times \frac{1\text{ yd}}{3\ \cancel{\text{ft}}} = \frac{16{,}896\text{ yd}}{3} = \mathbf{5{,}632\text{ yd}}$$

Homework

Using a calculator if you like, convert and round to the nearest hundredth:

1.	5 mi to yd	2.	4,224 ft to mi	3.	880 yd to mi
4.	90 in to yd	5.	1.7 yd to in	6.	2.4 mi to ft
7.	30 ft to yd	8.	2.7 ft to in	9.	30 in to ft
10.	1 mi to in	11.	12,672 in to mi	12.	11 in to ft
13.	15,840 ft to mi	14.	13 mi to yd	15.	44 ft to yd
16.	66 in to yd	17.	1.5 yd to in	18.	2 mi to in

19.	0.9 ft to in	20.	0.7 mi to ft	21.	0.5 yd to ft
22.	33 in to yd	23.	1.7 yd to ft	24.	1.5 mi to in
25.	14 in to ft	26.	7 ft to yd	27.	32 mi to yd
28.	1.7 mi to ft	29.	21 ft to yd	30.	16,352 yd to mi
31.	13 in to ft	32.	1.9 yd to ft	33.	69,696 in to mi

❑ *A Giant Length Unit*



An **astronomical unit** (AU) is the average distance from the Earth to the Sun, about 92,956,000 miles. This unit of length helps simplify calculations. The distance from the Sun to Pluto, for example, is 39.53 AU, which is a much easier number to work with compared to 3,674,590,210 miles. A **light-year** is even a longer distance — a much longer distance. Look it up.

❑ *Another Instance of Cross-Canceling Units*

You may have learned that multiplying the average rate (speed) that an object travels by the amount of time it travels gives us the distance it travels:

$$\boldsymbol{rt = d}$$

For instance, at a speed of 32 mph and a time of 5 hr, the distance is

$$rt = (32 \text{ mph})(5 \text{ hr}) = 160 \text{ mi}$$

But instead of writing "miles per hour" as mph, we can write it more mathematically as $\frac{\text{mi}}{\text{hr}}$. Now the same problem can be written like this:

$$rt = 32\,\frac{\text{mi}}{\text{hr}} \times 5\,\text{hr} = 32\,\frac{\text{mi}}{\text{hr}} \times 5\frac{\text{hr}}{1} = 32\,\frac{\text{mi}}{\cancel{\text{hr}}} \times 5\frac{\cancel{\text{hr}}}{1} = 160\,\text{mi}$$

We get the final unit in miles, just as distance (in this problem) should be, by cross-canceling the unit "hr".

❑ *The Basic Facts of Liquid Volume in the U.S. System: fluid ounces (fl oz) cups (c), cints (pt), quarts (q), and gallons gal)*

8 fl oz = 1 c
2 c = 1 pt
2 pt = 1 qt
4 qt = 1 gal

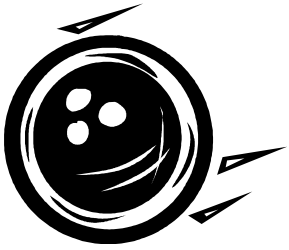
The word **gallon** comes from the Latin word *galleta*, meaning "bucket." And the term **cup** comes from the Latin word *cupa*, meaning "tub."



Here are some conversion factors that we will need in the homework:

$$\frac{1\text{ gal}}{4\text{ qt}} \qquad \frac{1\text{ qt}}{2\text{ pt}} \qquad \frac{2\text{ c}}{1\text{ pt}} \qquad \frac{8\text{ fl oz}}{1\text{ c}}$$

Remember that each fraction is a conversion factor because the numerator and the denominator are equal to each other (for example, 1 gal = 4 qt). This, of course, means that each conversion factor is ultimately equal to 1.



To help you understand that weight and volume are completely different concepts, consider that a volleyball and a bowling ball have about the same volume (they're about the same size), but they certainly do not have the same weight.



Conversion Examples:

EXAMPLE 1: **Convert 100 cups to quarts.**

Solution: Cups are smaller than quarts, so we expect that 100 cups will produce a smaller number of quarts. Using the appropriate conversion factors, we proceed as follows:

$$\frac{100\ \cancel{c}}{1} \times \frac{1\ \cancel{pt}}{2\ \cancel{c}} \times \frac{1\ qt}{2\ \cancel{pt}} = \frac{100}{4}\ qt = \mathbf{25\ qt}$$

EXAMPLE 2: **Convert 3.4 gallons to fluid ounces.**

Solution: Gallons are big compared to fluid ounces, so we expect that 3.4 gallons will convert to a whole lot of fluid ounces.

$$\frac{3.4\ \cancel{gal}}{1} \times \frac{4\ qt}{1\ \cancel{gal}} \times \frac{2\ \cancel{pt}}{1\ qt} \times \frac{2\ \cancel{c}}{1\ \cancel{pt}} \times \frac{8\ fl\ oz}{1\ \cancel{c}} = \mathbf{435.2\ fl\ oz}$$

Homework

Convert and round to the nearest hundredth, using a calculator if you like:

34.	7.1 qt to pt	35.	8.8 pt to c	36.	5.5 gal to c
37.	6.7 gal to qt	38.	24 c to gal	39.	68 fl oz to pt
40.	33 c to fl oz	41.	2.7 qt to c	42.	0.9 qt to fl oz
43.	6.6 gal to pt	44.	0.7 gal to fl oz	45.	71 pt to qt
46.	1.2 pt to fl oz	47.	4.9 c to pt	48.	5.2 pt to c
49.	34 fl oz to c	50.	407 pt to qt	51.	3.4 qt to gal
52.	55 c to fl oz	53.	0.6 gal to fl oz	54.	60 fl oz to pt

55. 8.8 gal to qt
56. 2.8 c to pt
57. 7.5 qt to c
58. 0.9 qt to fl oz
59. 484 pt to gal
60. 9.5 gal to c
61. 1.9 pt to fl oz
62. 8.8 gal to pt
63. 3.7 qt to pt
64. 2.5 gal to pt
65. 117 pt to qt
66. 2.3 c to pt
67. 4.1 gal to c
68. 4.5 qt to gal
69. 2.8 c to qt

Solutions

1. 8,800 yd
2. 0.8 mi
3. 0.5 mi
4. 2.5 yd
5. 61.2 in
6. 12,672 ft
7. 10 yd
8. 32.4 in
9. 2.5 ft
10. 63,360 in
11. 0.2 mi
12. 0.92 ft
13. 3 mi
14. 22,880 yd
15. 14.67 yd
16. 1.83 yd
17. 54 in
18. 126,720 in
19. 10.8 in
20. 3,696 ft
21. 1.5 ft
22. 0.92 yd
23. 5.1 ft
24. 95,040 in
25. 1.17 ft
26. 2.33 yd
27. 56,320 yd
28. 8,976 ft
29. 7 yd
30. 9.29 mi
31. 1.08 ft
32. 5.7 ft
33. 1.1 mi
34. 14.2 pt
35. 17.6 c
36. 88 c
37. 26.8 qt
38. 1.5 gal
39. 4.25 pt
40. 264 fl oz
41. 10.8 c
42. 28.8 fl oz

43.	52.8 pt	**44**.	89.6 fl oz	**45**.	35.5 qt
46.	19.2 fl oz	**47**.	2.45 pt	**48**.	10.4 c
49.	4.25 c	**50**.	203.5 qt	**51**.	0.85 gal
52.	440 fl oz	**53**.	76.8 fl oz	**54**.	3.75 pt
55.	35.2 qt	**56**.	1.4 pt	**57**.	30 c
58.	28.8 fl oz	**59**.	60.5 gal	**60**.	152 c
61.	30.4 fl oz	**62**.	70.4 pt	**63**.	7.4 pt
64.	20 pt	**65**.	58.5 qt	**66**.	1.15 pt
67.	65.6 c	**68**.	1.125 gal	**69**.	0.7 qt

“The highest activity a human being can attain is learning for understanding, because to understand is to be free.”

– Baruch Spinoza

VARIABLES AND IDENTITIES

❑ *INTRODUCTION*



First we need to know that there are many ways to indicate **multiplication**; for example, the product of 5 and 7 can be written in a variety of ways:

5×7 $\quad$ $5 \cdot 7$ $\quad$ $5(7)$ $\quad$ $(5)7$ $\quad$ $(5)(7)$

Each product is **35**

Repeated multiplication of the same number has its own terminology and notation. To ***square*** a number means to multiply that number by itself. For example, the "square of 5" is $5 \cdot 5 = 25$, and we can write $5^2 = 25$. This is read either "5 squared" or "5 to the second power" or sometimes just "5 to the second." In general, the square of n is written n^2. In the expression n^2, we call n the **base** and 2 the **exponent** (or **power)**.

Numbers that are multiplied together are called ***factors***.

As for what operations we do first, second, and so forth, we might recall from Algebra that one rule contained in the Order of Operations tells us that multiplication is done before addition and subtraction. Another part of the Order of Operations tells us that parentheses are always done first.

Here are three examples:

$7 + 2 \times 3 = 7 + 6 = \mathbf{13}$ $\quad$ Multiply before adding

$(3 + 10)(9 - 7) = (13)(2) = \mathbf{26}$ $\quad$ Parentheses before anything

$(8 + 2)^3 = 10^3 = 10 \cdot 10 \cdot 10 = \mathbf{1000}$ $\quad$ Parentheses before anything

❑ *IDENTITIES*

We continue with the following important definition:

An ***identity*** in algebra is a statement
that two expressions are equal.

If there are any letters in the expressions, then the two expressions must be equal no matter what numbers the letters might represent.

EXAMPLE 1: **Prove that $a + b = b + a$ is an identity.**

Solution: This statement is true because if a and b are replaced with any numbers at all, the two expressions will be equal. For instance, if we let $a = 10$ and $b = 5$, then the statement becomes

$$10 + 5 = 5 + 10$$

which is certainly true because each side of the identity is equal to 15. Since there are infinitely many numbers that a and b could be, we could never really prove that it's an identity with a slew of examples, so we're happy with a few examples that demonstrate its truth. In short, $a + b = b + a$ is an identity.

> There's a fancy word we use to describe the fact that addition can be done in either order. What has a judge done when she **commutes** a death sentence and changes it to life in prison? She has *reversed* the original sentence.
>
> When your parent **commutes** to work, they go in the morning and then *reverse* direction to get home in the evening, ending up where they started.
>
> Thus, to commute means to switch, or reverse. Since the statement $a + b = b + a$ is a switching of the order in which we add the a and the b, we say that addition is a ***commutative*** (accent on the '*mut*') operation.

Note #1: There's another commutative property; it's for multiplication. By using a pair of (different) numbers, convince yourself that $xy = yx$ is a true statement for any values of x and y. In short, multiplication is also a commutative operation.

Note #2: There would be no need for any mention of the commutative properties of addition and multiplication were it not for the fact that there are operations which are not commutative. For example, consider the division problem $\frac{6}{3}$, which equals 2. If we "commute" the top and bottom, we get the division problem $\frac{3}{6}$, which equals $\frac{1}{2}$. Since switching the two numbers in the division problem resulted in different answers, we conclude that division is not a commutative operation. Any ideas about whether subtraction is, or is not, a commutative operation?

EXAMPLE 2: **Prove that $(x + 5)(x - 5) = x^2 - 25$ is an identity.**

Solution: Let's plug the value $x = 8$ (I just chose this number off the top of my head) into both sides of this statement:

$(x + 5)(x - 5)$	$x^2 - 25$
$(\mathbf{8} + 5)(\mathbf{8} - 5)$	$\mathbf{8}^2 - 25$
$(13)(3)$	$64 - 25$
39	**39** ✓

Both sides resulted in the same number. Although we used only one value of x to test the statement, it truly is an identity; any number you use for x will result in equal numbers. Later in the course, we'll see a way to prove conclusively that $(x + 5)(x - 5) = x^2 - 25$ without using any number values of x at all!

On the other hand, to prove that something is not an identity, we need only give one example where the statement fails. Consider the next example, showing how we demonstrate that something is not an identity.

EXAMPLE 3: **Prove that $(x + 3)(x + 2) = x^2 + 6$ is not an identity.**

Solution: Let $x = 10$ (It's as good a number as any). Then, working each side of the statement separately, we obtain:

$(x + 3)(x + 2)$	$x^2 + 6$
$(\mathbf{10} + 3)(\mathbf{10} + 2)$	$\mathbf{10}^2 + 6$
$(13)(12)$	$100 + 6$
156	**106** ✘

An example which makes a statement fail is called a ***counterexample***.

We conclude that $(x + 3)(x + 2) = x^2 + 6$ is not an identity.

EXAMPLE 4: **Prove that $(x + 3)^2 = x^2 + 6x + 9$ is an identity.**

Solution: 7 is a nice number. Let's use it to test the statement:

$(x + 3)^2$	$x^2 + 6x + 9$	
$(\mathbf{7} + 3)^2$	$\mathbf{7}^2 + 6(\mathbf{7}) + 9$	
10^2	$49 + 42 + 9$	(multiply before adding)
100	**100** ✓	

A Final Reminder: Proving that an identity is true would theoretically require that every possible number value for each letter be tested. Since this is impossible to carry out, we are happy in this chapter to pick some numbers out of thin air and test them. If they work, we will assume that we have a genuine identity.

On the other hand, proving that an identity is false is relatively easy. We find one ***counterexample***, and that conclusively proves the

identity false, since that one exception would certainly prove that the identity can't be true for all numbers. By the way, try to avoid using the numbers 0, 1, and 2 for your experiments; these numbers can easily lead to bogus conclusions, as you will see in the following homework.

Homework

1. By using numbers, show that the statement $a(b + c) = ab + c$ is false (that is, show that it's not an identity).

   Note the *Order of Operations*: When you calculate the left side, do the parentheses first by adding *b* and *c*, and then multiply by *a*. On the right side, multiply and then add.

2. By using numbers, give some evidence that the statement $a(b + c) = ab + ac$ is true (that is, show that it's an identity).

3. You were advised above to avoid using the numbers 0, 1, or 2 to test whether a statement is an identity. Here's why:

   A. Show that if $x = 0$, the statement $4x = 3x$ is true.

   Now show that the statement $4x = 3x$ is not an identity.

   B. Show that if $n = 1$, the statement $n^2 = n^3$ is true.

   Now show that the statement $n^2 = n^3$ is not an identity.

   C. Show that if $a = 2$, the statement $a^2 = 2a$ is true.

   Now show that the statement $a^2 = 2a$ is not an identity.

4. Prove that the following is not an identity:

   $$(a + b)^2 = a^2 + b^2$$

5. Give some evidence that the following is an identity:

$$(a + b)^2 = a^2 + 2ab + b^2$$

6. Is $(a + b)(a - b) = a^2 - b^2$ an identity? Prove your contention.

7. Consider the statement: $(x + y)^3 = x^3 + y^3$. Decide if it's an identity, and give your reasoning.

Determine whether or not each statement is an identity:

8. $(x + 3)(x - 3) = x^2 - 6$ (choose x to be bigger than 3)

9. $(x + 3)(x - 3) = 2x - 9$ (choose x to be bigger than 3)

10. $(x + 3)(x - 3) = x^2 - 9$ (choose x to be bigger than 3)

11. $(a + b)(a - b) = 2a - 2b$ (choose a to be bigger than b)

12. $(a + b)(a - b) = a^2 + b^2$ (choose a to be bigger than b)

13. $(a + b)(a - b) = (a - b)^2$ (choose a to be bigger than b)

14. $(a + b)(a - b) = a^2 - b^2$ (choose a to be bigger than b)

15. $(n + 3)(n + 2) = 2n + 6$

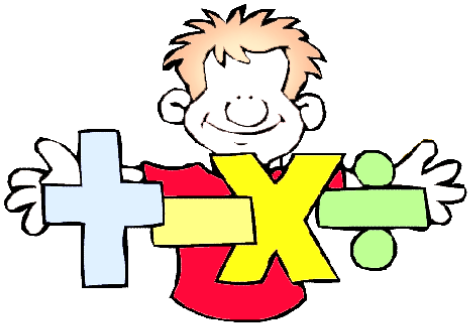
16. $(n + 3)(n + 2) = n^2 + 6$

17. $(n + 3)(n + 2) = n^2 + 5n + 6$

18. $(a + 5)^2 = 2a + 10$

19. $(a + 5)^2 = a^2 + 25$

20. $(a + 5)^2 = a^2 + 10a + 25$



❑ *CONSTANTS AND VARIABLES*

Quantities like 7, –80, π, and $\frac{2}{3}$ are called ***constants*** because their values never change. But quantities like x, y, N, and σ (Greek sigma) are called ***variables*** because we don't know their values, and so they could be just about any number at all. But the point is, although we write variables in algebra using letters of some alphabet (usually English, sometimes Greek), they're not really letters — *they're just symbols to represent numbers that we don't know*.

Sometimes we don't even care to find the value of a variable; for example, when we say that "the sum of a and b is written $a + b$," we're trying to make a statement that is valid for <u>any</u> numerical values of a and b.

Other times, for instance in the equation "$x + 9 = 11$," x is certainly a variable, but the goal is to figure out what x is, and in this case x turns out to be the constant 2.

Review Problems

21. Determine whether each statement is an identity, and prove your conclusion. Remember our agreement that you should not use 0, 1 or 2 for the values of your variables:

    a. $(w + 10)^2 = w^2 + 100$

    b. $(x + 1)(x + 2) = x^2 + 3x + 2$

    c. $x(y + z) = xy + z$

    d. $x(y + z) = xy + xz$

    e. $(a + b)(a - b) = a^2 - b^2$ (choose a bigger than b)

Solutions

1. For instance, let $a = 2$, $b = 3$, and $c = 4$. Then

$$a(b + c) = 2(3 + 4) = 2(7) = 14;$$

but, $ab + c = 2(3) + 4 = 6 + 4 = 10.$ ☹

2. Using the same numbers as above:

$$a(b + c) = 2(3 + 4) = 2(7) = 14;$$

and, $ab + ac = 2(3) + 2(4) = 6 + 8 = 14$ ☺

Note: Just because these numbers made the formula work doesn't necessarily mean that the formula works all the time (although it actually does work all the time). All we've done at this point is give some justification for the formula.

3. a. Since 4(0) is 0, and 3(0) is 0, it's true that $4x = 3x$ when $x = 0$. But if we choose x to be 5, for instance, 4(5) = 20, while 3(5) = 15. Hence, $4x = 3x$ is not an identity, because it's not true for all values of x. I'd like to see what you come up with for the rest of the problem.

4. Choose $a = 3$ and $b = 4$. (Choose any numbers you want, but avoid using 0, 1, and 2.)

$(a + b)^2$	$a^2 + b^2$
$(3 + 4)^2$	$3^2 + 4^2$
7^2	$9 + 16$
49	25 ☹

5. Choose a pair of numbers for a and b. Then test those numbers in the statement and show that you get the same result for each side.

6. Test this theory as in the previous problem.

7. It's not an identity. For example, if we let $x = 2$ and $y = 3$, we get

$$(x + y)^3 = (2 + 3)^3 = 5^3 = 125;$$

but, $x^3 + y^3 = 2^3 + 3^3 = 8 + 27 = 35.$

8. Let $x = 5$, for instance. Then

$(x + 3)(x - 3) = (\mathbf{5} + 3)(\mathbf{5} - 3) = (8)(2) = 16$

$x^2 - 6 = \mathbf{5}^2 - 6 = 25 - 6 = 19$ ☹

The statement is therefore not an identity.

9. Not an identity

10. Let $x = 10$, for example. Then

$(x + 3)(x - 3) = (\mathbf{10} + 3)(\mathbf{10} - 3) = (13)(7) = 91$

$x^2 - 9 = \mathbf{10}^2 - 9 = 100 - 9 = 91$ ☺

This situation (letting $x = 10$) does not prove it conclusively; after all, maybe it works for $x = 10$ but not for some other value of x. Indeed, we should try the expression for other values of x — the more x's we use, the more sure we can be of our conclusion. But for this chapter, one good example is good enough.

11. No **12**. No **13**. No **14**. Yes **15**. No **16**. No

17. Yes **18**. No **19**. No **20**. Yes

21. a. No b. Yes c. No d. Yes e. Yes

Question: "How does a math teacher scold her child?"

Answer: "If I've told you *n* times, I've told you *n* + 1 times!"

"EDUCATION IS THE KEY TO UNLOCK THE GOLDEN DOOR OF FREEDOM."



– George Washington Carver

THE NUMBER ZERO

❑ PROPERTIES OF ZERO

$N + 0 = N$ for any number N.

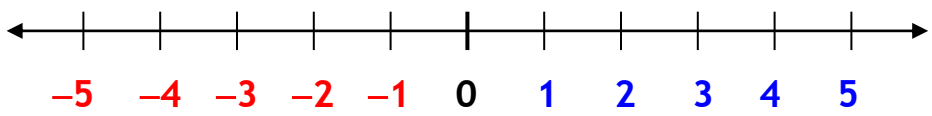
$N \cdot 0 = 0$ for any number N.

The ***opposite*** of 0 is 0.

0 has no ***reciprocal***. [If it did, it would be $\frac{1}{0}$, which is *undefined*, as described on Page 2.]

❑ THE NUMBER LINE

Here's a picture of the positive and negative numbers (and **zero**) called the ***number line***. It is also called the real line because all real numbers can be located somewhere on the line. So, even though I've labeled just a few integers on the line, all kinds of numbers, like $\frac{2}{3}$, π, $-\sqrt{7}$, 1.90977, and –2.37777. . . are all there somewhere.



The ***positive*** numbers are to the right of zero, while the ***negative*** numbers are to the left of zero. The positive number 4 could be written +4, but it's not necessary — we know that the plus sign is understood. By the way, 0 is neither positive nor negative; it's considered neutral, and is called the ***origin*** of the number line.

❑ *Division and Those Pesky Zeros*

One of the most important facts in all of mathematics is that the denominator (bottom) of a fraction can NEVER be zero.

Sometimes this is phrased

"Never divide by zero."



This is the result of dividing by zero.

What's the big deal? Why can't we do it?

Recall from the discussion of dividing signed numbers that

$$\frac{56}{7} = 8 \quad \text{because } 8 \times 7 = 56.$$

We don't have to blindly accept the fact that you should never divide by 0. Let's put zeros in fractions and see what happens.

Zero on the Top

How shall we interpret the division problem

$$\frac{0}{7} = ???$$

What number times 7 yields an answer of 0? Well, 0 works; that is,

$$\frac{0}{7} = 0 \quad \text{because } 0 \cdot 7 = 0.$$

Moreover, no other number besides 0 will work (confirm this yourself).

Zero on the Bottom

Now let's put a zero on the bottom (but not the top) and see what happens:

$$\frac{9}{0} = ???$$

Your first guess might be 0; let's check it out:

$\frac{9}{0} = 0$ would be true only if $0 \cdot 0 = 9$, which it isn't.

How about we try an answer of 9?

$\frac{9}{0} = 9$ which is checked by seeing if $9 \cdot 0 = 9$. Sorry.

Could the answer be π? Nope; $\pi \cdot 0 = 0$, not 9.

In fact, any number we conjure up as a potential answer will have to multiply with 0 to make a product of 9. But this is impossible, since any number times 0 is always 0, never 9. In short, *no number* in the whole world will work as the answer to this division problem.

Zero on the Top AND the Bottom

Now for an even stranger problem with division and zeros:

$\frac{0}{0} = ???$

We can try 0: $\frac{0}{0} = 0$ which seems reasonable, since $0 \cdot 0 = 0$.

Let's try an answer of 5: $\frac{0}{0} = 5$, which amazingly! — works out too, because $5 \cdot 0 = 0$.

Could π possibly work? $\frac{0}{0} = \pi$ — still true, since $\pi \cdot 0$ is surely 0.

That's three answers for this division problem: 0, 5, and π.

Is there any end to this madness? Apparently not, since any number we think up will multiply with 0 to make a product of 0. In short, *every number* in the whole world will work in this division problem.

SUMMARY:

1) Zero on the top of a fraction is perfectly okay, as long as the bottom is NOT zero. The answer to this kind of division problem is always zero. For example, $\frac{0}{7} = 0$.

2) There is no answer to the division problem $\frac{9}{0}$. Clearly, we can never work a problem like this.

3) There are infinitely many answers to the division problem $\frac{0}{0}$. This may be a student's dream come true, but in mathematics we don't want a division problem with trillions of answers.

Each of the two kinds of division problems with a zero in the denominator leads to a major conundrum, so we summarize cases 2) and 3) by stating that

DIVISION BY ZERO IS UNDEFINED!

And one more summary,

$$\frac{0}{7} = 0$$

$$\frac{9}{0} \text{ is undefined}$$

$$\frac{0}{0} \text{ is undefined}$$

"Black holes are where God divided by zero."

Steven Wright

Note: The two division problems with zero in the denominator may be undefined, but they're undefined for different reasons. Your teacher may require you to understand this.

❑ ZERO AS AN EXPONENT

We've already learned that anything to the zero power is 1 (as long as it's not zero to the zero). We reached this conclusion after we learned that $2^0 = 1$, and figured it might be true for any base. Now we try to verify this; that is, what is x^0? Consider the expression

$$x^3x^0 \qquad \text{where we assume } x \neq 0.$$

To figure out the meaning of x^0, we can use the first law of exponents to calculate

$$x^3x^0 = x^{3+0} = x^3$$

That is,

$$x^3x^0 = x^3$$

Now "isolate" the x^0, since that's what we're trying to find the value of. We do this by dividing each side of the equation by x^3:

$$\frac{x^3x^0}{x^3} = \frac{x^3}{x^3},$$

which implies that

$$x^0 = 1,$$

and we're done:

Any number (except 0) raised to the zero power is 1.

EXAMPLE:

A.	$(x - 3y + z)^0 = \mathbf{1}$	(any quantity ($\neq 0$) to the zero power is 1)
B.	$(abc)^0 = \mathbf{1}$	(any quantity ($\neq 0$) to the zero power is 1)
C.	$a + b^0 = \boldsymbol{a + 1}$	(the exponent is on the b only)
D.	$uw^0 = u(1) = \boldsymbol{u}$	(the exponent is on the w only)
E.	$(-187)^0 = \mathbf{1}$	(the exponent is on the –187)
F.	$-14^0 = \mathbf{-1}$	(the exponent in on the 14, not on the minus sign)